数理解析研究所講究録
1089 巻 1999 年 92-107

INTERMEDIATE PROCESSES BETWEEN BRANCHING PROCESSES AND FLEMING-VIOT PROCESSES

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ABSTRACT. Perkins [6] showed that the normalized binary branching process is a time inhomogeneous Fleming-Viot process. In the present paper we extend this result for jump-type branching processes. We show that the normalized jump-type branching process is a probability measure-valued process which will be called a "jump-type Fleming-Viot process". We also consider the intermediate processes between jump-type branching processes and Fleming-Viot processes, which are called "jump-type branching Fleming-Viot(-like) processes". In order to show the uniqueness we use the Perkins-type relations between these processes and jump-type Fleming-Viot processes.

1. INTRODUCTION

The *measure-valued branching processes* are of a typical example of population models, which are obtained as the limits of some suitable scaled branching particle systems (cf. Ethier and Kurtz [3] or Dawson [1]). Note that in the branching particle systems each particle moves and branches independently. In particular, in the binary branching case, each particle moves independently and dies at random time, and produces 0 or two offsprings with probability $1/2$.

On the other hand, the *Fleming-Viot process* is well-known as a typical model in the theory of population genetics introduced by Fleming and Viot [5] and investigated by many authors, e.g., Ethier and Kurtz [3], [4], Dawson [1], Donnelly and Kurtz [2] and so on. This is a probability measure-valued diffusion which is obtained as an infinite population limit of normalized empirical measure of a discrete genetic model with mutations (the simplest model is the *Moran particle system*, in which at each random time a pair of particles is selected and one particle jumps to the location of another particle.

These processes have evidently different properties. The branching process has no interaction, in particular, the binary branching process suffers extinction. On the other hand, the Fleming-Viot process has interaction and the total mass process is constant in time. However, there is a relationship between the binary branching process and the Fleming-Viot process. Perkins [6] established that the conditional law of the binary branching process given the total mass process is a time inhomogeneous Fleming-Viot process (see the last of this section with $\theta = 1$). However for the general measure-valued branching processes, such relations are not yet obtained.

Our aim of this paper is to obtain the Perkins-type relations for jump-type branching processes. In order to do it, we introduce a probability measure-valued process which will be called the *jump-type Fleming-Viot process* and give the Perkins-type relation between the jump-type branching process and the jump-type Fleming-Viot process. This result suggest that jump-type Fleming-Viot processes are very useful to built a larger class of measure-valued jump-type processes. We give two examples of such processes, which are the intermediate processes between branching processes and Fleming-Viot processes.

The paper is organized as follows.

In §2 we show the well-posedness of the martingale problem for the time inhomogeneous jump-type Fleming-Viot process. We also show that this process can be obtained as a scaling limit of a generalized Moran particle system.

In §3 we consider a simple model of branching particle systems with interaction. That is, at each random time one particle dies and produces j particles with probability q_j ($j = 0, 1, 2, \dots$) at the location of another particle. This model is called the *sampling branching particle system* or, simply, the *SB particle system*. We show that the same scaling limit exists, as in the branching case, and it is unique in the sense of the martingale problem. We call the limit process as the *jump-type branching Fleming-Viot process*. We also show that another scaling limit exists uniquely and the total mass process is an absorbing Lévy process. This limit process is called the *jump-type branching Fleming-Viot-like process*. Under the assumption that these limit processes are unique, the existence and the weak convergence can be shown by using the general theory described in [3]. However, for the uniqueness, it seems to be difficult to show by using the known methods. So we use the above Perkins-type result, that is, the normalized jump-type branching Fleming-Viot(-like) process is the time inhomogeneous jump-type Fleming-Viot process. The uniqueness of the jump-type branching Fleming-Viot process follows from the uniqueness of its total mass process and from the time inhomogeneous jump-type Fleming-Viot process.

Let S be a compact metric space, and set $\mathbf{D} = \mathbf{D}([0,\infty) \to S)$ be a path space of right continuous functions with left-hand limit. Let $(w(t), P_x)_{t\geq 0, x\in S}$ be a S-valued time homogeneous Borel strong Markov process starting from x with sample paths in $\mathbf{D}$. We denote the transition semi-group by (P_t) and the generator by A with a domain $D(A) \subset (C(S), \|\cdot\|)$, where $C(S)$ is a family of continuous functions on S and $\|\cdot\| = \|\cdot\|_\infty$ denotes the supremum norm. We suppose that this semi-group is a conservative Feller semi-group, i.e., a strongly continuous contraction conservative semi-group on $C(S)$.

Let $\mathcal{M}_F = \mathcal{M}_F(S)$ be a family of finite Radon measures on S with the weak topology, that is, $\mu_n \to \mu$ in $\mathcal{M}_F \iff \langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for every $f \in C(S)$, where $\langle \mu, f\rangle = \int f d\mu$. Then, $\mathcal{M}_F$ is a Polish space, i.e., complete separable metrizable space. The family of probability measures on S, $\mathcal{M}_1 = \mathcal{M}_1(S) \subset \mathcal{M}_F$, is a compact metric space (cf. Chap. 3 of [3]). For $\mu \in \mathcal{M}_F \setminus \{0\}$, we always denote $\overline{\mu} = \mu/\langle \mu, 1\rangle$.

It is well-known that if $(Z_t, \mathbf{P}_\mu)$ is a binary branching process starting from $\mu \in \mathcal{M}_F \setminus \{0\}$, then it is an $\mathcal{M}_F$-valued diffusion satisfying that $Z_t = Z_{t\wedge\tau_0}$ ($\tau_0 \equiv \inf\{s; Z_s = 0\} < \infty$ a.s.), and $\langle Z_t, f\rangle$ ($f \in D(A)$) has the following semi-martingale representation:

$$\langle Z_t, f\rangle = \langle Z_0, f\rangle + \int_0^t \langle Z_s, Af\rangle + M_t(f), \tag{1.1}$$

where $\{M(f)_t\}$ is a continuous martingale with quadratic variation

$$\langle\!\langle M(f)\rangle\!\rangle_t = \gamma \int_0^t \langle Z_s, f^2\rangle ds \quad (\gamma > 0).$$

If $(Y_t, \mathbf{P}_\mu^{(A,\gamma)})$ is a *Fleming-Viot process* ($\mu \in \mathcal{M}_1, \gamma > 0$), then it is an $\mathcal{M}_1$-valued diffusion and $\langle Y_t, f\rangle$ has the same type semi-martingale representation as in (1.1) with

$$\langle\!\langle M(f)\rangle\!\rangle_t = \gamma \int_0^t \left(\langle Y_s, f^2\rangle - \langle Y_s, f\rangle^2\right) ds.$$

If we state our results in the continuous case (i.e., the binary branching case), then they are as follows. Consider two kinds of scaled sampling binary branching particle systems $\{Z_{n,t}\}$. Then each converges weakly to an $\mathcal{M}_F$-valued processes $\{Z_t\}$ such that $Z_t = Z_{t\wedge\tau_0}$, which has the same semi-martingale representation as in (1.1). One is the binary branching Fleming-Viot process and, with a branching probability $\theta \in [0,1]$ at the same location, the quadratic variation part is given as,

$$\langle\!\langle M(f)\rangle\!\rangle_t = \gamma\int_0^t \left[(3-2\theta)\langle Z_s, f^2\rangle - 2(1-\theta)\langle Z_s, f\rangle\langle \overline{Z_s}, f\rangle\right] I(s<\tau_0)ds,$$

where $\tau_0 \equiv \inf\{s; Z_s = 0\}$. Another one is the binary branching Fleming-Viot-like process with

$$\langle\!\langle M(f)\rangle\!\rangle_t = \gamma\int_0^t \left[(3-2\theta)\langle \overline{Z_s}, f^2\rangle - 2(1-\theta)\langle \overline{Z_s}, f\rangle^2\right] I(s<\tau_0)ds$$

(in this case the total mass process is an absorbing Brownian motion in $(0,\infty)$). Moreover let $\mathbf{Q}_y = \mathbf{P}_\mu \circ (\langle Z_\cdot, 1\rangle)^{-1}$ with $y = \langle \mu, 1\rangle > 0$ ($\mu \in \mathcal{M}_F \setminus \{0\}$) and set $g \in C_+ \iff g : [0,\infty) \to [0,\infty)$; g is continuous and there exists $\tau_g \in (0,\infty]$ such that $g>0$ on $[0,\tau_g)$, $g = 0$ on $[\tau_g, \infty)$. Then, with $a = \gamma(3-2\theta)$,

$$\mathbf{P}_\mu\left(\overline{Z_\cdot} \in B \mid \langle Z_\cdot, 1\rangle = g\right) = \mathbf{P}_{\overline{\mu}}^{(A,g/a)}(Y_\cdot \in B), \quad \text{for } Q_y\text{-a.a. } g \in C_+,$$

where $(Y_t, \mathbf{P}_{\overline{\mu}}^{(A,g)})$ is a time inhomogeneous Fleming-Viot process such that $Y_t = Y_{\tau_g}$ for $t \geq \tau_g$ and that the martingale part $\{M_t^g(f)\}$ has quadratic variation

$$\langle\!\langle M^g(f)\rangle\!\rangle_t = \int_0^t g(s)^{-1}\left(\langle Y_s, f^2\rangle - \langle Y_s, f\rangle^2\right) I(s<\tau_g)ds.$$

We extend these results for the jump-type branching Fleming-Viot(-like) processes in Theorem 3.1 and Theorem 3.2.

