<table>
<thead>
<tr>
<th>Title</th>
<th>A Version of Evans-Perkins Type Stochastic Representation Formula for Historical Superprocesses (Stochastic Analysis on Measure-Valued Stochastic Processes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Doku, Isamu</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1089: 42-60</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62859">http://hdl.handle.net/2433/62859</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Version of Evans-Perkins Type Stochastic Representation Formula for Historical Superprocesses*

Isamu DÔKU (道工 勇)
Department of Mathematics, Saitama University
Urawa 338-8570, Japan

1 Introduction

The purpose of this article is to introduce a version of Evans-Perkins type stochastic representation formula for a generalized \( \{ \gamma, a, b, g \} \)-historical superprocess (see the definition in §2). Here by Evans-Perkins type formula we mean an explicit stochastic integral representation for historical functional of a certain class, which is similar to and is a historical process counterpart of Itô-Clark formula (e.g. [U95, p.42]) in elementary stochastic calculus. The key idea of demonstration of the Itô-Clark type formula for historical superprocess is to derive a variant of stochastic integration by parts with respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two reasons why their integration by parts formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In addition, a generalization of formula itself is of independent interest, and it is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any \( L^2 \) functional of superprocess may be represented as a constant \( C_0 \) plus a stochastic integral with respect to the associated orthogonal martingale measure \( M \) (e.g. [EP94] ). Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional

*Research supported in part by JMESC Grant-in-Aids SR(C) 07640280 and CR(A) 09304022.
of the so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95].

2 Notation and Preliminaries

Let \( C = C^d = C([0, \infty), \mathbb{R}^d) \) denote the space of \( \mathbb{R}^d \)-valued continuous paths on \( \mathbb{R}_+ = [0, \infty) \) with the compact-open topology. \( C = \mathcal{B}(C) \) is its Borel \( \sigma \)-field and

\[
C_t = \mathcal{B}_t(C) = \sigma(y(s), s \leq t)
\]
denotes its canonical filtration. For \( y, w \in C^d \) and \( s \geq 0 \), we define the stopped path by \( y^s(t) = y(t \wedge s) \) and let

\[
y/s/w = \begin{cases} y(t), & \text{for } t < s, \\ w(t - s), & \text{for } t \geq s. \end{cases}
\]

\( M_F(C) \) is the space of finite measures on \( C \) with the topology of weak convergence and we define

\[
M_F(C)^t := \{ m \in M_F(C); y = y^t, \text{ } m - a.s. \text{ } y \}, \quad t \geq 0.
\]

If \( P_x \) denotes Wiener measure on \( (C, \mathcal{B}(C)) \) starting at \( x \), \( \tau \geq 0 \), and \( m \in M_F(C)^\tau \), define \( P_{\tau, m} \in M_F(C) \) by

\[
P_{\tau, m}(A) := \int_C P_{y(\tau)}(\{w; y/\tau/w \in A\}) dm(y).
\]

Let

\[
\Omega_H[\tau, \infty) := \{ H \in C([\tau, \infty), M_F(C)); H_t \in M_F(C)^t, \forall t \geq \tau \},
\]

and put \( \Omega_H := \Omega_H[0, \infty) \). We write \( \mathcal{H} \) for the totality of Borel sets of \( \Omega_H \). We use the notation \( H_t(\omega) = \omega(t) \) for \( \omega \in \Omega_H \) as for the canonical realization of historical process.

Fix \( 0 \leq t_1 < \cdots < t_n \) and \( \psi \in C^2_b(\mathbb{R}^{nd}) \). For \( y \in C \) we set

\[
\overline{y}(t) = (y(t \wedge t_1), \cdots, y(t \wedge t_n)),
\]

\[
\overline{\psi}(y) = \overline{\psi}(t_1, \cdots, t_n)(y) = \dot{\psi}(y(t_1), \cdots, y(t_n)),
\]

and \( \tilde{\psi}(t,y) = \overline{\psi}(y^t) \). \( \psi_i \) (resp. \( \psi_{ij} \)) stands for the first (resp. second) order partials \( \partial_i \psi \) (resp. \( \partial_{ij} \psi \)) of \( \psi \). \( \nabla \overline{\psi} : [0, \infty) \times C \rightarrow \mathbb{R}^d \) is the \( (C_t) \)-predictable process whose \( i \)-th component at \( (t, y) \) is given by

\[
\sum_{i=0}^{n-1} \mathbf{I}(t < t_{i+1}) \psi_{id+i}(\tilde{y}(t)).
\]

While, for \( 1 \leq i, j \leq d \), \( \tilde{\psi}_{ij} : [0, \infty) \times C \rightarrow \mathbb{R} \) is the \( (C_t) \)-predictable process defined by

\[
\tilde{\psi}_{ij}(t,y) := \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \mathbf{I}(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i} \partial_{ld+j}(\tilde{y}(t)).
\]
Let us define the domains
\[
    D_0 := \bigcup_{n=1}^{\infty} \{ \tilde{\psi}(t_1, \cdots, t_n); 0 \leq t_1 < \cdots < t_n, \ \psi \in C^\infty_0(\mathbb{R}^{nd}) \} \cup \{1\},
\]
\[
    \tilde{D}_0 := \left\{ \tilde{\psi}; \ \tilde{\psi}(t, y) = \tilde{\psi}(y') \text{ for some } \tilde{\psi} \in D_0 \right\}.
\]

Let $\tilde{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq \tau}, \mathbb{P})$ be a filtered probability space and let $(\omega, y) = (\omega, y_1, \cdots, y_d)$ denote sample points in $\Omega = \Omega \times C^d$. Here $\tau \geq 0$ is fixed. When $f$ is a function on $[\tau, \infty) \times \tilde{\Omega}$ taking values in a normed linear space $(E, || ||)$, then a bounded $(\mathcal{F}_t)$-stopping time $T$ is a reducing time for if and only if
\[
    I(\tau < t \leq T) ||f(t, \omega, y)||
\]
is uniformly bounded. In addition we say that a sequence $\{T_n\}$ reduces $f$ if and only if each $T_n$ reduces $f$ and $T_n \not\to \infty$ holds $\mathbb{P}$-a.s. We say that $f$ is locally bounded if such a sequence $\{T_n\}$ exists. We assume that
\[
    (LB) \ \gamma \in [0, \infty), a \in S^d, b \in \mathbb{R}^d \text{ and } g \in \mathbb{R} \text{ are } (\hat{\mathcal{F}}_t^*)\text{-predictable processes on } [\tau, \infty) \times \tilde{\Omega}
\]
such that $\Lambda = (\gamma, a, b, g^{-1} \mathbb{I}(g \neq 0))$ is locally bounded.

