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Brownian snake による 測度値分枝拡散過程の消滅法

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1 Introduction

ここで論じることは, J. Warren ([War]) の仕事が動機となっている. 彼は, $\theta \geq 0$ として, 微分作用素

$$L_\theta = x\frac{d^2}{dx^2} - \theta x \frac{d}{dx} \tag{1}$$

が定める $[0,\infty)$ 上の拡散過程 (L_θ − 拡散過程と呼ぶ. これは SDE (15) の一意解の定める拡散過程と云ってもよい) を考えると, 0 から出発する L_θ − 拡散過程 $\mu^\theta(t)$ が, 0 から出発する L_0 − 拡散過程 $\mu^0(t)$ より, ある種の消滅法で得られることを示した.

L_θ − 拡散過程は, 1 次元の連続状態分枝拡散過程 (CB-diffusion) と呼ばれ, Galton-Watson 分枝過程の極限拡散過程を表すものとして Lamperti によって導入された (cf. [L], [KW], [SW]). 特に L_0 − 拡散過程は Feller によって導入されたので, Feller 拡散過程と呼ばれることが多い.

$\mu^{(0)}(t)$ は，反射壁ブラウン運動の局所時間を用いた所謂 Ray-Knight 表現を持つことはよく知られている。Warren は，$\mu^{(\theta)}(t)$ $(\theta>0)$ もあるずれを持った反射壁ブラウン運動によってRay-Knight表現を持ち，そしてこの2つのRay-Knight表現の間には，第3の反射壁ブラウン運動（境界0で滞留を持つ粘性壁反射ブラウン運動）が介在してその関係が与えられることを示した。これは,粘性壁ブラウン運動の構造に関する1つの興味ある結果としての興味もある。

一方，L_θ-拡散過程は,測度値分枝拡散過程，所謂 super-diffusion (cf. [Da], [Dy]) の特別な場合であり，実際それは状態空間が唯1点よりなる場合である（このとき，この上の非負測度の全体は $[0,\infty)$ と同一視される)。そこで Warrenの結果をより一般の場合，ここでは，例えば多様体M上である微分作用素Lで生成される拡散過程（L-拡散過程）$\{\xi(t), P_x\}$ をその underlying process として持ち，その branching mechanism が各々，

$$\psi(x, z) = -z^2 \qquad (2)$$

$$\psi(x, z) = -z^2 - \theta(x)z \qquad (3)$$

で与えられる super-diffusion $\mu(t)$, $\mu'(t)$ の場合に，Warren の結果の一般化を試みる．その際の鍵は，Warren の場合に反射壁ブラウン運動の役割を果すものとして，Le Gall によって導入された Brownian snake の概念 (cf. [L 1]) を用いることである。

2 Reflecting Brownian motions and CB-diffusions (Warren's results)

We introduce the following three different reflecting Brownian motions R, S, Λ on the positive half line $[0, \infty)$:

(i) The standard reflecting Brownian motion $R = (R_t)$ starting at 0: it is well-known that R is given, from a standard Brownian motion $B = (B_t)$ on $\mathbf{R}$ with $B_0 = 0$, by

$$R_t = B_t - \inf_{0 \le s \le t} B_s. \tag{4}$$

(ii) For $\theta \ge 0$, let $S = (S_t)$ be the reflecting Brownian motion starting at 0 with a constant drift $\theta/2$ towards the origin: S is given, from a standard Brownian motion $B = (B_t)$ on $\mathbf{R}$ with $B_0 = 0$, by

$$S_t = \left[B_t - \frac{\theta}{2}t\right] - \inf_{0 \le s \le t} \left[B_s - \frac{\theta}{2}s\right]. \tag{5}$$

(iii) For $\theta \ge 0$, let $\Lambda = (\Lambda_t)$ be the reflecting Brownian motion starting at 0 with a sticky boundary at 0 with the rate $2/\theta$: it is characterized by the following SDE for a *nonnegative* and continuous process $\Lambda = (\Lambda_t)$ defined on a probability space with a filtration $\mathbf{F}$; Λ is $\mathbf{F}$-adapted, $B = (B_t)$ is a standard $\mathbf{F}$-Brownian motion with $B_0 = 0$ and they satisfy

$$\Lambda_t = \int_0^t 1_{\{\Lambda_s > 0\}} dB_s + \frac{\theta}{2} \int_0^t 1_{\{\Lambda_s = 0\}} ds. \tag{6}$$

It is well-known that a solution Λ exists and is unique in the law sense. Actually it is even known that the joint process (Λ, B) is unique in the law sense for any solution of (6), although the solution Λ can never be a strong solution. According to [War], this fact was first remarked by R. J. Chitashvili.

When we would emphasize the parameter θ, we write $S^{(\theta)}$ and $\Lambda^{(\theta)}$ for S and Λ, respectively. When $\theta = 0$, then S coincides with R and Λ is trivial, i.e. $\Lambda_t \equiv 0$.