2. Jump-type Fleming-Viot processes and generalized Moran particle systems

The following give the definition of *time inhomogeneous jump-type Fleming-Viot processes* $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$. Let

$$D_+ \equiv \Big\{g : [0,\infty) \to [0,\infty);\ g \text{ is right-continuous and has left-hand limit, and there is } \tau_g \in (0,\infty] \text{ such that } g>0 \text{ on } [0,\tau_g),\ g=0 \text{ on } [\tau_g,\infty)\Big\}.$$

THEOREM 2.1. *Let $\mu \in \mathcal{M}_1$, $g \in D_+$. For $\omega \in \mathbf{D} \equiv \mathbf{D}([0,\infty) \to \mathcal{M}_1)$, set $Y_t(\omega) = \omega(t)$. Let $\nu(du)$ be a measure on $(0,\infty)$ satisfying that*

$$\int_0^\infty \left(u \wedge u^2\right)\nu(du) < \infty. \tag{2.1}$$

Fix $a \geq 0$. Then there is a unique distribution $\mathbf{P}_\mu^{(A,g,a,\nu)}$ on $\mathbf{D}$ satisfying the following:

(i) $Y_0 = \mu$, $Y_t = Y_{\tau_g -}$ $(t \geq \tau_g)$, $\mathbf{P}_\mu^{(A,g,a,\nu)}$*-a.s.,*

(ii) *For $f \in D(A)$,*

$$M_t(f) \equiv \langle Y_t, f\rangle - \langle Y_0, f\rangle - \int_0^t \langle Y_s, Af\rangle I(s<\tau_g)ds$$

is decomposed as $M_t(f) = M_t^c(f) + M_t^d(f)$, *where* $\{M_t^c(f)\}$ *is a continuous* L^2*-martingale with quadratic variation*

$$\langle\langle M^c(f)\rangle\rangle_t = a\int_0^t g(s)^{-1}\left[\langle Y_s, f^2\rangle - \langle Y_s, f\rangle^2\right] I(s<\tau_g)ds$$

and $\{M_t^d(f)\}$ *is a pure discontinuous martingale such that*

$$M^d(f)_t = \int_0^t \int_{\mathcal{M}_F} \frac{\langle \eta, 1\rangle/g(s-)}{1+\langle \eta, 1\rangle/g(s-)}\langle \overline{\eta} - Y_{s-}, f\rangle I(s<\tau_g)\widetilde{N}(ds, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ *is the martingale measure with compensator*

$$\widehat{N}(ds, d\eta) = ds\int_S Y_s(dx)\int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta).$$

REMARK 2.1. If $\nu(du) = \nu_\alpha^\beta(du) \equiv \alpha u^{-2-\beta}du$ ($\alpha \geq 0$, $0<\beta<1$), then the pure discontinuous martingale part is given as

$$M^d(f)_t = \int_0^t \int_{\mathcal{M}_F} \frac{\langle \eta, 1\rangle}{1+\langle \eta, 1\rangle}\langle \overline{\eta} - Y_{s-}, f\rangle I(s<\tau_g)\widetilde{N^g}(ds, d\eta),$$

where $\widetilde{N^g}(ds, d\eta)$ is the martingale measure with compensator

$$\widehat{N^g}(ds, d\eta) = dsg(s)^{-\beta}\int_S Y_s(dx)\int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta).$$

Before proceeding to the proof of Theorem 2.1 we investigate this process.

Let D^n be the algebra generated by $\{f_1(x_1)\cdots f_n(x_n); f_i \in D(A), i = 1,\ldots,n\}$. For $h(x) = h(x_1,\ldots,x_n) \in C(S^n)$, let

$$P_t^n h(x) = \prod_{i=1}^n P_t^{(i)} h(x_1,\ldots,x_n), \quad \text{where } P_t^{(i)} = P_t \text{ acts on } x_i.$$

For $h(x) = h(x_1,\ldots,x_n) \in D^n$, let

$$A^{(n)}h(x) = \sum_{i=1}^n A_i h(x_1,\ldots,x_n), \quad \text{where } A_i = A \text{ acts on } x_i.$$

For $\mu \in \mathcal{M}_1$, $h \in C(S^n)$, set $F(\mu; h) \equiv \langle \mu^n, h\rangle$. We also denote as $F_h(\mu) = F_\mu(h) = F(\mu;h)$.

In the following we assume $\mu \in \mathcal{M}_1$ and $n \geq 2$. For $g \in D_+$, $a \geq 0$, let

$$\begin{aligned}\mathcal{L}_t^g F_h(\mu) &= \langle \mu^n, A^{(n)}h\rangle + \frac{a}{2g(t)}\sum_{j\neq k}\left\{\langle \mu^{n-1}, \Theta_{j;k}h\rangle - \langle \mu^n, h\rangle\right\}\\ &+ \sum_{m=2}^n B_{m,n}(g(t)) \sum_{(j_1,\{j_2,\ldots,j_m\})}\left\{\langle \mu^{n-m+1}, \Theta_{j_1;j_2,\ldots,j_m}h\rangle - \langle \mu^n, h\rangle\right\},\end{aligned}$$

where $\Theta_{j_1;j_2,\ldots,j_m}h$ is the operator changing variables $(x_{j_2},\ldots,x_{j_m})$ of h to x_{j_1}. The above $\sum_{(j_1,\{j_2,\ldots,j_m\})}$ denotes the summation about j_1 in $\{1,\ldots,n\}$ and about $(m-1)$-combinations $\{j_2,\ldots,j_m\}$ choosing from $\{1,\ldots,n\}\setminus\{j_1\}$. Moreover

$$B_{m,n}(g) \equiv \frac{1}{m}\int_0^\infty \left(\frac{1}{1+u/g}\right)^{n-m}\left(\frac{u/g}{1+u/g}\right)^m \nu(du).$$

In particular, if $\nu(du) = \nu_\alpha^\beta(du) \equiv \alpha u^{-2-\beta}du$ ($\alpha \geq 0, 0 < \beta < 1$), then noting that $m-1-\beta$, $n-m+\beta+1 > 0$ ($m = 2, \dots, n$), we have

$$B_{m,n}(g) = \frac{\alpha}{mg^\beta} B(m-1-\beta, n-m+\beta+1)$$

with the beta function

$$B(p,q) = \int_0^1 u^p(1-u)^q du \quad (p,q>0).$$

$\mathcal{L}_t^g$ is the generator of the time inhomogeneous jump-type Fleming-Viot process $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$. In fact, for simplicity of the notations, we only consider the case of $\nu = \nu_\alpha^\beta$ and let $g(t) \equiv g > 0$ be a constant and $a = 0$. Then for $t < \tau_g$, $(Y_t, \mathbf{P}_\mu^{(A,g,0,\nu)})$ satisfies that

$$dY_t(f) = Y_t(Af)dt + \int \frac{\langle \eta, 1\rangle}{1+\langle \eta, 1\rangle} \langle \overline{\eta} - Y_{t-}, f\rangle \widetilde{N^g}(dt, d\eta), \quad Y_0 = \mu,$$

where $\widetilde{N^g}(ds, d\eta)$ is the martingale measure with compensator

$$\widehat{N^g}(dt, d\eta) = dsg^{-\beta} \int_S Y_t(dx) \int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta).$$

