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LACUNARY SERIES AND THETA FUNCTIONS

A. SEBBAR.

In this paper we wish to explain some interesting relationship between some lacunary series and automorphic forms. These notes are an expanded version of a talk given at the RIMS Symposium on Complex Analysis and Microlocal Analysis in December. 1997.

Plan:

I. Lacunary Series, Special facts, Problems.

II. Modular Relations, asymptotic behaviour near the natural boundary.

III. References.

I. Lacunary Series. Special facts

This section is more analytical, we want to study the two power series

$$\chi(z) = \sum_{n\geq 0} z^{2^n}, \quad |z|<1$$

inside the unit disc and even that the unit circle is a natural boundary, we study the function $\chi(z)$ "outside" the unit disc. The idea depend on the Wigert's theorem (See [W] or [Le]).

Wigert theorem : If $g(z)$ is a function of class $(1,0)$, that is an entire function of at most zero type and of order one :

$$\max_\theta |g(re^{i\theta})| < e^{\varepsilon r} \quad , \quad r \geqslant r_0(\varepsilon)$$

then the function defined by the series

$$f(x) = C + \sum_{n\geqslant 1} g(n)\, x^n \qquad (|x|<1) \qquad (\text{I}.1)$$

and its analytic continuation is regular in the whole plane (including ∞) except at $x=1$. Conversely, if $f(x)$ is a function with the above regularity properties, then there is a function $g(z)$ of class $(1,0)$ such that (I.1) holds in the unit disc. Moreover if $g(z)$ is a polynomial, $f(x)$ is a rational function of $\frac{1}{1-x}$ and conversely.

A sligh extension : If $g(z)$ is a function of class $(1,0)$ and if

$$f(x) = \sum_{n\geqslant 0} g(n)\, x^n \quad , \quad |x|<1 \qquad (\text{I}.2)$$

then $f(\infty)=0$. If $f(x)$ is regular except at $x=1$, there is a function $g(z)$ of class $(1,0)$ such that (I.2) holds.

We can relate the growth conditions of g to the growth conditions of f in the case where the order of g is $\rho<1$. The following result is due to A. O. Gelfond [G] (see also L. Ehrenpreis [E]): with $\sigma = \frac{\rho}{1-\rho}$, $\rho<1$, we have the equivalence of the two statments :

$$|f(z)| < \exp\left(\frac{1}{|1-z|}\right)^{\sigma+\varepsilon} \quad , \quad \lim_{z\to 1} \varepsilon = 0 \quad ; \quad \varepsilon = \varepsilon(z)$$

$$|g(x)| < \exp\, |x|^{\rho+\varepsilon'} \qquad \lim_{|x|\to+\infty} \varepsilon' = 0 \quad ; \quad \varepsilon' = \varepsilon'(x)$$

There is another interpretation of Wigert's theorem, by considering differential operators of infinite order.

if $g(x)$ is an entire function of type zero, that is:

$$\forall \varepsilon > 0, \ \exists C_\varepsilon > 0, \ |g(x)| \le C_\varepsilon \exp(\varepsilon |x|)$$

then $g(D)$, $D = \frac{d}{dx}$, is a well defined differential operator of infinite order. With $z = e^\zeta$, $|z| < 1$, $\mathrm{Re}\,\zeta < 0$ and $g(x) = \sum_{n \ge 0} b_n x^n$, $x \in \mathbb{C}$, we have:

$$\sum_{n \ge 0} g(n) z^n = \sum_{n \ge 0} g(n) e^{n\zeta} = \sum_{n \ge 0} \sum_{m \ge 0} b_m n^m e^{n\zeta}$$

$$= \sum_{m \ge 0} b_m \left(\frac{d}{d\zeta}\right)^m \frac{1}{1 - e^\zeta} = g(D) \frac{1}{1 - e^\zeta} = g(D) \frac{1}{1 - z} ; \ D = \frac{d}{d\zeta}$$

By the general properties of differential operators of infinite order, $g(D) \frac{1}{1 - e^\zeta}$ extends analytically to all $\mathbb{C}_\zeta - 2i\pi\mathbb{Z}$, so that $\sum_{n \ge 0} g(n) z^n$ extends analytically (as a uniform function) to all $\mathbb{C}_z - \{1\}$. This is precisely what the Wigert's theorem means.