We give several apparently different but essentially equivalent ways of defining a joint law of (R, S, Λ) for given constant $\theta \ge 0$. They are given by the following three theorems:

Theorem 2.1. *Let S and Λ be as above and assume that they are mutually independent. Define $A = (A_t)$ by*

$$A_t = \int_0^t 1_{\{\Lambda_s = 0\}} ds, \quad t \ge 0 \tag{7}$$

and $R' = (R'_t)$ by

$$R'_t = \Lambda_t + S_{A_t}. \tag{8}$$

Then, $R' \stackrel{d}{=} R$.

If we set $R := R'$ in Theorem 2.1, then this determines uniquely a joint law of (R, S, Λ). We call this *the joint law of Theorem 2.1.*

Theorem 2.2. *Let Λ satisfy the SDE (6) and, using the same Brownian motion $B = (B_t)$ in (6), define $R = (R_t)$ by (4). Then the joint law of (R, Λ) is uniquely determined. Define A by (7) and set*

$$S'_t = (R - \Lambda)_{A_t^{-1}} = R_{A_t^{-1}}, \tag{9}$$

where $A_t^{-1} = \inf\{u \mid A_u > t\}$, so that $t \mapsto A_t^{-1}$ is the right continuous inverse of $t \mapsto A_t$. Then $S' \stackrel{d}{=} S$ and, S' and Λ are mutually independent.

If we set $S := S'$ in Theorem 2.2, then the joint law (R, S, Λ) is uniquely determined and it coincides with the joint law of Theorem 2.1.

Theorem 2.3. *Let $R = (R_t)$ be given as above. Let $\kappa = (\kappa_t)$ be a measurable $\{0,1\}$-valued process with the following conditional law given R so that the law of the joint process (R, κ) is uniquely determined: $\kappa_0 = 1$, a.s. and, for $0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$,*

$$P(\kappa_{t_1} = 1, \kappa_{t_2} = 1, \ldots, \kappa_{t_{n-1}} = 1, \kappa_{t_n} = 1|R) = e^{-M[0,t_1]}e^{-M[t_1,t_2]} \cdots e^{-M[t_{n-1},t_n]}, \tag{10}$$

where

$$M[s,t] = \theta(R_t - \min_{s \le u \le t} R_u), \quad 0 \le s \le t. \tag{11}$$

Set

$$A'_t = \int_0^t \kappa_s ds, \quad (A')_t^{-1} = \inf\{\, u \mid A_u > t \,\}$$

and define $S' = (S'_t)$ and $\Lambda' = (\Lambda'_t)$ by

$$S'_t = R_{(A')_t^{-1}}, \quad \Lambda'_t = R_t - S'_{A'_t}.$$

Then $S' \stackrel{d}{=} S$ and $\Lambda' \stackrel{d}{=} \Lambda$, and S' and Λ' are independent. Furthermore,

$$\kappa_t = 1_{\{\Lambda'_t = 0\}} \quad \text{for almost all } t, \text{a.s..} \tag{12}$$

If we set $S := S'$ and $\Lambda := \Lambda'$ in Theorem 2.3, then the joint law (R, S, Λ) is uniquely determined and it coincides with the joint law of Theorem 2.1. Thus we have seen that there are three apparently different ways of determining the *same* joint law of (R, S, Λ).

The joint process (Λ, R) can be given explicitly as follows. Let

$$\Sigma^2 = \left\{(\lambda, x) \in \mathbf{R}^2 | x \ge 0, 0 \le \lambda \le x\right\}.$$

Theorem 2.4. *(Λ, R) is a time homogeneous diffusion on Σ^2 with $\Lambda_0 = R_0 = 0$ having the transition probability given by*

$$p(t, (\lambda, x), d\lambda' dx') = \iint_{0 \le a < b < \infty} \Theta_t^x(da, db) q_{a,b}^{(\lambda,x)}(d\lambda' dx')$$

where

$$
\begin{aligned}
& q_{a,b}^{(\lambda,x)}(d\lambda' dx') && (13)\\
= \; & 1_{\{x-\lambda<a\}} \cdot \delta_{x'-x+\lambda}(d\lambda') \cdot \delta_b(dx') \\
+ \; & 1_{\{x-\lambda\geq a\}} \cdot 1_{\{0<\lambda'<x'-a\}} \cdot \theta \cdot e^{-\theta(x'-\lambda'-a)} \cdot d\lambda' \cdot \delta_b(dx') \\
+ \; & 1_{\{x-\lambda\geq a\}} \cdot e^{-\theta(x'-a)} \cdot \delta_0(d\lambda') \cdot \delta_b(dx')
\end{aligned}
$$

and $\Theta_t^x(da, db) = P_x(\min_{0\leq s\leq t} R(s) \in da, R(t) \in db)$, P_x *being the probability law governing the standard reflecting Brownian motion* $R = (R(t))$ *with* $R(0) = x$*: It is given explicitly by*

$$
\begin{aligned}
\Theta_t^x(da, db) \;&=\; \frac{2(x+b-2a)}{\sqrt{2\pi t^3}} e^{-\frac{(x+b-2a)^2}{2t}} 1_{\{0<a<b\wedge x\}} da db && (14)\\
&+\; \sqrt{\frac{2}{\pi t}} e^{-\frac{(x+b)^2}{2t}} 1_{\{0<b\}} \delta_0(da) db.
\end{aligned}
$$