Notice that the above assumption implies that $g$ is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For $\tau \geq 0$ and $m \in M_F(C)^\tau$, we define
\[
    A_{\tau, m, \tilde{\psi}}(t, y) \equiv A(\tilde{\psi})(t, y) := \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(t, y)\tilde{\psi}_{ij}(t, y) + b(t, \omega, y) \cdot \nabla \tilde{\psi}(t, y) + g(t, \omega, y)\tilde{\psi}(y')
\]
for $\tilde{\psi} \in D_0$. We write $\langle \mu, f \rangle$ or sometimes $\mu(f)$ for the integral $\int f \, d\mu$ when $\mu$ is a measure and $f$ is a suitable $\mu$-integrable function. Suggested by [DkTn98], we may define

**Definition 1 (cf. [P95], §2)** A predictable process $K = \{K_t, t \geq \tau\}$ on $\tilde{\Omega}$ with sample paths a.s. in $\Omega_H[\tau, \infty)$ is a generalized $\{\gamma, a, b, g\}$-historical process (GHP) (or $(A, -\gamma \lambda^2/2)$-historical process) if and only if $K_t \in M_F(C)^\tau$ for all $t \geq \tau$, a.s. and $P[K_t(1)] < \infty$, and if there exists a probability measure $\mathcal{P}$ on $\Omega_H[\tau, \infty)$ such that it satisfies the martingale problem (MP) with initial data $\{\tau, m\}$ and $\{\gamma, a, b, g\}$: for $\forall \tilde{\psi} \in D_0$,
\[
    Z_t(\tilde{\psi}) = \langle K_t, \tilde{\psi} \rangle - \langle m, \tilde{\psi} \rangle - \int_{\tau}^{t} \langle K_s, A(\tilde{\psi})(s) \rangle \, ds, \quad t \geq \tau,
\]
is a continuous $(\mathcal{F}_t)$-local martingale satisfying $Z_\tau(\tilde{\psi}) = 0$ and
\[
    \langle Z(\tilde{\psi}) \rangle_t = \int_{\tau}^{t} \gamma(s, \omega, y)\tilde{\psi}(y)^2K_s(dy)ds, \quad \forall t \geq \tau, \text{ a.s.}
\]

**Remark.** The existence and uniqueness of the law of $K$ is essentially due to [F88] (cf. [DIP89]).
Set \( T_s = [s, \infty) \), and in particular \( T_0 = [\tau, \infty) \). Define \( C(M_{F}(C)) := C(T_0; M_{F}(C)) \), and we write \( C(t) = (\tau, t] \times C \) for the integral domain. When \( \mathcal{F} \) is the \( \sigma \)-field or the usual filtration, then \( f \in \mathcal{F} \) indicates that the function \( f \) is \( \mathcal{F} \)-measurable and \( P(\mathcal{F}) \) is the totality of \( \mathcal{F} \)-predictable functions, and \( bP(\mathcal{F}) \) denotes the whole space of functions that are all bounded elements of \( \mathcal{F} \). We use the symbol \( U(M_{F}(C)) \) for an admissible subset of the space \( C(C(M_{F}(C)); \mathbb{R}) \); more precisely \( U(M_{F}(C)) \) is the totality of real valued continuous functions \( F \) on \( C(M_{F}(C)) \) such that for some compactly supported finite measure \( L(dt) \) on \( T_0 \), the estimate
\[
|\Delta F(h, g)| \leq \int_{T_0} g(t, C) L(dt)
\]
holds for all \( h, g \in C(M_{F}(C)) \), where we define \( \Delta F(x, y) := F(x+y) - F(x) \).

3 Predictable Representation Property

Let \( \{T_N\} \) be a reducing sequence. Take a sequence \( \{\bar{\psi}_n\} \), \( \bar{\psi}_n \in D_0 \) such that \( \bar{\psi}_n \) converges bounded pointwise (bp for short) to \( \psi \), namely,
\[
\bar{\psi}_n \rightarrow \psi, \quad \text{bp} \quad (n \rightarrow \infty).
\]

An application of dominated convergence theorem together with the local boundedness of \( \gamma \) implies that
\[
\langle Z(\bar{\psi}_n - \bar{\psi}_m) \rangle_t \rightarrow 0 \quad \text{as} \quad n, m \rightarrow \infty
\]
for all \( t \geq \tau \), a.s. Therefore we obtain

**Proposition 1** There is an a.s. continuous adapted process \( \{Z_t(\psi); t \geq \tau\} \) such that
\[
\sup_{\tau \leq t \leq N} \left| Z_t(\bar{\psi}_n) - Z_t(\psi) \right| \rightarrow 0
\]
holds in probability ( w.r.t. \( P \) ) as \( n \rightarrow \infty \) for all \( N > \tau \).

To proceed our discussion, we need the following lemmas.

**Lemma 1** (cf. Corollary 2.2, p.11, [P95]) Let \( T \) be a reducing time for \((\gamma, g)\). Then we have
\[
(a) \quad 0 < P[K_T(1)] \leq P[\sup_{\tau \leq t \leq T} |K_t(1)| + \langle Z(1) \rangle_T] < \infty.
(b) \quad \text{If} \quad P[K_\tau(1)^p] < \infty \quad \text{for} \quad p \in \mathbb{N}, \quad \text{then}
\]
\[
P \left\{ \left( \sup_{\tau \leq t \leq T} |K_t(1)| \right)^p + \langle Z(1) \rangle_T^p \right\} < \infty.
\]

**Lemma 2** (cf. [EP94, p.123]) \( D_0 \) is dense in \( b\mathcal{B}(C) \) relative to the bounded pointwise convergence topology.
We may use Lemma 1 to obtain
\[ \sup_{\tau \leq t \leq T_{N}} |Z_{t}(\psi_{n}) - Z_{t}(\psi)| \to 0 \quad \text{in} \quad L^{2} \]
as \( n \to \infty \), for \( \forall N \in \mathbb{N} \). Clearly \( Z_{t}(\psi) \) is a continuous \((\mathcal{F}_{t})\)-local martingale whose quadratic variation process is given by
\[ \langle Z(\psi) \rangle_{t} = \int_{\tau}^{t} \int_{C} \gamma(s, \omega, y) \psi(y) 2K_{s}(dy) dS. \]  
By virtue of Lemma 2, it is a routine work to show that this \( Z_{t} \) extends to an orthogonal martingale measure
\[ \{ Z_{t}(\psi); t \geq \tau, \psi \in b\mathcal{B}(C) \}. \]
Consequently, the mapping \( t \mapsto Z_{t}(\psi) \) is a continuous local martingale satisfying Eq. (3) for each \( \psi \in b\mathcal{B}(C) \), and \( \psi \mapsto Z_{t\wedge T_{N}}(\psi) \) is an \( L^{2} \)-valued measure on \( \mathcal{B}(C) \) for each \( t \geq \tau \), \( N \in \mathbb{N} \). By a trivial localization argument, we may define the stochastic integral
\[ Z_{t}(\psi) = \int_{\tau}^{t} \int \psi(s, \omega, y) dM(s, y) \]  
( \( \exists \) an orthogonal martingale measure \( M = M^{K} \) in the sense of Walsh [W86, Chapter 2] ) such that
\[ \langle Z(\psi) \rangle_{t} = \int_{\tau}^{t} \langle K_{s}, \gamma(s, \omega) \psi(s, \omega)^{2} \rangle ds, \]  
\( \forall t \geq \tau \), a.s., as long as \( \psi \) belongs to \( L^{2}_{X}(K, \mathbb{P}) \). Here \( L^{2}_{X}(K, \mathbb{P}) \) denotes the \( L^{2} \) space of \((\mathcal{F}_{t} \times C)_{t \geq \tau}\)-predictable functions \( f \) and
\[ \int_{\tau}^{t} \int \gamma(s, y) f(s, y)^{2} K_{s}(dy) ds < \infty \]  
for \( \forall t \geq \tau \), \( \mathbb{P} \)-a.s.