Thus by using Ito's formula the generator $\mathcal{L}^g$ is expressed as (note that μ is a probability measure)

$$\begin{aligned} \mathcal{L}^g F_h(\mu) &= \langle \mu^n, A^{(n)}h\rangle + g^{-\beta}\int_0^\infty \nu(du) \\ &\quad \Big[\int \mu(dx)\Big\langle \Big(\frac{1}{1+u}\mu + \frac{u}{1+u}\delta_x\Big)^n, h\Big\rangle - \langle \mu^n, h\rangle\Big] \\ &= \langle \mu^n, A^{(n)}h\rangle + g^{-\beta}\int_0^\infty \nu(du)\Bigg[\sum_{m=0}^n \Big(\frac{1}{1+u}\Big)^{n-m}\Big(\frac{u}{1+u}\Big)^m \\ &\quad \sum_{\{j_1,\dots,j_m\}} \int \mu(dx)\langle \mu^{n-m}, \Theta_{x,\{j_1,\dots,j_m\}}h\rangle - \langle \mu^n, h\rangle\Bigg], \end{aligned}$$

where $\langle \mu^{n-1}(\delta_x - \mu)_j, h\rangle$ denotes the integration of h in the variable x_j by $\delta_x - \mu$ and in the other variables by μ^{n-1}. $\sum_{\{j_1,\dots,j_m\}}$ denotes the summation about m-combinations $\{j_1,\dots,j_m\}$ choosing from $\{1,\dots,n\}$ and $\Theta_{x,\{j_1,\dots,j_m\}}h$ is the operator changing variables $(x_{j_1},\dots,x_{j_m})$ of h to x. Moreover the summation about $m = 0, 1$ of the last second term is expressed as

$$\begin{aligned} &\sum_{m=0}^1 \Big(\frac{1}{1+u}\Big)^{n-m}\Big(\frac{u}{1+u}\Big)^m \sum_{\{j_1,\dots,j_m\}} \int \mu(dx)\langle \mu^{n-m}, \Theta_{x,\{j_1,\dots,j_m\}}h\rangle - \langle \mu^n, h\rangle \\ &\qquad = \Big[\Big(\frac{1}{1+u}\Big)^n + \Big(\frac{1}{1+u}\Big)^{n-1}\Big(\frac{u}{1+u}\Big)n - 1\Big]\langle \mu^n, h\rangle \\ &\qquad = -\sum_{m=2}^n \binom{n}{m}\Big(\frac{1}{1+u}\Big)^{n-m}\Big(\frac{u}{1+u}\Big)^m \langle \mu^n, h\rangle. \end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}^g F_h(\mu) &= \langle \mu^n, A^{(n)}h\rangle + \alpha g^{-\beta}\sum_{m=2}^{n} B(m-1-\beta, n-m+\beta+1) \\ &\quad \sum_{\{j_1,\dots,j_m\}} \left[\int \mu(dx)\langle \mu^{n-m}, \Theta_{x,\{j_1,\dots,j_m\}}h\rangle - \langle \mu^n, h\rangle\right] \\ &= \langle \mu^n, A^{(n)}h\rangle + \alpha g^{-\beta}\sum_{m=2}^{n} B(m-1-\beta, n-m+\beta+1) \\ &\quad \frac{1}{m}\sum_{(j_1,\{j_2,\dots,j_m\})} \left[\langle \mu^{n-m+1}, \Theta_{j_1;j_2,\dots,j_m}h\rangle - \langle \mu^n, h\rangle\right].\end{aligned}$$

In order to show the uniqueness of the solution $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$ to the martingale problem, we use a function-valued dual process: for $g \in D_+$, fix $0 \le T < \tau_g$.and for each $0 \le s \le T$, let $\mathcal{G}_s^g F_\mu(h) = \mathcal{L}_{T-s}^g F_h(\mu)$. Moreover set

$$\gamma_{2,n}^0(s) = \frac{a}{2g(T-s)} + B_{m,n}(g(T-s)), \gamma_{m,n}^0(s) = B_{m,n}(g(T-s))(m = 3,\dots,n)$$

and $\gamma_{m,n}(s) = m\binom{n}{m}\gamma_{m,n}^0(s)$. We consider the following function-valued dual process $(H_s, \mathbf{Q}_h) = (H_s^T, \mathbf{Q}_h^T)$, $0 \le s \le T$ ($H_0 = h \in C(S^n)$ $\mathbf{Q}_h$-a.s.), with generator $\mathcal{G}_s^g F_\mu(h)$.

(i) If H_s jumps at $s = t$, then the process jumps form $h \in C(S^n)$ to $\Theta_{j_1;j_2,\dots,j_m}h \in C(S^{n-m+1})$ at rate $\gamma_{m,n}^0(t)$ independently for $m = 2,\dots,n$.

(ii) Between jumps H_s is deterministic and evolves according to the semi-group (P_t^n) with generator $A^{(n)}$.

(iii) After jumping to the space $C(S)$, the process is deterministic and evolves according to (P_t).

For $0 \le r \le t \le T$, set $\gamma_{m,n}(r,t) \equiv \int_r^t \gamma_{m,n}(s)ds$ and $x = (x_1,\dots,x_n)$. Then $V_{r,t}h(x) = V_{r,t}^{T,n}h(x) \equiv \mathbf{Q}_f^T[H_t \mid H_r = h(x)]$ $(h \in C(S^n), f \in C(S^N), N \ge n)$ satisfies the following:

$$\begin{aligned}V_{r,t}h(x) &= \exp\left[-\sum_{m=2}^{n}\gamma_{m,n}(r,t)\right] P_{t-r}^n h(x) + \sum_{m=2}^{n}\exp\left[-\sum_{k\neq m;2\le k\le n}\gamma_{k,n}(r,t)\right] \\ &\quad \sum_{(j_1,\{j_2,\dots,j_m\})}\int_r^t ds\gamma_{m,n}^0(s)\exp\left[-\gamma_{m,n}(r,s)\right] P_{s-r}^n\left(\Theta_{j_1;j_2,\dots,j_m}(V_{s,t}h)\right)(x).\end{aligned}$$

REMARK 2.2. This $(V_{r,t})$ is the transition semi-group of the *time inhomogeneous generalized Moran particle system.* Recall that the Moran particle system is a model such that a pair of particles is selected at random time and one particle jumps to the location of another particle. However this generalized Moran particle system is a model such that particles more than one are selected at random time and they jump at the same time to the location of one of them.

Proof of Theorem 2.1. We first mention independent particle system. Let $\mu = \mu^{(n)} = \sum_{k=1}^n \delta_{x_k}$ on S. Let $(X_t^0, \mathbf{P}_\mu^0)$ be an independent Markov particle system starting from μ associated with the motion process $(w(t), P_x)$, i.e., for independent motions $(w_k(t), P_{x_k}) \overset{(d)}{=}$

$(w(t), P_{x_k})$,

$$X_t^0 = \sum_{k=1}^n \delta_{w_k(t)} \quad \text{and} \quad \mathbf{P}_\mu^0 = \bigotimes_{k=1}^n P_{x_k}.$$

For any nonnegative bounded function f on S such that $1-\exp[-f] \in D(A)$, the generator $\mathcal{L}^0$ of this particle system is given by the following:

$$\mathcal{L}^0 e^{-\langle \cdot, f\rangle}(\mu) = -\langle \mu, e^f A(1-e^{-f})\rangle e^{-\langle \mu, f\rangle}.$$

Let $n \geq 1$. We consider the n-scaled particle system $Z_{n,t}^0 = X_t^{0,(n)}/n$. If $\mu_n = \mu_n^{(n)} \equiv \mu^{(n)}/n \to \mu \in \mathcal{M}_1$, then $Z_{n,t}^0 \Rightarrow Z_t^0 \equiv \mu P_t$: the deterministic process as $n \to \infty$ by a dynamical law of large numbers. The generator of Z_t^0 is given as

$$\mathcal{L}^{Z^0} e^{-\langle \cdot, f\rangle}(\mu) = -\langle \mu, Af\rangle e^{-\langle \mu, f\rangle}.$$

In fact the generator of $Z_{n,t}^0$ is given as

$$\begin{aligned} \mathcal{L}_n^{Z^0} e^{-\langle \cdot, f\rangle}(\mu_n) &= \mathcal{L}^0 e^{-\langle \cdot, f/n\rangle}(n\mu_n) \\ &= -\langle \mu_n, ne^{f/n} A(1-e^{-f/n})\rangle e^{-\langle \mu_n, f\rangle}. \end{aligned}$$

For $f \in D(A)$ such that $f \geq 0$, $\|f\| < 1$ and $n = 1, 2, \ldots$, if we set $f_n = -n\log(1-f/n)$, then

$$0 \leq f_n - f \leq \frac{1}{n}\frac{f^2}{1-f/n} \leq \frac{\|f\|^2}{n-\|f\|} \to 0 \quad (n \to \infty).$$

(note that $x < -\log(1-x) < x/(1-x)$ for $0 < x < 1$). Moreover

$$ne^{f_n/n} A(1-e^{-f_n/n}) = \frac{n}{1-f/n} A(f/n) = \frac{Af}{1-f/n}.$$

Hence we have as $n \to \infty$, $\exp[-\langle \mu_n, f_n\rangle] \to \exp[-\langle \mu, f\rangle]$ and

$$\begin{aligned} \mathcal{L}_n^{Z^0} e^{-\langle \cdot, f_n\rangle}(\mu_n) &= -\langle \mu_n, \frac{Af}{1-f/n}\rangle e^{-\langle \mu_n, f\rangle} \\ &\to -\langle \mu, Af\rangle e^{-\langle \mu, f\rangle}. \end{aligned}$$

Fix any $0 < T < \tau_g$. We first prove the existence and uniqueness in $\mathbf{D}_T \equiv \mathbf{D}([0,T] \to \mathcal{M}_1)$.