We can think of the equality $\sum_{n \ge 0} g(n) z^n = g(D) \frac{1}{1 - z}$, ($D = \frac{d}{d\zeta}$, $z = e^\zeta$) as a <u>representation theorem</u>. Following some ideas of [J], we can give more general (and more abstract) representations in terms of the Poisson Kernel of the disc. These representations are different formulations of Wigert's theorem. Consider again a power series $f(z) = \sum_{p \ge 0} a_p z^p$, with $a_p = g(p)$ where g is an entire function of exponential type zero. Then if $g(w) = \sum_{n \ge 0} b_n w^n$:

i) $\lim_{n \to +\infty} (n! \, |b_n|)^{1/n} = 0$

and

ii) $f(z) = P(D) \, H_r(\theta)$

Where:

$$P(D) = \frac{1}{2} \sum_{n \geq 0} (-i)^n b_n \left(\frac{d}{dz}\right)^n$$

and

$$H_r(\theta - t) = \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} = \frac{e^{it} + z}{e^{it} - z}, \quad z = re^{i\theta}$$

$P(D)$ is now a differential operator of infinite order with constant coefficients. To give an idea of the proof of ii), we introduce the sequence $\varphi_n(z) = \mathcal{D}^n \frac{1}{1-z}$, $\mathcal{D} = z\frac{d}{dz}$, then

$$\varphi_n(z) = \sum_{p \geq 0} p^n z^p, \quad |z| < 1, \quad n \geq 0$$

and for $|z| < 1$: $f(z) = \sum_{n \geq 0} b_n \varphi_n(z)$

Now:

$$H_r(\theta) = \frac{1+z}{1-z} = 2\varphi(z) - 1$$

$$H_r^{(n)}(\theta) = \left(\frac{d}{d\theta}\right)^n H_r(\theta) = 2 i^n \varphi_n(re^{i\theta}), \quad n \geq 1$$

then:

$$\begin{aligned} f(z) &= \frac{b_0}{1-z} + \sum_{n \geq 1} \frac{b_n}{2} i^{-n} H_r^{(n)}(\theta) \\ &= \frac{b_0}{2}\left(H_r(\theta) + 1\right) + \sum_{n \geq 1} \frac{b_n}{2} i^{-n} H_r^{(n)}(\theta) \\ &= \frac{b_0}{2} + \sum_{n \geq 0} \frac{b_n (-i)^n}{2} H_r^{(n)}(\theta) \end{aligned}$$

In [J], the following theorem is proved:

Theorem (Johnson): Let f be analytic in the disc $\{|z|<1\}$, if $f(r,\theta) = O\left(\log(1-r)^{-1}\right)$, then there is two functions of bounded variation on $[0, 2\pi]$ α_0 and α_1 such that:

$$f(r,\theta) = \int_0^{2\pi} P_r(\theta - t)\, d\alpha_0(t) + \int_0^{2\pi} P_r'(\theta - t)\, d\alpha_1(t)$$

Where:

$$P_r(\theta - t) = \mathrm{Re}\left(\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\right).$$

Moreover, this theorem cannot be improved. There is a function f analytic in the unit disc, which is $O(\log(1-r)^{-1})$ and which is not a Poisson-Stieltjes integral [J].

The preceding remarks suggest the following question: <u>What kind of functionals on the circle are induced by entire functions</u> $f(z)=\sum_{n\geq 0} g(n)z^n$, <u>when g is an entire function of exponential type zero</u> ?

The Wigert's theorem and the estimates of A. O Gelfond give some answer to this question: If we start with a power series $f(z)=\sum g(n)z^n$, $|g(z)|=O\left(e^{c'|z|^\rho}\right)$, $\rho<1$ then:

$$f(z)=F\left(\frac{1}{1-z}\right)$$

where $F(\zeta)$ is an entire function of satisfying:

$$|F(\zeta)|=O\left(\exp c\,|\zeta|^{\frac{\rho}{1-\rho}}\right).$$

It is known that if an entire function $F(\zeta)=\sum_{n\geq 0} A_n\zeta^n$ satisfies $M(r)=\sup_{|\zeta|=r}|F(\zeta)|\leq e^{h(r)}$, where $rh'(r)$ is monotonic and if $k(r)$ is the function inverse to $rh'(r)$, then $|A_n|\leq \dfrac{e^{h(k(n))}}{k(n)^n}$.