For $\theta \geq 0$ and $\gamma \in [0,\infty)$, let $\mu^{(\theta)} = (\mu^{(\theta)}(t), P_\gamma)$ be the CB-diffusion on $[0,\infty)$ with $\mu^{(\theta)}(0) = \gamma$ which is generated by the differential operator L_θ given by (1). The origin 0 is necessarily a trap. Equivalently, $\mu^{(\theta)}$ is given by the unique strong solution to the following SDE:

$$
d\mu(t) = \sqrt{2\mu(t) \vee 0} \cdot dB(t) - \theta\mu(t)dt, \quad \mu(0) = \gamma \tag{15}
$$

where $B = (B(t))$ is the Brownian motion with $B(0) = 0$. The connection of these CB-diffusions with reflecting Brownian motions R, S and Λ can be stated in the following theorem of the Ray-Knight type.

Theorem 2.5. *Let* $R = (R_t)$, $S = (S_t)$ *and* $\Lambda = (\Lambda_t)$ *be reflecting Brownian motions as above and determine the joint law of* (R, S, Λ) *by Theorem 2.1, or equivalently, by Theorem 2.2 or Theorem 2.3. Let* $l(t,a)$ *and* $l'(t,a)$ *be the local time or sojourn time density of* R *and* S*, respectively:*

$$
l(t,a) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon]}(R_s) ds, \quad l'(t,a) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon]}(S_s) ds. \tag{16}
$$

Define, for $\gamma \in [0,\infty)$, *continuous and nonnegative processes* $\mu = (\mu(t))$ *and* $\mu' = (\mu'(t))$ *by*

$$
\mu(t) = l(l^{-1}(\gamma,0), t) \quad \text{and} \quad \mu'(t) = l'(l'^{-1}(\gamma,0), t) \tag{17}
$$

where $l^{-1}(\gamma,0) = \inf\{t | l(t,0) > \gamma\}$ *and* $l'^{-1}(\gamma,0) = \inf\{t | l'(t,0) > \gamma\}$. *Then, we have the following facts:*

(i) $\mu \stackrel{d}{=}$ *the CB-diffusion* $\mu^{(0)}$ *starting at* γ.

(ii) $\mu' \stackrel{d}{=}$ *the CB-diffusion* $\mu^{(\theta)}$ *starting at* γ.

(iii) It holds that

$$\mu'(t) = \int_0^{l^{-1}(\gamma,0)} \kappa(s) \cdot l(ds,t) \tag{18}$$

where

$$\kappa(s) = 1_{\{\Lambda_s=0\}}. \tag{19}$$

Since

$$\mu(t) = \int_0^{l^{-1}(\gamma,0)} l(ds,t),$$

we may say that μ' is obtained from μ by a killing determined by a $\{0,1\}$-valued process $\kappa(s)$. This is what we called a "killing operation" in Introduction.

The results of this section are essentially due to Warren ([War]). We amplified them a little bit by adding Theorems 2.3 and 2.4 which are not stated explicitly in [War]. These results will be extended to the case of general super-diffusions in Section 4. In the next section, we recall the notion of Brownian snakes which will play a fundamental role in this extension.

3 Brownian snakes

Throughout this paper, let $\xi = \{\xi(t), P_x\}$ be a nice diffusion process on a nice manifold M generated by a diffusion operator L with $L1 = 0$. We call ξ the L-diffusion. This L-diffusion has been considered as the underlying diffusion of super-diffusions in Introduction.

In this section, we recall the notion of Brownian ξ-snake due to Le Gall [L 1]. It is defined as a diffusion process with values *in the space of stopped paths* in M so that we introduce, first of all, the following notations for several spaces of continuous paths in M and continuous *stopped paths* in M:

(i) for $x \in M$, $W_x(M) = \{w \in C([0,\infty) \to M) \mid w(0) = x\}$,

(ii) $W(M) = \cup_{x\in M} W_x(M)$,

(iii) for $x \in M$ and $t \geq 0$,

$$\mathbf{W}_x^{(t)}(M) = \{\mathbf{w} = (w,t) \mid w \in W_x(M) \text{ such that } w(s) \equiv w(s \wedge t)\ \},$$

(iv) $\mathbf{W}_x^{stop}(M) = \cup_{t\geq 0} \mathbf{W}_x^{(t)}(M)$,

(v) $\mathbf{W}^{stop}(M) = \cup_{x\in M} \mathbf{W}_x^{stop}(M)$.