For the uniqueness, we consider the above function-valued dual process $(H_s^T, \mathbf{Q}_h^T)$ $(0 \leq s \leq T)$ corresponding to any solution $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$ to the martingale problem described in Theorem 2.1. For all $0 < r < t \leq T$, $\mu \in \mathcal{M}_1$, $f \in D^n$ $(n \in \mathbf{N})$, it holds that

$$\mathbf{P}_\mu^{(A,g,a,\nu)}[F(Y_t; h) \mid Y_r] = \mathbf{Q}_h^t[F(\eta; H_{t-r}^t)]|_{\eta=Y_r}.$$

This result can be easily checked by Th. 5.5.2 in [1] (we take r as t_1, t as t_1+t and set $\sigma = t_1 + t$, $\beta \equiv 0$ in the theorem). Hence the uniqueness of the solution $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$ in $\mathbf{D}_T$ follows. From this we can also see that the transition semi-group $(\mathbf{T}_{r,t})$ of the time inhomogeneous jump-type Fleming-Viot process $(Y_t, \mathbf{P}_\mu^{(A,g,a,\nu)})$ is given by

$$\mathbf{T}_{r,t} F_h(\mu) = F_{V_{0,t-r}^{t,n} h}(\mu) = \int_{S^k} (V_{0,t-r}^{t,n} h)(x) \mu^n(dx) \quad \text{if } h \in C(S^n).$$

The existence of the jump-type Fleming-Viot process can be shown by the same way as in case of the Fleming-Viot processes (refer to §5, §2 in [1]). In fact, for each integer n, let $\mathcal{M}_1^{(n)} = \mathcal{M}_1^{(n)}(S)$ be a family of counting measures on S of the form $\eta_n = \sum_{k=1}^n \delta_{x_k}/n$. We always assume that $\mu_n \in \mathcal{M}_1^{(n)} \to \mu \in \mathcal{M}_1$ and let $\mathcal{L}_{n,t}^g$ be the generator of scaled generalized Moran particle system $(Y_{n,t}, \mathbf{P}_{n,\mu_n})$. Moreover let $f_n = -n\log(1-f/n)$ for $f \in D(A)$ such that $\|f\| < 1$ and $\inf f > 0$, It is possible to show that for each $T < \tau_g$,

$$\lim_{n\to\infty} \sup_{t\le T} \sup_{\eta\in\mathcal{M}_1^{(n)}} |\mathcal{L}_{n,t}^g e^{-\langle\cdot,f_n\rangle}(\eta) - \mathcal{L}_t^g e^{-\langle\cdot,f\rangle}(\eta)| = 0. \tag{2.2}$$

In fact, for simplicity, we consider the case of $A = 0$, $a = 0$ and $\nu = \nu_\alpha^\beta$. Let $g(t) \equiv g > 0$ be a constant and we omit the notation "t". Note that for $\eta = \sum_j \delta_{x_j}/n \in \mathcal{M}_1^{(n)}$,

$$\begin{aligned}
&\mathcal{L}_n^g e^{-\langle\cdot,f\rangle}(\eta)\\
&= \sum_{m=2}^n \gamma_{m,n}^0 \sum_{j_1,\{j_2,\dots,j_m\}} \left\{\exp\left[-\frac{1}{n}\sum_{i=2}^m (f(x_{j_1}) - f(x_{j_i}))\right] - 1\right\} e^{-\langle\eta,f\rangle}\\
&= \sum_{m=2}^n \frac{\gamma_{m,n}^0}{n^{m-1}} \binom{n-1}{m-1} \sum_{j_1,\dots,j_m} \sum_{k=2}^\infty \frac{(-1)^k}{k!} \left[\frac{1}{n}\sum_{i=2}^m (f(x_{j_1}) - f(x_{j_i}))\right]^k e^{-\langle\eta,f\rangle}
\end{aligned}$$

(the first moment ($k=1$) is zero) and

$$\begin{aligned}
\mathcal{L}^g e^{-\langle\cdot,f\rangle}(\eta) &= g^{-\beta} \int \eta(dx) \int_0^\infty \nu(du)\\
&\quad \left\{\exp\left[-\frac{u}{1+u}\langle\delta_x - \eta, f\rangle\right] - 1 + \frac{u}{1+u}\langle\delta_x - \eta, f\rangle\right\} e^{-\langle\eta,f\rangle}\\
&= g^{-\beta} \sum_{k=2}^\infty \frac{(-1)^k}{k!} \int_0^\infty \nu(du) \left(\frac{u}{1+u}\right)^k \int \eta(dx)\langle\delta_x - \eta, f\rangle^k e^{-\langle\eta,f\rangle}.
\end{aligned}$$

Thus by the definition of

$$\gamma_{m,n}^0 = \frac{1}{mg^\beta} \int_0^\infty \left(\frac{1}{1+u}\right)^{n-m} \left(\frac{u}{1+u}\right)^m \nu(du),$$

it is enough to show that for each $k \ge 2$,

$$\begin{aligned}
&\sum_{m=k}^n \frac{1}{mn^{k+m-1}} \binom{n-1}{m-1} \left(\frac{1}{1+u}\right)^{n-m} \left(\frac{u}{1+u}\right)^m \sum_{j_1,\dots,j_m} \left[\sum_{i=2}^m (f(x_{j_1}) - f(x_{j_i}))\right]^k\\
&\qquad \to \left(\frac{u}{1+u}\right)^k \int \eta(dx)\langle\delta_x - \eta, f\rangle^k
\end{aligned}$$

uniformly in $\eta = \sum_j \delta_{x_j}/n \in \mathcal{M}_1^{(n)}$ as $n \to \infty$. Note that the main term of the expansion of $n^{-(m-1)}\sum_{j_2,\dots,j_m} \left[\sum_{i=2}^m (f(x_{j_1}) - f(x_{j_i}))\right]^k$ is

$$\begin{aligned}
&\sum_{l=0}^k (-1)^l \binom{m-1}{l} \sum_{k_2,\dots,k_m\ge 0; \sum k_i = k} k_2\cdots k_{l+1} \frac{k!}{k_2!\cdots k_m} f(x_{j_1})\langle\eta, f\rangle^l\\
&\approx m(m-1)\cdots(m-k+1) \sum_{l=0}^k (-1)^l \binom{k}{l} f(x_{j_1})\langle\eta, f\rangle^l
\end{aligned}$$

as $n \to \infty$. Moreover by using the relation

$$\begin{aligned}
\frac{1}{m}\binom{n-1}{m-1} m(m-1)\cdots(m-k+1) &= (n-1)(n-2)\cdots(n-k+1)\binom{n-k}{m-k}\\
&\approx n^{k-1}\binom{n-k}{m-k}
\end{aligned}$$

we can get the above result. Hence if we denote the transition semi-group of the empirical process of the generalized Moran particle system as $\mathbf{T}_{r,t}^{(n)}$, then by (2.2) we have

$$\lim_{n\to\infty} \sup_{0\le r\le t\le T} \sup_{\eta\in\mathcal{M}_1^{(n)}} |\mathbf{T}_{r,t}^{(n)} e^{-\langle\cdot,f_n\rangle}(\eta) - \mathbf{T}_{r,t} e^{-\langle\cdot,f_n\rangle}(\eta)| = 0$$

and by the Markov property of $\{Y_{n,t}\}_{t\le T}$ the convergence of the finite-dimensional distributions follows. Moreover since for $f \in D(A)$,

$$\langle Y_{n,t}, f\rangle - \langle Y_{n,0}, f\rangle - \int_0^t \langle Y_{n,s}, Af\rangle ds$$

is a $\mathbf{P}_{n,\mu_n}$-martingale and

$$\sup_n \mathbf{P}_{n,\mu_n}\left[\text{ess sup}_{t\le T}|\langle Y_{n,t}, Af\rangle|\right] \le \|Af\|,$$

$\{\langle Y_{n,t}, f\rangle\}$ is tight by Th. 9.4 in Chap. 3 (p 145) in [3]. Therefore by Th. 3.7.1 in [1] $(Y_{n,t}, \mathbf{P}_{n,\mu_n})_{t\le T}$ converges weakly to $(Y_t, \mathbf{P}_\mu)_{t\le T}$ in $\mathbf{D}_T$ (we denote $\mathbf{P}_\mu = \mathbf{P}_\mu^{(A,g,a,\nu)}$).