In our case, we obtain

$$F(\zeta)=\sum_{n\geq 0} A_n\zeta^n,\quad |a_n|\leq C\,\frac{1}{\Gamma\left(\left(\frac{1-\rho}{\rho}\right)n\right)}.$$

thus the series $f(z)=\sum_{n\geq 0} g(n)z^n$, $|z|<1$ defines a hyperfunction on the circle given by $\sum_{n\geq 0} A_n\frac{1}{(z-1)^n}$, that is:

$$S=\sum_{n\geq 0}\frac{A_{n+1}}{n!}\,\delta_1^{(n)}$$

where $\delta_1=\delta_{\theta=0}$ on the circle.

We can study any other point of the unit circle with the help of the following "Generalized Wigert's theorem" of J. Lehner [J]:

Theorem. Let $g_k(z)$, $k \geq 1$ be functions of class $(1,0)$ (that is of exponential type zero and of order one). Let (ε_k) be a sequence of (distinct) complex numbers of modulus one and let the series $g_k(n)$ verifies the following properties:

$$\sum_{k=0}^{\infty} |g_k(n)| \quad \text{converges for all } n$$

$$\limsup_n \left(\sum |g_k(n)|\right)^{1/|n|} \leq 1$$

Then if (α_k) is any bounded sequence, the series:

$$\sum_{k=0}^{\infty} \alpha_k \, \phi_k\left(\frac{z}{\varepsilon_k}\right) \quad , \quad \phi_k(z) = \sum_{n=0}^{\infty}{}' g_k(n) z^n$$

converges uniformly on every compact set disjoint from $\{|z|=1\}$ and defines two holomorphic functions:

$$G_1(z) = \sum_{n \geq 1} a_n z^n \quad , \quad |z| < 1$$

$$G_2(z) = - \sum{}' a_{-n} z^{-n} , \quad |z| > 1$$

Where for all n: $\quad a_n = \sum_{k \geq 0} \alpha_k \, \varepsilon_k^{-n} \, g_k(n)$.

The function $G_1(z)$ is analytic on the unit disc and from the functions ϕ_k, we can determine the functional included by $G_1(z)$ on the unit circle. We consider the function

$$li_j(z) = \sum_{n=0}^{\infty} n^j z \quad , \quad |z| < 1$$

(polylogarithm of "positive" weight, it is an elementary function) if $li_0(z) = \sum_{n\geq 1} z^n = \frac{z}{1-z}$ $(z \neq 1)$, then

$$li_j(z) = \theta^j \frac{z}{1-z} \quad ; \quad \theta = z\frac{d}{dz}$$

The expansion of $li_j(z)$ around the singular point 1 can be found by the use of Stirling numbers; Write as in [T]:

$$li_j(z) = (-1)^{j+1} \sum_{p=0}^{j} \frac{a_p^j}{(z-1)^{p+1}}$$

the coefficients a_p^j are positive integers. If we define σ_r^j by:

$$\sigma_r^j = \frac{(-1)^j}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} i^r \quad ; \quad \sigma_j^j = j!\,, \quad \sigma_j^{j-1} = \frac{j(j-1)}{2}, \ldots$$

then the coefficients a_p^j are given by:

$$a_p^j = (p+1)\,\sigma_j^{p+1} + p!\,\sigma_j^p$$

They satisfy:

$$a_p^j = (-1)^p \sum_{i=0}^{p} (-1)^i \binom{p}{i} (i+1)^j$$

and:

$$a_p^{j+1} = p\, a_{p-1}^j + (p+1)\, a_p^j .$$

All these calculations show that the hyperfunction on the circle induced by the series $li_j(z) = \sum_{n\geq 1} n^j z^n$, $|z|<1$ is:

$$T_j = (-1)^{j+1} \sum_{p=0}^{j} \frac{1}{p!}\, a_p^j\, \delta_1^{(p)}$$

where $\delta_1^{(p)}$ is the hyperfunction on the circle corresponding to the angle $\theta = 0$. Let $g_k(x) = \sum_{j\geq 0} C_{kj}\, x^j$, $k = 0,1,2,\ldots$ Under the hypothesis that:

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 : \sum_{k=0}^{\infty} |C_{kj}| \leq A_\varepsilon \left(\frac{\varepsilon}{(j+1)}\right)^j, \quad j = 0,1,2,\ldots$$

we have:

$$\phi_k(z) = \sum_{n=0}^{\infty} g_k(n)\, z^n = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} C_{kj}\, n^j = g_k(0) + \sum_{j\geq 0} C_{kj}\, li_j(z)$$

The hyperfunction induced by ϕ_k on the circle is then:

$$S_k = \sum_{j\geq 0} C_{kj}\, T_j \quad , \quad k = 0,1,2,\ldots$$

where T_j is as above.