We write \( f \in L^{2}(K, \mathbb{P}) \) (resp. \( L^{2}_{X}(K, \mathbb{P}) \) ) if, in addition,
\[ \mathbb{P} \left\{ \int_{\tau}^{t} \int \gamma(s, \omega, y) f(s, \omega, y)^{2} K_{s}(dy) ds \right\} < \infty, \quad \forall t > 0, \]
respectively,
\[ \mathbb{P} \left\{ \int_{\tau}^{\infty} \int \gamma(s, \omega, y) f(s, \omega, y)^{2} K_{s}(dy) ds \right\} < \infty. \]

**Theorem 1 (Predictable Representation Property)** If \( V \in L^{2}(\Omega, \mathcal{F}, \mathbb{P}) \), then there is an \( f \) in \( L^{2}_{X}(K, \mathbb{P}) \) such that
\[ V = \mathbb{P}[V] + \int_{\tau}^{\infty} f(s, \omega, y) dM^{K}(s, y), \quad \mathbb{P} \text{–a.s.} \]  
The proof of Theorem 1 will be given in the succeeding section.

**Remark.** The predictable representation property was proved by Evans-Perkins (1994) [EP94, Theorem 1.1] for the \((Y, -\lambda^{2}/2)\)-superprocess with a Hunt process \( Y \) as its underlying process. In [EP95] a variant of the stochastic integral representation formula of the above type was proved for the \((Y, -\lambda^{2}/2)\)-historical process with a Markov process \( Y \).
4 Proof of Theorem 1

If \( f \in bB(C) \), then the moment \( \mathbb{P}[K_t(f)] \) is uniformly bounded as \( t \) ranges over a compact subset of \([\tau, \infty)\). We have the following explicit formula for the moment, namely,

**Lemma 3** \( \mathbb{P}[K_t(f)] = P_{\tau,\nu}[f(Y^t)] \) holds for every \( f \) in \( bB(C) \) under \( \nu \in M_F(C)^\mathcal{T} \), where \( Y^t \) is the corresponding stopped path-valued process.

We set \( \hat{E} := \{(s, y) \in [\tau, \infty) \times C; \ y^s = y \} \) and define a measure \( Q_{s,y} \) on \((C, C)\) by

\[
Q_{s,y}(A) := P_{y(s)}\{w \in C; (y/s/w) \in A \}, \quad A \in C, \quad (s, y) \in \hat{E}.
\]

Then a similar argument as in [F88] (cf. Theorem 2.1.3, [DP91]) allows us to show

**Proposition 2** Assume that \( T_{s,t}f(y) := P_{y(s)}[f(y/s/Y^{t-s})] \) satisfies the semigroup property for \((s, y) \in \hat{E}, t \geq s, \) and \( f \in bB(C) \). Then we have

\[
P\{\exp\{-\langle K_t, f \rangle\}\} = \exp\{-\langle m, V_{\tau,t}f \rangle\},
\]

for all \( f \in bpB(C) \) and \( m \in M_F(C) \). Moreover, \( \{V_{\tau,t}\} \) forms a semigroup on \( bpC \), and \( V_{s,t}f(y) \equiv \nu_{s,t}(y) \) is Borel measurable as a function of \((s, y, t)\) in \( \hat{E} \times [\tau, \infty) \) with \( t \geq s \), and is the unique solution of

\[
\nu_{s,t}(y) = P_{y(s)}[f(y/s/Y^{t-s})] - \frac{1}{2} \int_{\tau}^{t-s} P_{y(s)}[\gamma(u, \omega, y)\nu_{u+s,t}(y/s/Y^u)]du.
\]

**Proof of Lemma 3.** According to the same discussion as in Theorem 2.1.5 [DP91, p.19], we can deduce from Proposition 2 that under \( \nu \in M_F(C)^\mathcal{T} \)

\[
\mathbb{P}[\langle K_t, f \rangle] = \langle \nu, G_{\tau,t}f \rangle, \quad ---(*)
\]

where \( G_{s,t}f(y) = Q_{s,y}[f(Y_t)] \). A simple computation reads

\[
\langle \nu, G_{\tau,t}f \rangle = \int_C Q_{\tau,y}(f(Y_t))\nu(dy)
\]

\[
= \int_C \left\{ \int_C f(Y_t)P_{y(\tau)}\{w \in C; (y/\tau/w) \in d\zeta\} \right\} \nu(dy)
\]

\[
= \int_C f(Y^t) \int_C P_{y(\tau)}\{(y/\tau/Y) \in d\zeta\} \nu(dy)
\]

\[
= \int_C f(Y^t)P_{\tau,\nu}(dy) = P_{\tau,\nu}[f(Y^t)],
\]

because we made use of the Fubini theorem in the second line. By \((*)\), this concludes the proof. Q.E.D.

Suggested by the argument [MP92, pp.331-332] (also see [EP95, pp.1779-1780]), we define

\[
P^{\tau,\nu} := \{ \varphi \in b(B([\tau, \infty)) \times C); \varphi(t, y) = \varphi(t, y^t) \quad \text{for all} \quad t \geq \tau, \}
\]

the map \( t \mapsto \varphi(t, Y) \) is \( P^{\tau,\nu} \)-a.s. right continuous, \( \forall t \geq \tau \).
under $\nu \in M_F(C)^t$, and $\tilde{F}^{\tau,\nu}$ is the set of bounded functions $\psi$ in $B([\tau, \infty)) \times \mathcal{F} \times C$ such that
$$\psi(\cdot, \omega, \cdot) \in \tilde{F}^{\tau,\nu}, \ P - \text{a.s.,}$$
and the condition (C) is compatible with the definition of $K$ in §2.