For $\mathbf{w} = (w,t) \in \mathbf{W}^{stop}(M)$, we set $|\mathbf{w}| = t$ and call it the *lifetime* of $\mathbf{w}$. Thus we may think of $\mathbf{w} \in \mathbf{W}^{stop}(M)$ a continuous path on M stopped at its own lifetime $|\mathbf{w}|$. We endow a metric on $\mathbf{W}^{stop}(M)$ by

$$d(\mathbf{w}_1, \mathbf{w}_2) = ||\mathbf{w}_1| - |\mathbf{w}_2|| + \max_s \rho(w_1(s), w_2(s))$$

where ρ is a suitable metric on M. Then, $\mathbf{W}^{stop}(M)$ is a Polish space and so is also $\mathbf{W}^{stop}_x(M)$ as its closed subspace.

3.1 Snakes with deterministic lifetimes

Let x be given and fixed. For each $0 \le a \le b$ and $\mathbf{w} = (w, |\mathbf{w}|) \in \mathbf{W}^{stop}_x(M)$ such that $a \le |\mathbf{w}|$, define a Borel probability $Q^{\mathbf{w}}_{a,b}(d\mathbf{w}')$ on $\mathbf{W}^{stop}_x(M)$ by the following property:

(i) $|\mathbf{w}'| = b$ for $Q^{\mathbf{w}}_{a,b}$-a.a. $\mathbf{w}'$,

(ii) $w'(s) = w(s), s \in [0, a]$, for $Q^{\mathbf{w}}_{a,b}$-a.a. $\mathbf{w}'$,

(iii) under $Q^{\mathbf{w}}_{a,b}$, the shifted path $\{(w')^+_a(s) = w'(a+s), s \ge 0\}$ is equally distributed as the stopped path $\{\xi(s \wedge (b-a)), s \ge 0\}$ under $P_{w(a)}$.

Let $\zeta(t)$ be a nonnegative continuous function of $t \in [0, \infty)$ such that $\zeta(0) = 0$. Define, for each $0 \le t < t'$ and $\mathbf{w} \in \mathbf{W}^{stop}_x(M)$, a Borel probability $P(t, \mathbf{w}; t', d\mathbf{w}')$ on $\mathbf{W}^{stop}_x(M)$ by

$$P(t, \mathbf{w}; t', d\mathbf{w}') = Q^{\mathbf{w}}_{m^\zeta[t,t'],\zeta(t')}(d\mathbf{w}') \tag{20}$$

where

$$m^\zeta[t, t'] = \min_{t \le u \le t'} \zeta(u).$$

It is easy to see that the family $\{P(t, \mathbf{w}; t', d\mathbf{w}')\}$ satisfies the Chapman-Kolmogorov equation so that it defines a family of transition probabilities on $\mathbf{W}^{stop}_x(M)$. Then, by the Kolmogorov extension theorem, we can construct a time inhomogeneous Markov process $\mathbf{X} = \{\mathbf{X}^t = (X^t(\cdot), \zeta(t))\}$ on $\mathbf{W}^{stop}_x(M)$ such that $\mathbf{X}^0 = \mathbf{x}$ where $\mathbf{x}$ is the *constant path* at x: $\mathbf{x} = (\{x(\cdot) \equiv x\}, 0)$. Note that $|\mathbf{X}^t| \equiv \zeta(t)$. If $\zeta(t)$ is Hölder-continuous, then it can be shown that a continuous modification in t of $\mathbf{X}^t$ exists (cf. Le Gall [L 1]). In the following, we always assume that $\zeta(t)$ is Hölder-continuous so that $\mathbf{X}^t$ is continuous in t, a.s..

Definition 3.1. *The $\mathbf{W}^{stop}_x(M)$-valued continuous process $\mathbf{X} = (\mathbf{X}^t)$ is called the ξ-snake starting at $x \in M$ with the lifetime function $\zeta(t)$. Its law on $C([0,\infty) \to \mathbf{W}^{stop}_x(M))$ is denoted by $\mathbf{P}^\zeta_x$.*

We can easily see that the following three properties characterize the ξ-snake starting at $x \in M$ with the life time function $\zeta(t)$:

(i) $|\mathbf{X}^t| \equiv \zeta(t)$ and, for each $t \in [0, \infty)$,

$$X^t : s \in [0, \infty) \mapsto X^t(s) \in M$$

is an L-diffusion such that $X^t(0) = x$ and stopped at time $\zeta(t)$,

(ii) for each $0 \le t < t'$,

$$X^{t'}(s) = X^t(s), \quad s \in \left[0, m^\zeta[t, t']\right],$$

(iii) for each $0 \le t < t'$, $\{X^{t'}(s); s \ge m^{\zeta}[t,t']\}$ and $\{X^u(\cdot); u \le t\}$ are independent given $X^{t'}(m^{\zeta}[t,t'])$.

3.2 Brownian snakes

In the following, we denote by $RBM^x([0,\infty))$ a reflecting Brownian motion $R = (R(t))$ on $[0,\infty)$ with $R(0) = x$.