II. The Lacunary Series $\chi(z) = \sum'_{n\geq 0} z^{2^n}$, $|z|<1$

Our problem here is to determine the functional induced by the function χ on the Unit circle. The function χ has many interesting physical and dynamical properties [MS]. It is lacunary and it is related to the Jacobi theta function

$$\vartheta_3(z) = 1 + 2\sum_{n\geq 1} z^{n^2}, \quad |z|<1.$$

More precisely, the functional induced by the function χ on the Unit circle will be determined from an identity relating χ to ϑ_3^4, the fourth power of the theta function.

Let us start with a remark: If $(m_n)_{n\geq 0}$ is a sequence of positive real numbers with $\lim_{n\to+\infty} m_n = +\infty$, we have for $|x| \neq 1$:

$$\frac{1+x^{m_n}}{1-x^{m_n}} = \frac{1+x^{m_0}}{1-x^{m_0}} + \sum_{\nu=1}^{n}\left\{\frac{1+x^{m_\nu}}{1-x^{m_\nu}} - \frac{1+x^{m_{\nu-1}}}{1-x^{m_{\nu-1}}}\right\}$$

$$= \frac{1+x^{m_0}}{1-x^{m_0}} + \sum_{\nu=1}^{n} \frac{2x^{m_{\nu-1}}\left(x^{m_\nu - m_{\nu-1}} - 1\right)}{(x^{m_\nu}-1)(x^{m_{\nu-1}}-1)}$$

In particular if $m_\nu = 2^\nu$, $\nu\geq 0$, then

$$\frac{1+x}{1-x} + \frac{2x}{x^2-1} + \frac{2x^2}{x^4-1} + \frac{2x^4}{x^8-1} + \cdots = \psi(x)$$

where

$$\psi(x) = 1 \text{ if } |x|<1, \quad \psi(x) = -1 \text{ if } |x|>1$$

or:

$$\sum_{\nu\geq 0} \frac{1}{x^{2^\nu} - x^{-2^\nu}} = \varphi(x) = \begin{cases} \frac{x}{x-1}, & |x|<1 \\ \frac{1}{x-1}, & |x|>1 \end{cases}$$

if we introduce the Möbius function μ, the last equality becomes for $|x|<1$

$$\sum_{\nu\geq 0} \frac{x^{2^\nu}}{x^{2^\nu\cdot 2} - 1} = \frac{x}{x-1}, \quad \chi(x) = \sum_{n\geq 1}{}' \mu(n)\,\frac{x^n}{1-x^n}$$

($\sum'_{n\geq 1}$ means that the sum is over odd integers). Hence we obtain a Lambert's series for our lacunary series $\chi(z)$. But this is not enough to give the explicit functional induced by $\chi(z)$ on the unit circle.

The analysis of the function $\chi(z)$ near the unit circle depend on the functional equation satisfied by it. In fact:

$$\chi(z) = \chi(z^2) + z \qquad (FE)_0$$

From $(FE)_0$ we will prove:

Proposition. <u>The function</u> $\chi(z)$ <u>satisfies the estimates:</u>

i) $|\chi(z)| = O(\log(1-|z|))$

ii) $|\chi'(z)| = O((1-|z|)^{-1})$.

The point ii) means that the function $\chi(z)$ belongs to the space $\mathcal{B}$ of Bloch functions, that is:

$$\mathcal{B} = \left\{ f \text{ holomorphic for } |z|<1,\ \sup_{|z|<1} (1-|z|^2)\,|f'(z)| < \infty \right\}$$

This space is a Banach Space and plays an important role in conformal geometry.