(C) For $H_t \in M_F(C)^t$, $P$-a.s. for all $t \geq \tau$ with $Y$ as its corresponding path-valued process, and for all $\varphi \in F^{\tau,\nu}$,
$$M_t(\varphi) := \langle H_t, \varphi(t, \cdot) \rangle - \langle \nu, \varphi(\tau, \cdot) \rangle - \int_{(\tau, t]} \langle H_s \psi(s, \omega, \cdot) \rangle dS,$$
$t \geq \tau$, under $\nu \in M_F(C)^t$,
is a continuous $(\mathcal{F}_t)_{t \geq \tau}$ martingale for which $M_\tau(\varphi) = 0$ and
$$\langle M(\varphi) \rangle_t = \int_{(\tau, t]} \int_C \gamma(s, \omega, y) \varphi(s, y)^2 H_s(dy)ds.$$  

Let $A^{\tau,\nu}$ denote the set of pairs $$(\varphi, \psi) \in F^{\tau,\nu} \times \tilde{F}^{\tau,\nu}$$ such that
$$Z_t := \varphi(t, Y) - \varphi(\tau, Y) - \int_{[\tau, t]} \psi(s, Y)ds, \ t \geq \tau,$$
is a $(\tilde{C}_t^\nu)_{t \geq \tau}$-martingale under $P_{\tau,\nu}$, where $\tilde{C}_t^\nu$ is the $\sigma$-field generated by $C_{t+}$ and the $P_{\tau,\nu}$-null sets in $C$.

**Proposition 3** There exists for each $n \in \mathbb{N}$ a function $g_n = g_n(t, \omega, y)$ in $bP(C_t \times \mathcal{F}_t)$ such that
$$V = P[V] + \lim_{n \to \infty} \int_{[\tau, \infty)} \int_C g_n(s, \omega, y) dM^K(s, y),$$
with $L^2(P)$-convergence.

**Proof.** Recall the condition (C). By virtue of Theorem 2 and Proposition 2 of Jacod (1977) [J77] (e.g. [EP94, p.124] or [EP95, p.1796]), we can deduce that for each $n \in \mathbb{N}$ there exist suitable pairs
$$(\varphi^1_n, \psi^1_n), \ldots, (\varphi^N_n, \psi^N_n) \in A^{\tau,\nu},$$
(relative to $K_t$), $\xi^1_n, \ldots, \xi^N_n \in bP(\mathcal{F}_t)$, and $\{t_n\}_n \subset (\tau, \infty)$ such that $t_n \nearrow \infty$ (as $n \to \infty$) and
$$V = P[V] + \lim_{n \to \infty} \int_{[\tau, t_n]} \int_C \sum_k \xi^k_n(s, \omega) \varphi^k_n(s, y) dM^K(s, y),$$
where the convergence is in $L^2(P)$. Moreover, we can choose a bounded $(\mathcal{C}_t)_{t \geq \tau}$-predictable function $\eta$ such that
$$\int \int_{C(t)} \xi(s, \omega) \varphi(s, y) dM^K(s, y) = \int \int_{C(t)} \xi(s, \omega) \eta(s, y) dM^K(s, y), \quad P - \text{a.s.,} \quad \forall t \geq \tau,$$
for each $(\varphi, \psi) \in A^{\tau,m}$ and each $\xi$ in $bP(\mathcal{F}_t)$, and also that the $y$-section
\[
\{(s, y) \in [\tau, \infty) \times C; \varphi(s, y) \neq \eta(s, y)\}
\]
is a countable set. By the property of stochastic integral and the Fubini type theorem, we readily obtain
\[
P\left[\int_\tau^t \int_{C(t)} \xi \varphi dM^K - \int_\tau^t \int_{C(t)} \xi \eta dM^K \right]^2 = P\left[\int_\tau^t \int_{C(t)} \gamma(s) \{\varphi(s, y) - \eta(s, y)\}^2 dK_s ds\right]
\leq C_0 \cdot P\left[\int_\tau^t \int_C \{\varphi(s, y) - \eta(s, y)\}^2 K_s(dy) ds\right]
= C_0 \int_\tau^t P\left[K_s(|\varphi_s - \eta_s|^2)\right] ds.
\]
for some constant $C_0$. By Lemma 3, the last term in the above can be replaced by
\[
\int_\tau^t P_{\tau,m} \left[|\varphi(s, Y^s) - \eta(s, Y^s)|^2\right] ds,
\]
which, indeed, becomes null if we apply the Fubini theorem again because we employed the condition
\[
\int_{(\tau,t]} \{\varphi(s, Y) - \eta(s, Y)\}^2 ds = 0, \quad \forall t > \tau, \ P_{\tau,m} - a.s.
\]
So that, by making use of the above-mentioned $\eta$, we have only to set
\[
g_n(s, \omega, y) = \sum_k \xi_k(s, \omega) \eta_n^k(s, y)
\]
for each $n$. This completes the proof. Q.E.D.

By virtue of the arguments in the proof of Proposition 3, we have that
\[
0 = \lim_{n,k \to \infty} \left\|\int_{C(\infty)} g_n(s, \cdot, y) dM^K(s, y) - \int_{C(\infty)} g_k(s, \cdot, y) dM^K(s, y)\right\|_{L^2(P)}^2
= \lim_{n,k \to \infty} ||g_n(\cdot) - g_k(\cdot)||_{L^2(K, P)}^2.
\]
Hence there exists a limit function $f$ in $L^2(K, P)$ such that
\[
0 = \lim_{n \to \infty} P\left[\int_\tau^t \int_C \gamma(s, \omega, y) \{g_n(s, \omega, y) - f(s, \omega, y)\}^2 K_s(dy) ds\right]
= \lim_{n \to \infty} P\left[\int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y) - \int_{C(\infty)} f(s, \omega, y) dM^K(s, y)\right]^2.
\]
Immediately this implies from Proposition 3 that
\[
V = P[V] + \lim_{n \to \infty} \int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y)
= \int_{C(\infty)} f(s, \omega, y) dM^K(s, y),
\]
which completes the proof of Theorem 1.
5 Canonical Measure and Campbell Measure

For $y \in D = D(\mathbb{R}_+; \mathbb{R}^d)$, we define $y^t-(s)$ as $y(s)$ itself if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s,y)$ is a $\sigma$-finite measure on $C(M_F(D))$ such that

$$Q\left(s, y^{s-}; \{h \in C(M_F(D)) ; \tau \leq t \leq s, h(t) \neq 0\}\right) = 0,$$

which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ [D93] associated with the law of $K_t = K(t)$ and the path restriction mapping $\pi$ (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory (1991) (e.g. Theorem 2.2.3(p.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here $R$ is characterized by

$$\log^p_{s,\delta_{y}}[\exp\langle K_{t}, -\varphi\rangle] = \int_{M_{F}}(M_{F}(c))([\exp\langle K_{t}, -\varphi\rangle] - 1R(S, t, y;d\zeta)$$