Definition 3.2. *The* Brownian ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at $x \in M$ *is a* $\mathbf{W}_x^{stop}(M)$-*valued continuous process with the law on* $C([0,\infty) \to \mathbf{W}_x^{stop}(M))$ *given by*

$$\mathbf{P}_x(\cdot) = \int_{C([0,\infty)\to[0,\infty))} \mathbf{P}_x^{\zeta}(\cdot) P^R(d\zeta) \tag{21}$$

where P^R *is the law on* $C([0,\infty) \to [0,\infty))$ *of* $RBM^0([0,\infty))$.

It is obvious that $\mathbf{X}^0 = \mathbf{x}$, a.s..

Proposition 3.1. *(Le Gall ([L 1]))* $\mathbf{X} = (\mathbf{X}^t)$ *is a time homogeneous diffusion on* $\mathbf{W}_x^{stop}(M)$ *with the transition probability*

$$P(t, \mathbf{w}, d\mathbf{w}') = \iint_{0 \le a \le b < \infty} \Theta_t^{|\mathbf{w}|}(da, db) Q_{a,b}^{\mathbf{w}}(d\mathbf{w}') \tag{22}$$

where $\Theta_t^{|\mathbf{w}|}(da, db)$ *is the joint law of* $(\min_{0\le s\le t} R(s), R(t))$, $R(t)$ *being* $RBM^{|\mathbf{w}|}([0,\infty))$; *explicitly,*

$$\begin{aligned} \Theta_t^{|\mathbf{w}|}(da, db) &= \frac{2(|\mathbf{w}| + b - 2a)}{\sqrt{2\pi t^3}} e^{-\frac{(|\mathbf{w}|+b-2a)^2}{2t}} 1_{\{0<a<b\wedge|\mathbf{w}|\}} \\ &+ \sqrt{\frac{2}{\pi t}} e^{-\frac{(|\mathbf{w}|+b)^2}{2t}} 1_{\{0<b\}} \delta_0(da) db. \end{aligned} \tag{23}$$

The lifetime process $\zeta(t) := |\mathbf{X}^t|$ *is a* $RBM^0([0,\infty))$ *and, conditioned on the process* $\zeta = (\zeta(t))$, *it is the* ξ-*snake with the deterministic lifetime fuction* $\zeta(t)$.

3.3 The snake description of super-diffusion $\{\mu(t), \mathbf{P}_\mu\}$.

Let $x \in M$ and $\mathbf{X} = (\mathbf{X}^t)$ be the Brownian ξ-snake starting at x. Then $|\mathbf{X}^t|$ is a $RBM^0([0,\infty))$. Let

$$l(t,a) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon)}(|\mathbf{X}^s|) ds \tag{24}$$

be its local time at $a \in [0,\infty)$.

Let $\mathcal{M}_F(M)$ be the space of all finite Borel measures on M with the topology of weak convergence and $C_b(M)$ be the space of all bounded continuous functions on M. Introduce the usual notation

$$\langle \mu, f \rangle = \int_M f(x)\mu(dx), \quad \mu \in \mathcal{M}_F(M),\ f \in C_b(M).$$

Let $(\mu(t), P_\mu)$ be the super-diffusion introduced in Introduction. Recall that this is given as follows:

(i) the underlying process $\{\xi(t), P_x\}$ being given by the L-diffusion,

(ii) the branching mechanism given by

$$\psi(x,z) = -c(x)z^2,$$

where $c(x)$ is a bounded and positive function on M,

so that the log-Laplace functional

$$u(t,x) = -\log \mathbf{E}_{\delta_x}[\exp(-\langle \mu(t), f\rangle)]$$

is the solution to the initial value problem

$$\frac{\partial u}{\partial t} = Lu + \psi(\cdot, u), \quad u(0+,\cdot) = f.$$

We assume that $c(x) \equiv 1$; a modification necessary to treat the general case of positive functions $c(x)$ has been studied in [Wat] (cf. [DS]).

Then, for $\gamma > 0$ and $x \in M$, the process $\mu(t)$ under $P_{\gamma\cdot\delta_x}$ can be constructed from the Brownian ξ-snake $\mathbf{X} = (\mathbf{X}^t)$ starting at x in the following way: Define $\mu(t) \in \mathcal{M}_F(M), t \geq 0$, by

$$\langle \mu(t), f\rangle = \int_0^{l^{-1}(\gamma,0)} f(\langle \mathbf{X}^s\rangle) l(ds,t), \quad f \in C_b(M), \tag{25}$$

where $\langle \mathbf{X}^t\rangle = X^t(|\mathbf{X}^t|) \in M$: the position of $\mathbf{X}^t$ stopped at its lifetime $|\mathbf{X}^t|$ and

$$l^{-1}(\gamma, 0) = \inf\{u \mid l(u,0) > \gamma\}.$$

Theorem 3.1. *(Le Gall [L 1])* *$\{\mu(t)\}$ defined by (25) is exactly the super-diffusion $\{\mu(t)\}$ under $P_{\gamma\cdot\delta_x}$.*

Let $\mu'(t)$ be another super-process with the same underlying process as $\mu(t)$ but with the branching mechanism now replaced by

$$\psi(x,z) = -z^2 - \theta(x)z.$$

It is intuitively obvious that the super diffusion $\mu'(t)$ is obtained from the super-diffusion $\mu(t)$ by eliminating or killing some of its "particles"; however, there is no picture of particles in the usual formulation of super processes as measure-valued processes. We can, however, apply the snake description (25) to realize a killing operation; in the next section, we discuss how we can modify the expression (25) for $\mu(t)$ to obtain $\mu'(t)$.