Thus $(Y_t, \mathbf{P}_\mu)_{t\le T}$ exists uniquely in $\mathbf{D}_T$ for all $T < \tau_g$.

In order to extend $T \to \infty$, we consider the stopped process $Y_{n,t}^{(k)} \equiv Y_{n,t\wedge(\tau_g-1/k)}$ for each fixed k. By the above argument for any $T' \ge \tau_g$, $(Y_{n,t}^{(k)}, \mathbf{P}_{n,\mu_n})_{t\le T'}$ converges weakly to $(Y_t^{(k)}, \mathbf{P}_\mu)_{t\le T'}$ as $n\to\infty$ and $Y_t^{(k)} = Y_t$ a.s. for $t \le \tau_g - 1/k$. The martingale part $\{M_t^{(k)}(f)\}$ of $\{\langle Y_t^{(k)}, f\rangle\}$ is given as

$$M_t^{(k)}(f) = \langle Y_t^{(k)}, f\rangle - \langle Y_0^{(k)}, f\rangle - \int_0^t \langle Y_s^{(k)}, Af\rangle I(s < \tau_g - 1/k) ds$$

and satisfies that

$$\sup_{t\le T', k\ge 1} |M_t^{(k)}(f)| \le 2\|f\| + T'\|A\|.$$

By Doob's maximal inequality

$$\mathbf{P}_\mu\left[\sup_{t\le T'} |M_t^{(k)}(f) - M_t^{(j)}(f)|^2\right] \le 4\mathbf{P}_\mu[|M_{T'}^{(k)}(f) - M_{T'}^{(j)}(f)|^2].$$

Since $\{M_{T'}^{(k)}(f)\}$ is a bounded martingale in k, the right-hand side converges to 0 as $j,k\to\infty$. Hence there is a suitable subsequence $\{k_j\}$ such that $\lim_{j\to\infty} M_\cdot^{(k_j)}(f) = M_\cdot^{(\infty)}(f)$ exists in $\mathbf{D}([0,T'] \to \mathbf{R})$ a.s. for all $T' \ge \tau_g$, and by the uniqueness it holds that $M_t^{(\infty)}(f) = M_t(f)$ for $t < \tau_g$. Therefore $M_{\tau_g-}(f)$ exists and it is possible to extend as $M_t(f) = M_{\tau_g-}(f)$ for $t \ge \tau_g$. This implies $Y_t = Y_{t\wedge\tau_g}$ for all t.

Finally the semi-martingale representation can be shown as in case of measure-valued branching processes (see the proof of Th. 6.1.3 in [1]). We complete the proof of Theorem 2.1.

□

3. Jump-type branching Fleming-Viot(-like) processes

Fix $N \ge 1$. Let $\mu^{(N)} = \sum_{k=1}^N \delta_{x_k}$. We first define sampling branching Markov particle systems $(X_t, \mathbf{Q}_{\mu^{(N)}})$. As in the proof of Theorem 2.1, we denote the independent particle system by $(X_t^0, \mathbf{P}_\mu^{(N)})$ associated with the motion process $(w(t), P_x)$ starting from $X_0^0 = \mu^{(N)}$. For each fixed $M = 0, 1, 2, \ldots$, let $\lambda = \lambda(M)$ be a nonnegative number, and if $M \ge 1$, then for $m = 1, \ldots, M$, $\{p_{m,k}^{(M)}\}_{1\le k\le M}$ be a probability. Also let $\{q_j\}_{j=0}^\infty$ be a probability.

We consider the following Markov particle system $(X_t, \mathbf{Q}_{\mu^{(N)}})$ starting from $\mu^{(N)}$: first N-particles move independently. After the independent $\lambda(N)$-exponential random time τ_1, one particle, for example, m-th particle is selected with probability $1/N$. At the same time another k-th particle is selected with probability $p_{m,k}^{(N)}$ (we admit the same particle can be selected). Then the m-th particle dies and produces j particles at the location of the k-th particle with probability q_j ($j = 0, 1, 2, \dots$). Note that the number of particles N changes to $\langle X_{\tau_1}, 1\rangle = N - 1 + j$. The resulting particles move independently. Again after the independent $\lambda(\langle X_{\tau_1}, 1\rangle)$-exponential random time τ_2, m'-th particle is selected with probability $1/\langle X_{\tau_1}, 1\rangle$ and at the same time $\tau_1 + \tau_2$, k'-th particle is selected with probability $p_{m',k'}^{(\langle X_{\tau_1},1\rangle)}$. Then the m'-th particle dies and produces j' particles at the location of the k'-th particle with probability $q_{j'}$ and the resulting particles move independently. These operations are continued. Of course if all particles die, then these operations will be stopped.

This particle system $(X_t, \mathbf{Q}_{\mu^{(N)}})$ is called the *sampling branching Markov particle system starting from $\mu^{(N)}$ associated with the motion process $(w(t), P_x)$, sampling rate function $\lambda = \lambda(M)$, sampling probability $\{p_{m,k}^{(M)}\}_{1\le k\le M}$ for $m = 1, \dots, M$ ($M = 1, 2, \dots$) and branching probability $\{q_j\}_{j\ge 0}$.*

Let $L_t(\mu^{(N)}) = \mathbf{Q}_{\mu^{(N)}}[\exp -\langle X_t, f\rangle]$ and $L_t^0(\mu^{(N)}) = \mathbf{P}^0_{\mu^{(N)}}[\exp -\langle X_t^0, f\rangle]$. Note that $N = \langle \mu^{(N)}, 1\rangle = \langle X_0^0, 1\rangle$, $\mathbf{P}^0_{\mu^{(N)}}$-a.s.. $L_t(\mu^{(N)})$ is the unique solution to the following equation:

$$\begin{aligned} L_t(\mu^{(N)}) &= e^{-\lambda(N)t} L_t^0(\mu^{(N)}) + \mathbf{P}^0_{\mu^{(N)}}\Bigg[\lambda(N)\int_0^t ds e^{-\lambda(N)s}\Bigg\{ I_{\{N=0\}} + \\ & I_{\{N\ge 1\}} \frac{1}{N}\sum_{m=1}^N\sum_{k=1}^N p_{m,k}^{(N)} \sum_{j=0}^\infty q_j L_{t-s}\left(X_s^0 - \delta_{w_m(s)} + j\delta_{w_k(s)}\right)\Bigg\}\Bigg]. \end{aligned}$$

Note that $L_t(0) = 1$. If we denote the generating function of $\{q_j\}$ as $G(z) = \sum_{j\ge 0} z^j q_j$, then the generator $\mathcal{L}$ is given as

$$\begin{aligned} \mathcal{L}e^{-\langle\cdot,f\rangle}(\mu^{(N)}) &= \mathcal{L}^0 e^{-\langle\cdot,f\rangle}(\mu^{(N)}) + \lambda(0) I_{\{N=0\}} \\ & + I_{\{N\ge 1\}} \frac{\lambda(N)}{N}\sum_{m=1}^N\sum_{k=1}^N p_{m,k}^{(N)}\left(e^{f(x_m)}G(e^{-f(x_k)}) - 1\right) e^{-\langle\mu^{(N)},f\rangle}. \end{aligned}$$

We set the domain of $\mathcal{L}$ by

$$\mathcal{D}_0(\mathcal{L}) \equiv \text{lin span}\left\{e^{-\langle\mu,f\rangle}; f = -\log(1-g), 0 \le g < 1, g \in D(A)\right\}.$$

Then it is easy to see that $(X_t, \mathbf{Q}_{\mu^{(N)}})$ is a Markov process with sample paths in $\mathbf{D} \equiv \mathbf{D}([0,\infty) \to \mathcal{M}_F(S))$ and the unique solution to the martingale problem for $(\mathcal{L}, \mathcal{D}_0(\mathcal{L}))$ on $\mathbf{D}$.