(cf. Lemma 1 in [Dk99c]; see also [DP91, Proposition 3.3, pp.38-39]). Let $F$ be a real valued Borel function on $C(M_F(C))$. Assume that

$$I_{s,y}^{Q}[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h, g)Q(s, y^{s-}; dg)$$

is well-defined and bounded below for all $s > \tau$, $y \in C$, and $h \in C(M_F(C))$. For a bounded $(\mathcal{F}_t)$-stopping time $T$, we define the Campbell measure $P_T$ associated with $K(t)$ by

$$P_T(A \times B) := \mathbf{P}(K(T, A) \cdot \mathbf{I}_{B}\{K(\tau)\})/m(C)$$

for any $A \times B \in (C \times \Omega, C \times \mathcal{F})$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_\tau = m$. Since the mapping $(s, y, \omega) \mapsto I_{s,y}^{Q}[\Delta F](K(\omega))$ is bounded below and measurable with respect to the product of the predictable $\sigma$-field associated with the filtration $(\mathcal{C}_t)$ and the $\sigma$-field $\mathcal{F}$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(\mathcal{C}_t \times \mathcal{F}_t)_{t\geq \tau}$-predictable function $\mathbf{Pr}[F](s, y, \omega) : (\tau, \infty) \times C \times \Omega \rightarrow \mathbb{R}$ such that

$$P_T\{I^{Q}[\Delta F](T, y)K(T, dy)\}/(C \times \mathcal{F})_{T} = \mathbf{Pr}[F](T, \omega, y)$$

holds $P_T$-a.s. for all bounded $(\mathcal{F}_t)$-predictable stopping times $T > s$. It is quite interesting to note that in particular

$$\mathbf{P} \int_C I^{Q}[\Delta F](T, y)K(T, dy) = \mathbf{P} \int_C \mathbf{Pr}[F](T, y)K(T, dy).$$

We shall introduce an approximation map. For each $l \in \mathbb{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j); 1 \leq j \leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \cdots < t^{(l)}(k[l]) < \infty,$

$$\lim_{l \rightarrow \infty} \{\sup_k \Delta t[l; k]\} = 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} t^{(l)}(k[l]) = +\infty.$$
The approximation map $W[l]$ from $C(M_F(C))$ into $C(M_F(C))$ is defined by
\[
W[l](g)(t) := \{Sb(t^0(i+1)) \cdot g(t^0(i)) - Sb(t^0(i)) \cdot g(t^0(i+1))\} \Delta t[l;i]^{-1}
\]
if $t \in [t^0(i), t^0(i+1))$, and := $g(t^0(k[l]))$ if $t \geq t^0(k[l])$, for any element $g$ of $C(M_F(C))$ with $Sb(k) = k - t$. Immediately we get

**Lemma 4** (cf. Lemma 4, [DK98a]) Let $F$ be an element of $C(C(M_F(C)); R)$. Then for all $g \in C(M_F(C))$
\[
\lim_{l \to \infty} (F \circ W[l])(g) = F(g).
\]

### 6 Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section. $C^t$ denotes the image of $C$ under the map: $y \mapsto y^t$. We define a measure $K^*[s, t]$ on $C^s$ by $K^*[s, t](F) := K_t(\{y : y^s \in F\})$.

Then the measure $K^*[s, t]$ is atomic with a finite set of atoms, and we write $L[s, t](C \subset C^s)$ for the locations of these atoms. For $s \in (a, b]$, let $\lambda_s[\phi]$ be the random measure on $C$ that places mass $\phi(s, y)$ at each point $y$ in $(L[b, c])^s = L[s, c]$. With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure $Q_N$ on $(\Omega, \mathcal{F})$ such that
\[
\frac{dQ_N}{dP}[\mathcal{F}_t] = \exp \left\{ \int_{\tau}^{t \wedge T_N} \int g^{-1}(s) \mathbb{I}(g(s) \neq 0) dM^K(s, y) \right. \right. \left. \left. - \frac{1}{2} \int_{\tau}^{t \wedge T_N} \int g^2 g^{-1}(s) \mathbb{I}(g(s) \neq 0) K_s(dy) ds \right\}.
\]

For brevity's sake we rather write $\mathcal{E}(t \wedge T_N)$ than the above. On this account, $K_{\wedge T_N}$ satisfies the martingale problem (MP)$[\gamma_N, a_N, b_N, 0]$ instead of (MP)$[\gamma, a, b, g]$, where we set $f_N := f \cdot \mathbb{I}(\tau < t \leq T_N)$. Moreover, for $s \in (a, b]$, $y \in C^s$, the symbol $\mathcal{M}[s, y]$ denotes the mapping of the set of functions $\{m : (\tau, \infty) \to M_F(C)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s, y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \geq s$.

Let us now introduce assumptions for our principal results.

(A.1) $g : [\tau, \infty) \times \Omega \times C \to \mathbb{R}$ is a $(\mathcal{F}_t \times C_t)^*$-predictable process such that $g \cdot \gamma^{-1} \cdot \mathbb{I}(g \neq 0)$ is locally bounded.

(A.2) For any predictable function $f$ on $[\tau, \infty) \times I \times C^* \times \Omega$, the counting measure $n^*$ satisfies
\[
P \int_{C^*} n^*((s, t] \times I) G_t(dx) = m(C^*)(t - s)
\]
where $G_t$ is a marked historical process corresponding to $K$ and $N_t$ is the martingale measure associated with $G_t$ (cf. §7 for details).
There exists a random measure $\Lambda_{\varphi}$ on $(\tau, \infty) \times C$ such that
\[ \int \int_{C(\infty)} f(s, y) \Lambda_{\varphi}(ds \otimes dy) = \int_{a+}^{b} \int_{C} f(S, y) \lambda_{S}[\varphi](dy) d_{S} \]
holds for any suitable predictable function $f$.

(4) $\Psi(s, y) \mathcal{E}(t \wedge T_{N})^{-1}$ is uniformly bounded in $s$, $K_{s}$-a.e. $y$, $\mathbb{Q}_{N}$-a.s.

There exists some constant $C_{0}(>0)$ such that
\[ \int \int_{C(t)} \Psi_{S}(s, y)^{2} \mathcal{E}(t \wedge T_{N})^{-2} \gamma(s, y) K_{s}(dy) ds \leq C_{0} \]
holds $\mathbb{Q}_{N}$-a.s., for all $t \geq \tau$.

Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

7 Stochastic Integration Formulae : Main Results

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let $K$ be a predictable measure-valued process whose law is specified by a general martingale problem $(\text{MP})[\tau, K_{\tau}, \gamma, a, b, g]$.