4 A killing operation on super-diffusions and subsnakes

Let $\{\mu(t), \mathbf{P}_\mu\}$ and $\{\mu'(t), \mathbf{P}'_\mu\}$ be super-diffusions as above. Then, for $x \in M$ and $\gamma > 0$, the measure-valued process $\mu(t)$ under $\mathbf{P}_{\gamma\cdot\delta_x}$ has the snake description given by (25). We would obtain the process $\mu'(t)$ under $\mathbf{P}'_{\gamma\cdot\delta_x}$ in the form:

$$\langle \mu'(t), f\rangle = \int_0^{l^{-1}(\gamma,0)} f(\langle \mathbf{X}^s\rangle)\kappa(s)l(ds,t), \quad f \in C_b(M), \tag{26}$$

where $\kappa(t)$ is a certain process *taking values* 0 *or* 1. So our problem is concerned with the definition and the characterization of this process $\kappa(t)$ in terms of the Brownian ξ-snake $\mathbf{X}$ and the function $\theta(x)$. Actually, we would associate to the snake $\mathbf{X}$ a certain *nonnegative and continuous* process $\lambda(t)$ with $\lambda(0) = 0$ so that the desired process $\kappa(t)$ is given by

$$\kappa(t) = 1_{\{\lambda(t)=0\}}, \quad t \geq 0. \tag{27}$$

We enlarge the stopped path space $\mathbf{W}_x^{stop}(M)$ to a larger space $[\mathbf{W}_x^{stop}(M)]$ defined by

$$\left[\mathbf{W}_x^{stop}(M)\right] = \left\{(\alpha, \mathbf{w}) \in [0,\infty) \times \mathbf{W}_x^{stop}(M) \mid 0 \leq \alpha \leq |\mathbf{w}|\right\}. \tag{28}$$

We endow it with the topology induced from the product topology of $[0,\infty) \times \mathbf{W}_x^{stop}(M)$. Given a bounded, nonnegative and continuous function $\theta = (\theta(x))$ on M, we define, for $t > 0$ and $(\alpha, \mathbf{w}) \in [\mathbf{W}_x^{stop}(M)]$, a Borel probability $\widehat{P}(t, (\alpha, \mathbf{w}), d\alpha' d\mathbf{w}')$ on $[\mathbf{W}_x^{stop}(M)]$ by

$$\widehat{P}(t, (\alpha, \mathbf{w}), d\alpha' d\mathbf{w}') = \iint_{0 \leq a \leq b < \infty} \theta_t^{|\mathbf{w}|}(da, db) q_{a,b}^{(\alpha,\mathbf{w})}(d\alpha' d\mathbf{w}') \tag{29}$$

where $q_{a,b}^{(\alpha,\mathbf{w})}(d\alpha' d\mathbf{w}')$, $0 \leq a \leq b < \infty$, is defined by

$$\begin{aligned} q_{a,b}^{(\alpha,\mathbf{w})}(d\alpha' d\mathbf{w}') \quad = \quad & 1_{\{\alpha<a\}}\delta_\alpha(d\alpha')Q_{a,b}^{\mathbf{w}}(d\mathbf{w}') \\ & +1_{\{\alpha\geq a\}}1_{\{a<\alpha'<|\mathbf{w}'|\}}\theta(w'(\alpha'))\exp\left[-\int_a^{\alpha'}\theta(w'(u))du\right] d\alpha' Q_{a,b}^{\mathbf{w}}(d\mathbf{w}') \\ & +1_{\{\alpha\geq a\}}\exp\left[-\int_a^{|\mathbf{w}'|}\theta(w'(u))du\right]\delta_{|\mathbf{w}'|}(d\alpha')Q_{a,b}^{\mathbf{w}}(d\mathbf{w}'). \end{aligned} \tag{30}$$

Theorem 4.1. *The family* $\{\widehat{P}(t, (\alpha, \mathbf{w}), d\alpha' d\mathbf{w}')\}$ *defines a system of transition probabilities on* $[\mathbf{W}_x^{stop}(M)]$ *and it determines a unique time homogeneous diffusion* $\widehat{\mathbf{X}} = (\alpha^t, \mathbf{X}^t)$ *on* $[\mathbf{W}_x^{stop}(M)]$.