What is nice here is that the proposition comes from a general picture. Consider the Weirstrass's non-differentiable function:

$$f_c(x) = \sum_{n\geq 0} 2^{cn} x^{2^n}, \quad 0<x<1.$$

then

$$f_c(x^2) = 2^{-c} f_c(x) - 2^{-c} x \qquad (FE)_c$$

This equation reduces to $(FE)_0$ when $c=0$, it is nonhomogeneous. Another solution of $(FE)_c$ is given by

$$g(x) = \sum_{m\geq 0} \frac{(-1)^{m-1}}{m!} \frac{(\log 1/x)^m}{2^{c+m}-1} \qquad 0<x<1$$

This series is an entire function of $\log 1/x$ and we can explain formally how we get it. We start from:

$$f(x) = \sum_{n\geq 0} 2^{cn} x^{2^n} = \sum_{n\geq 0} 2^{cn} e^{-2^n \log 1/x}, \quad 0<x<1$$

and we expand each exponential factor $e^{-2^n \log 1/x}$. Here c is a positive real number. We obtain formally:

$$f(x) = \sum_{n\geq 0} 2^{cn} \sum_{m\geq 0} \frac{(-1)^m}{m!} 2^{nm} (\log 1/x)^m$$

$$= \sum_{m\geq 0} \frac{(-1)^m}{m!} (\log 1/x)^m \sum_{n\geq 0} 2^{(c+m)n}$$

the last series is divergent but if we give it the sum:

$$\sum_{n\geq 0} 2^{(c+m)n} = \frac{1}{1-2^{c+m}}$$

we obtain the defined solution $g(x)$. We can justify all the calculations made here [MS]. Let $\varphi = f - g$, then

$$\varphi(x^2) = f(x^2) - g(x^2) = 2^{-c}(f(x) - g(x)) = 2^{-c}\varphi(x)$$

and with $\varphi(x) = e^{-c\frac{\log\log x}{\log 2}} A(x)$, we find that:

$$A(x^2) = A(x).$$

and $A(x)$ is a periodic function of the variable $\log\log 1/x$ of period $\log 2$. All this means that for $c>0$:

$$\sum_{n\geq 0} 2^{cn} x^{2^n} = \frac{1}{\log 2} \sum_{n\in\mathbb{Z}} \Gamma\left(c + \frac{2i\pi n}{\log 2}\right) (\log 1/x)^{-c-\frac{2i\pi n}{\log a}}$$

$$- \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!\,(2^{c+n}-1)}$$

For c real and not zero or a negative integer, this identity for the Weierstrass's function is well known [Li]. The case of $c=0$ is a limit case and it is discussed in details in [Ha], [MS]. The result is the following formula:

$$\sum_{n=0}^{\infty} x^{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\,(2^n-1)} \left(\log 1/x\right)^n - \frac{1}{\log 2} \log(\log 1/x)$$
$$+ \frac{1}{2} - \frac{\gamma}{\log 2} - \frac{1}{\log 2} \sum_{k\in\mathbb{Z}^*} \Gamma\left(\frac{-2i\,k\pi}{\log 2}\right)\left(\log\frac{1}{x}\right)^{\frac{2ik\pi}{\log 2}}$$

Where γ is the small constant of Euler. When x tends to unity, the sums:

$$\sum_{k\in\mathbb{Z}} \Gamma\left(\frac{-2i\,k\pi}{\log 2}\right)\left(\log 1/x\right)^{\frac{2ik\pi}{\log 2}} \; ; \; \sum_{n\in\mathbb{Z}} \Gamma\left(c+\frac{2i\pi n}{\log 2}\right)\left(\log 1/x\right)^{\frac{-2i\pi n}{\log 2}}$$

are oscillating between finite values, so we have the two behaviours:

$$\chi(x) \sim -\frac{1}{\log 2}\log(\log 1/x) \quad , \; x\to 1,\ 0<x<1$$
$$\chi'(x) \sim \left(x \log 1/x\right)^{-1} \sum_{n\in\mathbb{Z}} \Gamma\left(c+\frac{2i\pi n}{\log 2}\right)\left(\log 1/x\right)^{\frac{-2i\pi n}{\log 2}}$$

or in other terms:

$$\chi(x) \sim -\frac{1}{\log 2}\log(1-x) \qquad x\to 1,\ 0<x<1$$

$$|\chi'(x)| \leqslant C\,(1-x)^{-1} \qquad 0<x<1.$$

The proposition is proved.