Theorem 2 (cf. [Dk98b]) Assume that $\Phi : C(M_{F}(C)) \to R$ is a cylinder function with bounded representing function $\varphi : [M(C)]^{k} \to R$ and base $\tau < t(1) < \cdots < t(k)$, such that
\[ |\Delta \varphi(\alpha, \beta)| \leq c_{0} \sum_{j} \beta_{j}(C) \]
for some positive constant $c_{0}$, for all $\alpha, \beta = (\beta_{j}) \in [M(C)]^{k}$. Then for $t > \tau$
\[ P \left\{ \Phi(K) \int \int_{C(t)} \Psi(s, y) dM^{K}(s, y) \right\} = P \int \int_{C(t)} Pr[\Phi](s, y) \Psi(s, y) \gamma(s, y) K_{s}(dy) ds \]
holds where $\Psi$ is a bounded $(C \times \mathcal{F}_{t})_{t \geq \tau}$-predictable function, $K_{t}$ is a GHP, and $Pr[\Phi]$ is a predictable function determined by (9) in accordance with the given $\Phi$.

Remark 1. The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]).

Theorem 3 (Stochastic Integration By Parts) Let $F \in U(M_{F}(C))$. If $\Psi$ is an element of $b \mathcal{P}(C \times \mathcal{F}_{t})$, then for all $t > s$,
\[ P \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM^{K}(s, y) \right\} = P \int \int_{C(t)} Pr[F](s, y) \gamma(s, y) \Psi(s, y) K_{s}(dy) ds \]
(10)
Remark 2. Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional $F(K)$. As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it (cf. §3, [Dk98a]). We shall refer to this generalization in §9.

**Theorem 4 (A Variant of Evans-Perkins Type Formula)** Let $F \in U(M_F(C))$.

\[
F(K) = P[F(K)] + \int_{\tau^+}^{\infty} \int Pr[F](s, y) dM^K(s, y)
\]

where $Pr[F](s, y)$ is a $\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$-measurable version (relative to $P_T$) of

\[
P_T\left[ \int_{C(M_F(C))} \Delta F(K, h) Q(s, y^{s-}; dh) / (D \times \mathcal{F})_T \right].
\]

8 Marked Historical Processes and Girsanov-Dawson-Perkins Theorem

Set $I = [0, 1]$, $E^* = C \times I$ and $C^* = C(\mathbb{R}_+, E^*)$, and let $C^*$ (resp. $C^*_t$) be the Borel $\sigma$-field (resp. the canonical filtration) of $C^*$. Put $x = (y, n) \in E^*$. Let $G$ be the corresponding counterpart historical process of $K$ starting at $(\tau, \mu)$, defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, P^*)$. Suppose that $\varphi : (\tau, \infty) \times C \times \Omega \to I$ be an element of $\mathcal{P}(\mathcal{C}_t \times \mathcal{H}_t)$. Given any cadlag function $n : \mathbb{R}_+ \to I$, we can construct a $\sigma$-finite counting measure $n^*$ on $\mathbb{R}_+ \times I$ by assigning an atom of mass one to each point $(s, z)$ such that $n(s) - n(s-)$ is $z \neq 0$. Put

\[
A(t, x, \omega) := n^*(\{(s, z) : \varphi(s, y, \omega) > z\})
\]

and $B(t, x, \omega) = I\{A(t, x, \omega) = 0\}$. Then we can define an $M_F(C)$-valued process $K[\varphi](t)$ by

\[
K[\varphi; J](t) := \int_{C^*} I(J)(y) B(t, x) G_t(dX).
\]

Put

\[
I_1(\varphi, N) = \int \int_{C^*(t)} \varphi(s, y) dN(s, x), \quad \text{and} \quad I_2(\varphi, G) = \int \int_{C^*(t)} \gamma(s, y) \varphi(s, y)^2 G_s(dx) ds
\]

with $C^*(t) = (\tau, t] \times C^*$. Then we define

\[
\Lambda[\varphi](t) := \exp\{I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G)\}.
\]

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$-martingale. The new probability space $(\Omega, \mathcal{H}, P^*[\varphi])$ is defined by $P^*[\varphi]\{F\} := P^*\{F \cdot \Lambda[\varphi](t)\}$ (cf. [Dk98a]) for any $F \in b\mathcal{H}_t$ with

\[
\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t
\]

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

**Proposition 4** (cf. Theorem 5.1, p.1798, [EP95]) The law of $K[\varphi]$ under $P[\varphi]$ is equivalent to the law of $K$ under $P$. 
9 Sketch of Proofs of Main Theorems

§9.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §7, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function $L^*$ on $C(M_F(C))$ by

$$L^*[g] := \int_{T_0} g(t, C)L(dt) = \langle L, g(\cdot, C) \rangle.$$  

(16)

In connection with the measure $L$ (see §2), we introduce the finite measure $L(l) \equiv L(l, dt)$ which concentrates its mass on $\{t^{(l)}(j); 0 \leq j \leq k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, C) \rangle$ for $g \in C(M_F(C))$. Recall that

$$\int g(t, C) Q(s, y; dg) = \int \xi(C) R(s, t, y; d\xi) = 1$$

holds (cf. Lemma 3, [Dk99a]) with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:

$$\text{P} \int \int_{C(t)} \{Q(s, y^{s-})L^*[g]\} K_s(dy) dS = \lim_{l \rightarrow \infty} \text{P} \int \int_{C(t)} \{Q(s, y^{s-})(L^* \circ W[l])[g]\} K_s(dy) ds$$

holds with $g \in C(M_F(C))$ for all $t > \tau$, and

$$\text{P} \int \int_{C(t)} \text{Pr}[F](s, y) Z(s, y) K_s(dy) dS = \lim_{l \rightarrow \infty} \text{P} \int \int_{C(t)} \text{Pr}[F \circ W[l]](s, y) Z(s, y) K_s(dy) ds.$$  

(17)

holds for all $t > \tau$ if $Z \in P(C_t \times \mathcal{F}_t)$. Since, for each $n \geq 1$, $\text{P}\{K_t(C)^n\}$ is uniformly bounded on compact intervals, we can readily deduce that $\text{P}\{(L^* \circ W[l])[K]^n\}$ is bounded in $l$ for each $n \geq 1$. Moreover,

$$\text{P}\{F(K) \int \int_{C(t)} \Psi(s, y) dM(s, y)\} = \lim_{l \rightarrow \infty} \text{P}\{F \circ W[l]\}(K) \int \int_{C(t)} \Psi(s, y) dM(s, y)\}.$$

To complete the extension discussion in this section we have only to observe that $F \circ W[l]$ satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 4 in §8 is valid. Hence, combining the above results, we obtain

$$\text{P}\{F(K) \int \int_{C(t)} \Psi(s, y) dM\} = \lim_{l \rightarrow \infty} \text{P}\{F \circ W[l]\}(K) \int \int_{C(t)} \Psi(s, y) dM\}$$

$$= \lim_{l \rightarrow \infty} \text{P} \int \int_{C(t)} \text{Pr}[F \circ W[l]](s, y) \Psi(s, y) K_s(dy) ds$$

$$= \text{P} \int \int_{C(t)} \text{Pr}[F](s, y) \Psi(s, y) K_s(dy) ds,$$

which concludes Theorem 3.
§9.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk99a].