In the following, we assume $\alpha^0 = 0$ and $\mathbf{X}^0 = \mathbf{x}$:

Definition 4.1. *The diffusion* $\widehat{\mathbf{X}} = (\alpha^t, \mathbf{X}^t)$ *on* $[\mathbf{W}_x^{stop}(M)]$ *with* $\alpha^0 = 0$ *and* $\mathbf{X}^0 = \mathbf{x}$ *is called the* θ-subsnake *of the Brownian* ξ*-snake* $\mathbf{X}$ *starting at* x.

Obviously, the process $\mathbf{X} = (\mathbf{X}^t)$ defined by the second component of $\widehat{\mathbf{X}}$ is a Brownian ξ-snake starting at x.

Define

$$\lambda(t) = |\mathbf{X}^t| - \alpha^t, \quad t \geq 0. \tag{31}$$

Since $|\mathbf{X}^0| = 0$ and $\alpha^0 = 0$, we also have $\lambda(0) = 0$.

Definition 4.2. *The diffusion* $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ *on* $[\mathbf{W}_x^{stop}(M)]$ *is called the* second θ-subsnake *of the Brownian ξ-snake* $\mathbf{X}$ *starting at* x.

Theorem 4.2. *Let* $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ *be the second θ-subsnake and define the process* $\kappa(t)$ *by (27). Then the equation (26) determines the superdiffusion* $\mu'(t)$ *under* $\mathbf{P}'_{\gamma\cdot\delta_x}$.

Thus, the killing operation (26) to obtain $\mu'(t)$ from $\mu(t)$ through their snake descriptions can be determined by the second θ-subsnake $\widetilde{\mathbf{X}}$, or equivalently, by the θ-subsnake $\widehat{\mathbf{X}}$. So we would like to characterize these snakes in terms of the Brownian ξ-snake and the function θ in a much simpler way.

Theorem 4.3. *Let* $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ *be the second θ-subsnake of the Brownian ξ-snake starting at $x \in M$. Then, for* $0 < s_1 < s_2 < \ldots < s_{m-1} < s_m$ and $0 < t_1 < \ldots < t_n$, *we have*

$$\begin{aligned} & P\left(\lambda(s_1) = 0, \lambda(s_2) = 0, \ldots, \lambda(s_{m-1}) = 0, \lambda(s_m) = 0, \mathbf{X}^{t_1} \in d\mathbf{w}_1, \ldots, \mathbf{X}^{t_n} \in d\mathbf{w}_n\right) \\ = \; & E\left(e^{-M[0,s_1]} e^{-M[s_1,s_2]} \cdots e^{-M[s_{m-1},s_m]};\ \mathbf{X}^{t_1} \in d\mathbf{w}_1, \ldots, \mathbf{X}^{t_n} \in d\mathbf{w}_n\right) \end{aligned} \tag{32}$$

where

$$M[s,t] = \int_{\min_{s \leq u \leq t} |\mathbf{X}^u|}^{|\mathbf{X}^t|} \theta(X^t(u))du, \quad 0 \leq s < t. \tag{33}$$

In other words, conditioned on the Brownian ξ-snake $\mathbf{X}$, *the joint law of the* $\{0,1\}$-*valued process* $\kappa(t) = 1_{\{\lambda(t)=0\}}$ *is given by*

$$\begin{aligned} & P(\kappa(s_1) = 1, \kappa(s_2) = 1, \ldots, \kappa(s_{m-1}) = 1, \kappa(s_m) = 1 \ / \ \mathbf{X}) \\ = \; & e^{-M[0,s_1]} e^{-M[s_1,s_2]} \cdots e^{-M[s_{m-1},s_m]}. \end{aligned} \tag{34}$$

Theorem 4.4. *Let* $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ *be the second θ-subsnake of the Brownian ξ-snake starting at $x \in M$. Define*

$$A(t) = \int_0^t 1_{\{\lambda(s)=0\}} ds \left(= \int_0^t \kappa(s) ds\right) \tag{35}$$

and let $A^{-1}(t)$ be the right-continuous inverse of $t \mapsto A(t)$. Define further

$$S(t) = |\mathbf{X}^{A^{-1}(t)}|, \quad t \geq 0. \tag{36}$$

Then $S(t)$ is a continuous process and the following identity holds:

$$\alpha^t = S(A(t)), \quad t \geq 0 \quad \text{so that} \quad \lambda(t) = |\mathbf{X}^t| - S(A(t)), \quad t \geq 0. \tag{37}$$

Theorem 4.4 asserts that the process $(A(t), \mathbf{X}^t)$ determines the θ-subsnake $\widehat{\mathbf{X}}$ or $\widetilde{\mathbf{X}}$ so that we only need to obtain the process $(A(t), \mathbf{X}^t)$ in order to obtain the θ-subsnake $\widehat{\mathbf{X}}$ or $\widetilde{\mathbf{X}}$. By Theorem 4.3, we can obtain the process $(\kappa(t), \mathbf{X}^t)$ uniquely in the law sense and hence its measurable version, so that the process $(A(t), \mathbf{X}^t)$ can be obtained uniquely in the law sense.

Another characterization of the second θ-subsnake $\widehat{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ can be given by means of a stochastic differential equation (SDE) which is a natural generalization of SDE (6). First, we formulate a SDE.