Next we consider the scaled SB particle system $(Z_{n,t}, \mathbf{P}_{n,\mu_n})$ and show that the two kinds of scaling limit exist.

For $n, N = 1, 2, \dots$, set $\mu^{(N)} = \sum_{k=1}^N \delta_{x_k}$ and $\mu_n^{(N)} = \mu^{(N)}/n$. Suppose that $\mu_n^{(N)} \to \mu(\neq 0) \in \mathcal{M}_F$ as $n \to \infty$. Thus N depends on n and $N/n \to \langle \mu, 1\rangle \in (0, \infty)$ as $n \to \infty$. We consider the following.

Branching mechanism;

$$\Psi(v) \equiv \frac{c}{2}v^2 + \int_0^\infty \left[e^{-vu} - 1 + vu\right] \nu(du) \ (\geq 0),$$

where $c \geq 0$ is a constant and $\nu(du)$ is a measure on $(0, \infty)$ satisfying the condition (2.1). Then

$$\lim_{n\to\infty} \frac{\Psi'(n)}{n} = c$$

(in particular, we mainly consider the case, with some constants $\alpha \geq 0$, $0 < \beta < 1$,

$$\nu(du) = \nu_\alpha^\beta(du) \equiv \alpha \frac{du}{u^{2+\beta}} \quad \text{i.e.,} \quad \Psi(v) = \frac{c}{2}v^2 + \frac{\alpha\Gamma(1-\beta)}{\beta(1+\beta)} v^{1+\beta},$$

where Γ denotes the gamma function). Moreover set $a_n \equiv \Psi'(n) \geq 0$ and let

$$G_n(z) = \Psi(n(1-z))/(na_n) + z.$$

It is easy to check that this is a generating function and thus the branching probability $\{q_j^{(n)}\}$ is defined by $G_n(z) = \sum_j q_j^{(n)} z^j$.

Sampling rate functions; $\lambda_n(0) = 0$ and with $\gamma > 0$ for $M = 1, 2, \ldots$,

$$\text{(i) } \lambda_n(M) = \gamma M a_n, \qquad \text{(ii) } \lambda_n(M) = \gamma n a_n.$$

Sampling probability $\{p_{m,k}^{(M)}\}$; for each $M = 1, 2, \ldots$, let r_M be given in $[0,1]$, but $r_1 = 1$. Set $p_{1,1}^{(1)} = 1$, and if $M \geq 2$, then for each $m = 1, \cdots, M$,

$$p_{m,k}^{(M)} = \begin{cases} r_M & (k = m), \\ s_M = (1 - r_M)/(M-1) & (k \neq m). \end{cases}$$

Moreover let $\overline{r}_M = r_M - s_M$. For convenience, we further set $s_1 = 0$ and $\overline{r}_1 = 1$. In order to take the limit, we further assume that there exist constants $\theta \in [0,1]$ and $M_0 \geq 1$ such that

$$r_M = \frac{1}{M} + \theta\left(1 - \frac{1}{M}\right) \quad \text{for all integers } M \geq M_0. \tag{3.1}$$

In this case we have $\overline{r}_M \equiv \theta$ $(M \geq M_0)$.

Let $(X_t, \mathbf{Q}_{\mu^{(N)}})$ be the sampling branching Markov particle system with the sampling rate function λ_n and branching probability $\{q_j^{(n)}\}$ such that $X_0 = \mu^{(N)}$.

We define the scaled particle system $Z_n = \{Z_{n,t}\}_{t\geq 0}$ by $Z_{n,t} = X_t/n$ and denote its probability law by $\mathbf{P}_{n,\mu_n^{(N)}}$. The generator $\mathcal{L}_n^Z$ of $(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}})$ is given as

$$\begin{aligned} \mathcal{L}_n^Z e^{-\langle \cdot, f\rangle}(\mu_n^{(N)}) &= \mathcal{L}_n^{Z^0} e^{-\langle \cdot, f\rangle}(\mu_n^{(N)}) \\ &+ \frac{\lambda_n(N)}{N} \sum_{m=1}^N \sum_{k=1}^N p_{m,k}^{(N)} \left(e^{f(x_m)/n} G_n(e^{-f(x_k)/n}) - 1\right) e^{-\langle \mu_n^{(N)}, f\rangle}. \end{aligned}$$

We define an operator $\mathcal{L}^Z$ as follows. Recall that $\overline{\mu} \equiv \mu/\langle \mu, 1\rangle$ for $\mu \in \mathcal{M}_F \setminus \{0\}$.

(i) If $\lambda_n(M) = \gamma M a_n$, then

$$\mathcal{L}^Z e^{-\langle \cdot, f\rangle}(\mu) = \left[-\langle \mu, Af\rangle + \gamma\left\{\langle \mu, \Psi(f)\rangle + c(1-\theta)\left(\langle \mu, f^2\rangle - \langle \mu, f\rangle\langle \overline{\mu}, f\rangle\right)\right\}\right] e^{-\langle \mu, f\rangle}.$$

(ii) If $\lambda_n(M) = \gamma n a_n$, then

$$\mathcal{L}^Z e^{-\langle \cdot, f\rangle}(\mu) = \left[-\langle \mu, Af\rangle + \gamma\left\{\langle \overline{\mu}, \Psi(f)\rangle + c(1-\theta)\left(\langle \overline{\mu}, f^2\rangle - \langle \overline{\mu}, f\rangle^2\right)\right\}\right] e^{-\langle \mu, f\rangle}.$$

We also define the domain

$$\mathcal{D}_0 = \mathcal{D}_0(\mathcal{L}^Z) \equiv \text{lin span}\left\{e^{-\langle \mu, f\rangle}; f \in D(A), \|f\| < 1, \inf f > 0\right\}.$$

We have the following result.

THEOREM 3.1. *Suppose that* $\mu_n^{(N)} \to \mu(\neq 0) \in \mathcal{M}_F$ *as* $n \to \infty$ $(N = N(n))$ *and the condition* (3.1) *for* r_M *is satisfied with some* $\theta \in [0,1]$, $M_0 \geq 1$. *Then, corresponding to* $\lambda_n(M) = \gamma M a_n, \gamma n a_n$, *the scaled sampling branching process* $(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}})$ *with branching probability* $\{q_j^{(n)}\}$ *defined by* $\Psi(v)$ *converges to an* $\mathcal{M}_F$*-valued process* $(Z_t, \mathbf{P}_\mu)$ *weakly in* $\mathbf{D}([0,\infty), \mathcal{M}_F)$. *The limit process is the unique solution to the martingale problem for* $(\mathcal{L}^Z, \mathcal{D}_0, \mu)$ *satisfying that* $Z_t = Z_{t\wedge\tau_0}$ $(\tau_0 = \inf\{t; \langle Z_t, 1\rangle = 0\})$. *Moreover* $\langle Z_t, f\rangle$ $(f \in D(A))$ *has the following semi-martingale representation:*

$$\langle Z_t, f\rangle = \langle Z_0, f\rangle + \int_0^t \langle Z_s, Af\rangle ds + M_t^c(f) + M_t^d(f),$$

where $\{M_t^c(f)\}$ *is a continuous* L^2*-martingale with quadratic variation* $\langle\langle M^c(f)\rangle\rangle_t$ *such that*

(i) *if* $\lambda_n(M) = \gamma M a_n$, *then*

$$\langle\langle M^c(f)\rangle\rangle_t = \gamma c \int_0^t \left[\langle Z_s, f^2\rangle + 2(1-\theta)\left(\langle Z_s, f^2\rangle - \langle Z_s, f\rangle\langle \overline{Z_s}, f\rangle\right)\right] \quad (t < \tau_0),$$

(ii) *if* $\lambda_n(M) = \gamma n a_n$, *then*

$$\langle\langle M^c(f)\rangle\rangle_t = \gamma c \int_0^t \left[\langle \overline{Z_s}, f^2\rangle + 2(1-\theta)\left(\langle \overline{Z_s}, f^2\rangle - \langle \overline{Z_s}, f\rangle^2\right)\right] ds \quad (t < \tau_0).$$

Moreover

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_F} \langle \eta, f\rangle \widetilde{N}(ds, d\eta) \quad (t < \tau_0),$$

where $\widetilde{N}(ds, d\eta)$ *is a martingale measure with compensator*

$$\widehat{N}(ds, d\eta) = \begin{cases} \gamma ds \int_S Z_s(dx) \int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta) & (\lambda_n(M) = \gamma M a_n), \\ \gamma ds \int_S \overline{Z_s}(dx) \int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta) & (\lambda_n(M) = \gamma n a_n). \end{cases}$$

REMARK 3.1. In the binary branching case we have

$$\Psi(v) = \frac{1}{2}v^2 \quad (c = 1), \quad a_n = n, \quad G_n(z) = \frac{1}{2}(1 + z^2).$$

If $\lambda_n(M) = \gamma M n$ and $\theta = 1$, then

$$\mathcal{L}^Z e^{-\langle \cdot, f\rangle}(\mu) = \left[-\langle \mu, Af\rangle + \frac{\gamma}{2}\langle \mu, f^2\rangle\right] e^{-\langle \mu, f\rangle}.$$

The corresponding Markov process is the measure-valued binary branching process.