Thanks to (A.1), it suffices to verify the integral formula for a special \( \{\gamma_N, a_N, b_n, 0\} \)-historical process \( K_{\wedge T_N} \) under \( Q_N \) instead of the generalized \( K \) (GHP) with \( P \). Indeed, since \( dP = \mathcal{E}(t \wedge T_N)^{-1} dQ_N \), what we have to show is as follows:

\[(The \ Modified \ Stochastic \ Integration \ By \ Parts \ Formula)\]

\[
Q_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \cdot \Phi(K_{\wedge T_N}) \int_{C(t)} \Psi(s, y) dM(s, y) \right\} \\
= Q_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \int_{C(t)} \mathcal{P}r[\Phi](s, y) \gamma(s, y) \Psi(s, y) K_{s \wedge T_N}(dy) ds \right\}.
\]

Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for \( \varphi = \Psi \cdot \mathcal{E}^{-1} \). Hence, by the arguments on exponential martingale formalism for the historical process, \( \Lambda[\Psi \cdot \mathcal{E}^{-1}](t) \) is a \( \mathcal{H}_t \)-martingale and the measure \( Q_N[\Psi \cdot \mathcal{E}^{-1}] \) is given by \( Q_N[\cdot \Lambda[\Psi \cdot \mathcal{E}^{-1}]] \). Then it follows from Dawson’s Girsanov theorem (Proposition 3 in §8) that, for any positive \( \epsilon \),

\[
Q_N \{ \Phi(K_{\wedge T_N}) \} = Q_N[\epsilon \Psi \mathcal{E}^{-1}] \{ \Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}]) \}.
\]

Immediately,

\[
Q_N \left\{ \Phi(K_{\wedge T_N}) \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\
+ Q_N \left\{ \left( \Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}]) - \Phi(K_{\wedge T_N}) \right) \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\
= Q_N \left\{ \Phi(K_{\wedge T_N}) - \Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}]) \right\}.
\]

For simplicity we denote by \( I_1 \) (resp. \( I_2 \)) the first (resp. second) term at the left hand side of the above equality, and put

\[ I_3 = \text{the right hand side with the minus sign.} \]

Then we find that the convergence

\[
\epsilon^{-1} \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1) \to \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y), \quad Q_N - a.s. \quad (\epsilon \to 0)
\]

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

\[
\lim_{\epsilon \downarrow 0} \epsilon^{-1} I_1 = Q_N \left\{ \Phi(K_{\wedge T_N}) \cdot \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y) \right\}.
\]
Paying attention to the fact that
\[ \lim_{\epsilon \downarrow 0} K^* \epsilon \Psi \mathcal{E}^{-1} C \mathcal{E}^{-1}(t) = 0, \quad Q_N - a.s., \]
we can show that \( \lim_{\epsilon \downarrow 0} \epsilon^{-1} I_2 = 0 \), as well.

It remains to treat the third term \( I_3 \). In order to discuss the convergence of \( I_3 \) divided by \( \epsilon \), we need the following:

**Key Lemma** (cf. Lemma 12, [Dk99a])

\[
Q_N \int \left\{ \Phi(M[s, y] K_{\Lambda T_N}) - \Phi(K_{\Lambda T_N}) \right\} \lambda_{\Psi, \epsilon}^{-1}(ds \otimes dy) = -Q_N \int \int \Pr[\Phi] \gamma(s, y) \Psi(s, y) \mathcal{E}^{-1}(t \wedge T_N) dK_{s \wedge T_N}(y) ds.
\]

On the other hand, for \( \epsilon > 0 \) we have
\[
Q_N \left[ \Phi(K(\epsilon \varphi)) - \Phi(K) / \mathcal{F} \right] = \epsilon \cdot e^{-\epsilon \Lambda_{\varphi}(t, \infty) \times C} \int \int_{C(\infty)} \Phi(M[s, y] K - \Phi(K)) \lambda_{\varphi}(ds \otimes dy) + R(\epsilon, \Phi, \varphi) \quad (18)
\]
where the residue function \( R \) satisfies \( |R(\epsilon, \Phi, \varphi)| = o(\epsilon) \). From (18) we get the convergence
\[
\lim_{\epsilon \downarrow 0} \epsilon^{-1} I_3 = -Q_N \int \int_{C(0)} \Pr[\Phi] \gamma(s, y) \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge T_N} ds. \quad (19)
\]

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

§9.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any \( y \in C^s, R(s, t, y) \) denotes the canonical measure (cf §5) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually, \( R \) is a \( \sigma \)-finite measure such that
\[
R(s, t, y; M_F(C)) = r_{s,t}.
\]

Here the crucial point is that the total mass \( r_{s,t} \) does not depend on \( y \). So \( r_{s,t}^{-1} dR(s, t, y) \) becomes a probability measure. It is interesting to note that \( K_t \) is a sum of independent nonzero clusters with laws \( r_{s,t}^{-1} R(s, t, y; dh) \), conditional on \( L[s, t] \) (see §6). Furthermore, conditional on \( \mathcal{F}_s, L[s, t] \) can be regarded as a Poisson point process with intensity \( r_{s,t} \gamma(s) K_s \). This is one of the most important points for the computation in terms of clusters growing from the points of \( L[s, t_{l+1}] \) in what follows. We define a measure \( S \) by the following equation: for \( \forall g \in bB([M_F(C)]^{k-l} \rightarrow \mathbb{R}) \),
\[
\int g(\eta_{l+1}, \cdots, \eta_k) S_{s,t}(d\eta_{l+1} \otimes \cdots \otimes d\eta_k) = \int g(h(t_{l+1}), \cdots, h(t_k)) \cdot I\{h(t_{l+1}) \neq 0\} Q(s, y; dh)
\]
where $Q(s, y; dh)$ is a $\sigma$-finite measure on $C(M_F(C))$ (cf. Eq.(7) in §5). $S_{s,y}^*$ is the normalization of $S_{s,y}$, given by $dS_{s,y}^* := r_{s,t+1}^{-1}dS_{s,y}$. Moreover, we define

$$
\Xi(s; E) := \int \int \cdots \int \varphi(K(t_1), \ldots, K(t_l), \sum_{i=1}^{m} \eta_{l+1}^{i}, \ldots, \sum_{i=1}^{m} \eta_{l}^{i})
\times \prod_{i=1}^{m} S_{s,y}^*(d\eta_{l+1}^{i} \otimes \cdots \otimes d\eta_{l}^{i}),
$$

where $E = \{y_1, \ldots, y_m\} (\neq \emptyset)$.