On a suitable probability space equipped with a filtration $\mathbf{F} = \{\mathcal{F}_t\}$, we consider a continuous process $(\lambda(t), \mathbf{X}^t)$ on $[0, \infty) \times \mathbf{W}_x^{stop}$ with $\lambda(0) = 0$ and $\mathbf{X}^0 = \mathbf{x}$, which satisfies the following conditions:

(i) the process $(\lambda(t), \mathbf{X}^t)$ is $\mathbf{F}$-adaped and $\mathbf{X} = \{\mathbf{X}^t\}$ is a Brownian ξ-snake starting at x,

(ii) the Brownian motion $\{B(t)\}$ defined by

$$B(t) = |\mathbf{X}^t| - l(t, 0), \tag{38}$$

is an $\mathbf{F}$-Brownian motion in the sense that $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for every $0 \leq s < t$,

(iii) $\{\lambda(t)\}$ satisfies the following stochastic differential equation:

$$d\lambda(t) = 1_{\{\lambda(t)>0\}} dB(t) + \frac{1}{2} 1_{\{\lambda(t)=0\}} \cdot \theta(\langle \mathbf{X}^t \rangle) dt. \tag{39}$$

Such a process $(\lambda(t), \mathbf{X}^t)$ is called a solution of SDE (39), or an $\mathbf{F}$-solution of SDE (39) when we would refer to the filtration $\mathbf{F}$, with initial values $\lambda(0) = 0$ and $\mathbf{X}^0 = \mathbf{x}$.

Theorem 4.5. *Let $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ be the second θ-subsnake of the Brownian ξ-snake starting at $x \in M$. Then it is a solution to SDE (39). Furthermore, the uniqueness in law of solutions to SDE (39) holds so that $\widetilde{\mathbf{X}} = (\lambda(t), \mathbf{X}^t)$ is characterized completely by SDE (39).*

In the definition of SDE (39), we assumed that the second component $\mathbf{X}^t$ of a solution is a Brownian ξ-snake. If we rewrite a martingale problem for Brownian ξ-snake studied by Dhersin and Serlet ([DS]), we can also formulate a SDE for the joint process $(\lambda(t), \mathbf{X}^t)$ and characterize the second θ-subsnake by its solution.

A proof of Theorem 4.5 can be given by showing the following: Set, for a bounded and continuous function $F(\alpha, \mathbf{w})$ on $[\mathbf{W}_x^{stop}(M)]$ and $t \geq 0$,

$$H(t, (\lambda, \mathbf{w})) = \int_{[\mathbf{W}_x^{stop}(M)]} F(\alpha', \mathbf{w}') \widehat{P}(t, (|\mathbf{w}| - \lambda, \mathbf{w}), d\alpha' d\mathbf{w}')$$

where $\widehat{P}(t, (\alpha, \mathbf{w}), d\alpha' d\mathbf{w}')$ is defined by (29). Then for any $\mathbf{F}$-solution $(\lambda(t), \mathbf{X}^t)$ of SDE (39) and a fixed $T > 0$, $t \to H(T - t, (\lambda(t), \mathbf{X}^t))$ is an $\mathbf{F}$-martingale.

References

[Da] D. A. Dawson. Measure-valued Markov processes, École d'été de Probabilités de Saint-Flour, XXI-1991, **LNM 1541**, Springer(1993), 1-260.

[DS] J-S. Dhersin and L. Serlet. A stochastic calculus approach for the Brownian snake, preprint.

[Dy] E. B. Dynkin. Branching particle systems and superprocesses, Ann. Probab. **19**(1991), 1157-1194.

[F] W. Feller. Diffusion processes in genetics, *Proc. Second Berkeley Symp. Math. Statist. Prob.,* University of California Press(1951), Berkeley, 227-246.

[KW] K. Kawazu and S. Watanabe. Branching processes with immigration and related limit theorems, Teor. Veroyatnost. i. Primenen, **16**(1971), 34-51

[La] J. Lamperti. The limit of a sequence of a branching processes, Z. Wahrsch, verw. Gebiete **7**(1967), 271-288.

[Le 1] J. F. Le Gall. A class of path-valued Markov processes and its applications to superprocesses, Probab. Theory Relat. Fields **95**(1993), 24-46.

[Le 2] J. F. Le Gall. A path-valued Markov processes and its connections with partial differential equations, *Proceedings 1st European Congress of Mathematics,* Vol. II, Birkhäuser(1994), 185-212.

[SW] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes, Z. Wahrsch, verw. Gebiete **27**(1973), 37-46.

[War] J. Warren. Branching processes, the Ray-Knight theorem, and sticky Brownian motion, *Sém. Prob. XXXI*, **LNM 1655**, Springer(1997), 1-15.

[Wat] S. Watanabe. Branching diffusions (superdiffusions) and random snakes, *Trends in Prob. Relat. Analysis*, Proc. SAP'96, World Scientific(1997), 289-304.