We call the limit process $(Z_t, \mathbf{P}_\mu)$ as the *jump-type branching Fleming-Viot process* in case of $\lambda_n(M) = \gamma M a_n$, and as the *jump-type branching Fleming-Viot-like process* in case of $\lambda_n(M) = \gamma n a_n$.

The following is the extension of the Perkins's result in [6]. It is also used to prove the uniqueness of the solution $(Z_t, \mathbf{P}_\mu)$ to the above martingale problem.

THEOREM 3.2. *Let $\mu \in \mathcal{M}_F\backslash\{0\}$ and set $y = \langle \mu, 1\rangle$. For a given $\theta \in [0,1]$, let $(Z_t, \mathbf{P}_\mu)$ be a solution to the martingale problem for $(\mathcal{L}^Z, \mathcal{D}_0, \mu)$ on $\mathbf{D}([0,\infty), \mathcal{M}_F)$ described in Theorem 3.1. Set $x_t = \langle Z_t, 1\rangle$ and $\tau_0 = \inf\{t; x_t = 0\}$. Then x_t has a decomposition $x_t = y + x_t^c + x_t^d$, where $\{x_t^c\}$ is a continuous martingale starting from 0 with quadratic variation, for $t < \tau_0$,*

$$\langle\langle x^c \rangle\rangle_t = \begin{cases} \gamma c \int_0^t x_s ds & (\lambda_n(M) = \gamma M n), \\ \gamma c t & (\lambda_n(M) = \gamma n^2). \end{cases}$$

$\{x_t^d\}$ is a pure discontinuous martingale such that

$$x_t^d = \int_0^t \int_0^\infty u \tilde{n}(ds, du) \quad (t < \tau_0),$$

where $\tilde{n}(ds, du)$ is a martingale measure with compensator

$$\hat{n}(ds, du) = \begin{cases} \gamma ds \langle Z_s, 1\rangle \nu(du) & (\lambda_n(M) = \gamma M a_n), \\ \gamma ds \nu(du) & (\lambda_n(M) = \gamma n a_n). \end{cases}$$

Moreover if $Q_y = \mathbf{P}_\mu \circ x.^{-1}$, then with $a = \gamma c(3 - 2\theta)$

$$\mathbf{P}_\mu \left(\overline{Z}. \in B \mid \langle Z., 1\rangle = g \right) = \mathbf{P}_{\overline{\mu}}^{(A,g,a,\gamma\nu)}(Y. \in B), \qquad Q_y\text{-a.a. } g \in D_+,$$

where $(Y_t, \mathbf{P}_{\overline{\mu}}^{(A,g,a,\gamma\nu)})$ is a time inhomogeneous jump-type Fleming-Viot process described in Theorem 2.1.

Proof. In case of binary branching this result was shown by Perkins in [6]. In our sampling branching case the proof goes the same way. In the following we describe the different part of the computation formally in case of $\lambda_n(M) = \gamma M a_n$. For simplicity, we denote $Z_t(f) = \langle Z_t, f\rangle$, $|Z_t| = \langle Z_t, 1\rangle$. Recall that

$$dZ_t(f) = Z_t(Af)dt + dM_t^c(f) + dM_t^d(f), \quad Z_0(f) = \langle \mu, f\rangle,$$

where $\{M_t^c(f)\}$ is a continuous L^2-martingale with quadratic variation

$$d\langle\langle M^c(f)\rangle\rangle_t = |Z_t| \left(a\overline{Z_t}(f^2) - b\overline{Z_t}(f)^2 \right) dt$$

($a \equiv \gamma c(3 - 2\theta) > b \equiv 2\gamma c(1 - \theta) \geq 0$ with $\theta \in [0,1]$) and

$$dM_t^d(f) = \int_{\mathcal{M}_F} \langle \eta, f\rangle \widetilde{N}(dt, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ is the martingale measure with compensator

$$\widehat{N}(ds, d\eta) = \gamma ds \int_S Z_s(dx) \int_0^\infty \nu(du) \delta_{u\delta_x}(d\eta).$$

Thus

$$d\langle\langle x^c\rangle\rangle_t = d\langle\langle M^c(1)\rangle\rangle_t = (a - b)|Z_t|dt = \gamma c x_t dt.$$

Moreover for any Borel functions Φ on $(0,\infty)$ such that $\Phi(u) \leq C(u \wedge u^2)$, we have

$$\int_{\mathcal{M}_F} \Phi(|\eta|)\widehat{N}(dt, d\eta) = dt|Z_t| \int_0^\infty \Phi(u)\nu(du).$$

Hence x_t^d is given as in the theorem. Furthermore by using Ito's formula we have

$$\begin{aligned} d(1/|Z_t|) &= -d|Z_t|/|Z_t|^2 + d\langle\!\langle M^c(f)\rangle\!\rangle_t/|Z_t|^3 \\ &\quad + \int \left[\frac{1}{|Z_{t-}|+|\eta|} - \frac{1}{|Z_{t-}|}\right] \widetilde{N}(dt,d\eta) \\ &\quad + \int \left[\frac{1}{|Z_t|+|\eta|} - \frac{1}{|Z_t|} + \frac{|\eta|}{|Z_t|^2}\right] \widehat{N}(dt,d\eta), \end{aligned}$$

and noting that

$$d\langle\!\langle M^c(f), M^c(1)\rangle\!\rangle_t = (a-b)Z_t(f)dt = (a-b)|Z_t|\overline{Z_t}(f)dt,$$

$$d\langle\!\langle Z^c(f), (1/|Z|)^c\rangle\!\rangle_t = -d\langle\!\langle M^c(f), M^c(1)\rangle\!\rangle_t/|Z|^2 = -(a-b)\overline{Z_t}(f)/|Z_t|dt,$$

we also have

$$\begin{aligned} d\overline{Z_t}(f) &= \overline{Z_t}(Af)dt + \left\{\frac{dM_t^c(f)}{|Z_t|} - \frac{\overline{Z_t}(f)}{|Z_t|}dM_t^c(1)\right\} \\ &\quad + \int \langle \overline{\eta} - \overline{Z_{t-}}, f\rangle \frac{|\eta|/|Z_{t-}|}{1+|\eta|/|Z_{t-}|}\widetilde{N}(dt,d\eta). \end{aligned}$$

If we set

$$dU_t(f) = dM_t^c(f)/|Z_t| - [\overline{Z_t}(f)/|Z_t|]dM_t^c(1), \quad N_0(f) = 0,$$

then $\{U_t(f)\}$ is a continuous L^2-martingale with quadratic variation

$$d\langle\!\langle U(f)\rangle\!\rangle_t = a[\overline{Z_t}(f^2) - \overline{Z_t}(f)^2]|Z_t|^{-1}dt.$$

Hence

$$d\overline{Z_t}(f) = \overline{Z_t}(Af) + dU_t(f) + \int \langle \overline{\eta} - \overline{Z_{s-}}, f\rangle \frac{|\eta|/|Z_{s-}|}{1+|\eta|/|Z_{s-}|}\widetilde{N}(ds,d\eta), \quad \overline{Z_0} = \mu$$

(more exactly we should use stopping times $\tau_n = \inf\{t; |Z_t| \le 1/n\}$). Therefore, roughly speaking, under the condition $|Z_t| = g(t)$ we can get the desired distribution. It is the same in case of $\lambda_n(M) = \gamma n a_n$. □

REMARK 3.2. For $\mu \in \mathcal{M}_F \setminus \{0\}$, it is possible to construct the solution $(Z_t, \mathbf{P}_\mu)$ to the martingale problem $(\mathcal{L}^Z, \mathcal{D}_0, \mu)$ directly. In fact, for $g \in D_+$, let $(Y_t, \mathbf{P}_{\overline{\mu}}^{(A,g,a,\nu)})$ be the Fleming-Viot process described in Theorem 2.1 with $a = \gamma c(3-2\theta)$ and $Q_{\langle\mu,1\rangle}$ be a probability measure on D_+ such that under $Q_{\langle\mu,1\rangle}$, the canonical process $\{g(t)\}$ is the same as $\{x_t\}$ in Theorem 3.2. Then under $\mathbf{P}_\mu \equiv \int Q_{\langle\mu,1\rangle}(dg)\mathbf{P}_{\overline{\mu}}^{(A,g,a,\nu)}$, $Z_t \equiv g(t)Y_t$ is the desired jump-type branching Fleming-Viot(-like) process.