Take the mass $\varphi$ as $(\Psi \mathcal{E}^{-1})(s, y)$ at each point $y$ (cf. §6). For simplicity we set

$$
\Delta[\Phi](\mathcal{M}; s, y, K) := \Phi(\mathcal{M}[s, y]K_\wedge T_N) - \Phi(K_\wedge T_N).
$$

Recall the assumption (A.3). Immediately we can get

$$
Q_N \int \int_{C(\infty)} \Delta[\Phi](\mathcal{M}; s, y, K) \lambda_{\psi \mathcal{E}^{-1}}(ds \otimes dy)
= Q_N \int_{a+}^{b} \int_{C} \Delta[\Phi](\mathcal{M}; s, y, K) \lambda_{s}[\Psi \mathcal{E}^{-1}](dy)ds
= \int_{a+}^{b} dsQ_N \left\{ \sum_{y \in L[s,u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi \mathcal{E}^{-1})(s, y) \right\}.
$$

In the following calculation, we may take much advantage of those concepts such as i) the Markov property of $K_t$; ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location $L[s, t_{l+1}]$. Hence we can proceed with the computation. In fact,

$$
Q_N \left\{ \sum_{y \in L[s,u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi \mathcal{E}^{-1})(s, y) \right\}
= Q_N \left\{ \mathbb{P} \left[ \sum_{y \in L[s,u]} \mathbb{P}\{ \Delta[\Phi] \cdot \Psi \mathcal{E}^{-1} | \mathcal{F}_s \} \right] \mathcal{F}_s \right\}
= Q_N \left\{ \mathbb{P} \left[ \sum_{y \in L[s,u]} \{ \Xi(s; L[s, u] \setminus \{y\}) - \Xi(s; L[s, u]) \} \cdot \Psi \mathcal{E}^{-1} | \mathcal{F}_s \} \right\}.
$$

It is easy to see the following lemma.

**Lemma 5** The last expression of (20) is equivalent to

$$
Q_N \int_{C}(\Psi \mathcal{E}^{-1})(s, y) \cdot r_{s,t_{l+1}} \gamma(s, y)K_{s\wedge T_N}(dy) \left[ \exp \left( -r_{s,t_{l+1}} K_s(C) \right) \cdot \right.
$$

$$
\times \sum_{m=0}^{\infty} \frac{1}{m!} \int \int \cdots \int_{C} \{ \Xi(s; \{y_1, \ldots, y_m\}) - \Xi(s; \{y_1, \ldots, y_m, y\}) \} \cdot
$$

$$
\times (r_{s,t_{l+1}})^{m} K_s^{\otimes m}(dy_1, \ldots, dy_m).
$$
A simple computation implies that the integral expression in Lemma 5 is also equal to

\[
\mathbf{Q}_N \int_C (\Psi \mathcal{E}^{-1})(s, y) \gamma(s, y) K_s \Lambda \tau N(dy) \cdot \left[ \int \int \cdots \int_{[M_F(C)]^k-l} \right] \\
\times \mathbf{P} \{ \varphi(K(t_1), \cdots, K(t_k)) - \varphi(K(t_1), K(t_1), K(t_{l+1}) + \eta_{l+1}, \cdots, K(t_k) + \eta_k)|\mathcal{F}_s \} \\
\times r_{s,t_{l+1}} \cdot S_{s,y}^* (-d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \right].
\]

(21)

While, taking (7), (8) in §5, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

**Lemma 6** The following equality holds for all \( s, y \):

\[
\mathbf{P} \{ \Phi \} (s, y) = \int \int \cdots \int_{[M_F(C)]^k-l} \mathbf{P} \cdot \mathcal{E}^{-1} \cdot \gamma(s, y) K_s \Lambda \tau N(dy) \\
\times \{ \varphi(K(t_1), \cdots, K(t_k)) - \varphi(K(t_1), \cdots, K(t_k))|\mathcal{F}_s \}.
\]

Therefore, an application of the above assertion with Lemma 5 implies

\[
\mathbf{Q}_N \int_{C(t)} \mathbf{P} \{ \Phi \} (s, y) \cdot \gamma(s, y) K_s \Lambda \tau N(dy) ds \\
= \int_{\tau}^{t} \mathbf{P} \{ \Phi \} (s, y) \cdot \gamma(s, y) K_s \Lambda \tau N(dy) ds
\]

which completes the proof.

10 **Evans-Perkins Type Formula: Proof of Theorem 4**

Since \( \mathbf{P}[K_t(C)^2] \) is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity \( \mathbf{P}[F(K)^2] \). Therefore we can apply Theorem 1 (§3) for \( F(K) \) to obtain that

\[
F(K) = \mathbf{P}[F(K)] + \int_{\tau}^{\infty} \int_C f(s, y) dM^K(s, y), \mathbf{P} - \text{a.s.}
\]

(22)

holds for some \( f \) in \( L^2_{\infty}(K, \mathbf{P}) \). While, it follows from the covariance formula in the theory of stochastic integration that

\[
\mathbf{P} \left[ \left( \int \int_{C(\infty)} f(s, y) dM^K(s, y) \right) \left( \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right) \right]
\]

(23)

\[
= \mathbf{P} \left[ \int_{\tau}^{t} \int_C f(s, y) \Psi(s, y) \gamma(s, y) K_s(dy) ds \right]
\]

for all \( t > \tau \) and \( \Psi \) in \( b\mathcal{P}(C_t \times \mathcal{F}_t) \). Rewriting the left hand side of Eq.(23) we get

\[
\mathbf{P} \left[ F(K) \int_{\tau}^{t} \int_C \Psi(s, y) dM^K(s, y) \right]
\]

(24)
by employing the predictable representation property (22). Hence we may apply Theorem 3 (§7) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded \((C_\ell \times \mathcal{F}_\ell)\)-predictable functions. So that, from (23)

\[
P \int_t \int_{C(\ell)} f(s,y)\Psi(s,y)\gamma(s,y)dK_s ds = P \int_t \int_{C(\ell)} Pr[F](s,y)\Psi(s,y)\gamma(s,y)dK_s ds.
\]

On this account, the general theory of Hilbert spaces shows that

\[
P \int_{\tau}^t \int_C \{f(s,y) - Pr[F](s,y)\}^2 \gamma(s,y) K_s(dy) ds = 0.
\]

Therefore the uniqueness argument allows us to conclude that \(\int \int_{C(\ell)} f dM\) is equivalent to \(\int \int_{C(\ell)} Pr[F] dM\), \(P\)-a.s. Note that \(Pr[F](s,y)\) become null for \(K_s\)-a.s. \(y\), for any \(s > t\), by its construction, as long as we choose \(t\) largely enough for the support of \(m\) to be contained in \([	au,t]\). Consequently, the above integral \(\int P r[F] dM\) can be replaced by \(\int \int_{C(\infty)} Pr[F] dM\), which completes the proof. This goes quite similarly as in the proof of Theorem 2.5 in [EP95].

References