In order to prove Theorem 3.1 we apply the following result (it is a modified result of Cor. 8.16 in Chap. 4 (p236) in [3] to our case). For each integer n, let $\mathcal{M}_F^{(n)} = \mathcal{M}_F^{(n)}(S)$ be a family of counting measures on S of the form $\eta_n^M = \sum_{k=1}^M \delta_{x_k}/n$ for $M = 1, 2, \ldots$. Set

$$\begin{aligned} \mathcal{D}_0^{(n)} &= \mathcal{D}_0^{(n)}(\mathcal{L}_n^Z) \\ &\equiv \text{lin span}\left\{e^{-\langle\mu,f_n\rangle}; f_n = -n\log(1-f/n), f \in D(A), \|f\| < 1, \inf f > 0\right\}. \end{aligned}$$

LEMMA 3.1. *Let $\mu_n^{(N)} \to \mu(\neq 0) \in \mathcal{M}_F$ $(n \to \infty)$. Suppose that the martingale problem for $(\mathcal{L}^Z, \mathcal{D}_0, \mu)$ in $\mathbf{D}([0,\infty), \mathcal{M}_F)$ has at most one solution $(Z_t, \mathbf{P}_\mu)$. Suppose that for each n, $\{(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}})\}$ is a solution to the martingale problem for $(\mathcal{L}_n^Z, \mathcal{D}_0^{(n)}, \mu_n^{(N)})$ and that $\{(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}})\}$ satisfies the compact containment condition, that is, for every $\epsilon > 0$, $T > 0$, there is a compact set $K_{\epsilon,T} \subset \mathcal{M}_F$ such that*

$$\inf_n \mathbf{P}_{n,\mu_n^{(N)}}(Z_{n,t} \in K_{\epsilon,T} \ \text{ for } 0 \le t \le T) \ge 1-\epsilon. \tag{3.2}$$

Moreover for $f_n = -n\log(1-f/n)$ with $f \in D(A)$ such that $\|f\| < 1$ and $\inf f > 0$, if it holds that

$$\lim_{n\to\infty} \sup_{\eta \in \mathcal{M}_F^{(n)}} |\mathcal{L}_n^Z e^{-\langle \cdot, f_n\rangle}(\eta) - \mathcal{L}^Z e^{-\langle \cdot, f\rangle}(\eta)| = 0, \tag{3.3}$$

then $(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}}) \Rightarrow (Z_t, \mathbf{P}_\mu)$ in $\mathbf{D}([0,\infty), \mathcal{M}_F)$.

This result can be shown, if we take $\mathcal{M}_F$, $\mathcal{M}_F^{(n)}$, $\mathcal{M}_F^{(n)}$ as E, E_n, G_n of Ch. 4, Cor. 8.16 (p236) in [3], respectively.

Here we mention the following relations:

$$\begin{cases} Ns_N + \overline{r}_N - 1 = 0, \\ \sum_{m=1}^N \sum_{k=1}^N p_{m,k}^{(N)} (g(x_k) - g(x_m)) = 0, \\ \sum_{m=1}^N \sum_{k=1}^N p_{m,k}^{(N)} g_1(x_m) g_2(x_k) = \dfrac{1-\overline{r}_N}{N} \langle \mu^{(N)}, g_1\rangle \langle \mu^{(N)}, g_2\rangle + \overline{r}_N \langle \mu^{(N)}, g_1 g_2\rangle. \end{cases}$$

Proof of Theorem 3.1. By Theorem 3.2 the uniqueness follows from the uniqueness of the total mass process and from the jump-type Fleming-Viot process.

By using the above relations we have, with $f_n = -n\log(1-f/n)$ of Lemma 3.1,

$$\begin{aligned} &\mathcal{L}_n^Z e^{-\langle \cdot, f_n\rangle}(\mu_n^{(N)}) \\ &= \mathcal{L}_n^{Z^0} e^{-\langle \cdot, f_n\rangle}(\mu_n^{(N)}) \\ &\quad + \frac{n\lambda_n(N)}{N} \Bigg[(1-\overline{r}_N) \left(\langle \mu_N^{(N)}, (1-f/n)^{-1}\rangle \langle \mu_n^{(N)}, G_n(1-f/n)\rangle - \frac{N}{n} \right) \\ &\quad + \overline{r}_N \langle \mu_n^{(N)}, (1-f/n)^{-1} G_n(1-f/n) - 1\rangle \Bigg] e^{-\langle \mu_n^{(N)}, f_n\rangle} \\ &= -\langle \mu_n^{(N)}, Af\rangle e^{-\langle \mu_n^{(N)}, f_n\rangle} + \frac{n\lambda_n(N)}{N} \Bigg[\frac{1}{na_n} \langle \mu_n^{(N)}, \Psi(f)\rangle \\ &\quad + \frac{1-\overline{r}_N}{n^2} \left(\langle \mu_n^{(N)}, f^2\rangle - \langle \mu_n^{(N)}, f\rangle \langle \mu_n^{(N)}, f\rangle \right) \Bigg] e^{-\langle \mu_n^{(N)}, f_n\rangle} + \mathcal{R}_n(\mu_n^{(N)}), \end{aligned}$$

where $\mathcal{R}_n(\mu_n^{(N)})$ is the error term. Therefore noting that $\lim_{n\to\infty} \|f_n - f\| = 0$, $f_n \ge f \ge \inf f = \epsilon > 0$, $\|f_n\| \le C_f \equiv \|f\|/(1-\|f\|)$ $(\|f\| < 1)$, $\lim_{n\to\infty} a_n/n = c$ and (3.1), we can easily get $\lim_{n\to\infty} \sup_M |\mathcal{R}_n(\eta_n^{(N)})| = 0$ in both cases of $\lambda_n(M) = \gamma M a_n, \gamma n a_n$, and thus the condition (3.3) is fulfilled.

In order to prove that $(Z_{n,t}, \mathbf{P}_{n,\mu_n^{(N)}})$ satisfies the condition (3.2), it is enough to show that for each $x > 0$,

$$\mathbf{P}_{n,\mu_n^{(N)}}\left(\sup_{t\geq 0}\langle Z_{n,t}, 1\rangle > x\right) \leq \frac{\langle \mu_n^{(N)}, 1\rangle}{x}.$$

However this immediately follows from the following result.

LEMMA 3.2. *For each $\alpha > 0$, the following are $\mathbf{P}_{n,\mu_n^{(N)}}$-martingales.*

$$\begin{aligned} M_{n,t}(\alpha) &\equiv e^{-\alpha\langle Z_{n,t},1\rangle} - \int_0^t \mathcal{L}_n^Z e^{-\alpha\langle\cdot,1\rangle}(Z_{n,s})ds \\ &= \begin{cases} e^{-\alpha\langle Z_{n,t},1\rangle} - \gamma n a_n \left(e^{\alpha/n}G_n(e^{-\alpha/n}) - 1\right)\int_0^t \langle Z_{n,s}, 1\rangle e^{-\alpha\langle Z_{n,s},1\rangle}ds \\ \qquad\qquad (\lambda_n(M) = \gamma M a_n), \\ e^{-\alpha\langle Z_{n,t},1\rangle} - \gamma n a_n \left(e^{\alpha/n}G_n(e^{-\alpha/n}) - 1\right)\int_0^t e^{-\alpha\langle Z_{n,s},1\rangle}ds \\ \qquad\qquad (\lambda_n(M) = \gamma n a_n) \end{cases} \end{aligned}$$

and

$$\langle Z_{n,t}, 1\rangle = \lim_{\alpha\downarrow 0}\frac{1}{\alpha}(1 - M_{n,t}(\alpha)).$$

Therefore the weak convergence follows by Lemma 3.1. The semi-martingale representation can be shown as in the proof of Th. 6.1.3 in [1]. We complete the proof of Theorem 3.1. □

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