Remark: For the function χ', the last inequality can be obtained with a very simple considerations: Consider the double series expansion of $\frac{x}{1-x}\chi'(x)$:

$$\frac{x}{1-x}\,\chi'(x) = \sum_{n\geq 0} x^n \sum_{k\geq 0} 2^k x^{2^k} = \sum_{n\geq 1}\Big(\sum_{2^k\leq n} 2^k\Big) x^n$$

$$\leq 2\sum_{n\geq 1} n x^n = \frac{2x}{(1-x)^2}$$

hence

$$\chi'(x) \leq 2\,(1-x)^{-1} \qquad 0<x<1$$

But this inequality does not give the more precise information about the oscillating behaviour of $\chi'(x)$, when x tends to 1. In any case, if z is in the unit disc, then:

$$|\chi(z)| \leq \chi(|z|) \leq C_1 \log(1-|z|)^{-1}$$

$$|\chi'(z)| \leq \chi'(|z|) \leq C_2\,(1-|z|)^{-1}$$

where C_1, C_2 are positive constants.

Corollary : The function $\chi(z)$ is not bounded on any angle with center at the origin and with positive opening.

In fact, the behaviour of the function of χ at q and $q\,e^{-2i\pi k/2^s}$ are the same. The functional equation (EF) says that $\chi(q\,e^{-2i\pi k/2^s})$ and $\chi(q)$ are different by a finite sum of terms only. Furthermore, if $|q|=1$ the set $\{q\,e^{-2i\pi k/2^s},\ k,s \text{ integers}\}$ is dense in the unit circle. The result of the Corollary can be proved by the fact that $\chi(z)$ is not a rational function and its Taylor coefficients take only the value 0 or 1.

Our study of the function $\chi(z)$ will be continued from another point of view. We are going to establish some relations of χ with thêta functions.

For $0<k<1$, we define (the periods):

$$K(k) = K = \int_0^1 \{(1-t^2)(1-k^2t^2)\}^{-1/2}\, dt$$

$$K'(k) = K' = \int_0^1 \{(1-t^2)(1-k'^2 t^2)\}^{-1/2}\, dt \quad , \; k^2+k'^2=1$$

The function K is related to the hypergeometric function by $K = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1, k^2)$ and so $K(k)$ can be defined also for complex k, $|k|<1$. For $\tau = i\frac{K'}{K}$, $q = e^{-\pi K'/K}$ we define:

$$\vartheta_3(q) = \sum_{n=-\infty}^{+\infty} q^{n^2} = 1+2\sum_{n\geq 1} q^{n^2}$$

$$\vartheta_4(q) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} = 1+2\sum_{n\geq 1} (-1)^n q^{n^2}$$

then from the theory of thêta functions [H]:

$$\left(\frac{2K}{\pi}\right)^2 = \vartheta_3^4(q)$$

Now we formulate a theorem due to Jacobi:

Theorem. — <u>If</u> $\sigma_1(n)$ <u>is the sum of the divisors of n then</u>

$$\vartheta_3^4(q) = \frac{4K^2}{\pi^2} = 1+8\sum{}' \sigma_1(n)\{q^n+3q^{2n}+3q^{4n}+\dots\}$$

This expansion is a sequence of the Lambert's series of $\vartheta_3^4(q)$

$$\vartheta_3^4(q) = 1+8\sum_{m=1}^{\infty} \frac{m\, q^m}{1+(-1)^m q^m}$$

If we introduce the series:

$$F(q) = \sum{}' \sigma_1(n)\, q^n \quad , \; |q|<1$$

then with $\vartheta_2(q) = \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2}$:

$$\vartheta_2^4(q) = 16\, F(q)$$

and because $\vartheta_3^4(q) = \vartheta_2^4(q) + \vartheta_4^4(q)$, we deduce from the theorem of Jacobi the following equality:

$$\sum{}' \sigma_1(n)\, \chi(q^n) = \frac{1}{24}\left(2\vartheta_3^4(q) - \vartheta_4^4(q) - 1\right)$$
$$= \frac{1}{24}\left(2\vartheta_3^4(q) - \vartheta_3^4(-q) - 1\right)$$

($\sum'$ means the sum over odd integers). Our main result is:

Theorem - <u>Let g be the multiplicative function defined by:</u>

$$g(1) = 1 \quad , \qquad g(2^\alpha) = 0, \quad \alpha \geq 1$$

<u>and if p is a prime,</u> $p \geq 3$:

$$g(p) = -(1+p) \quad , \quad g(p^2) = p$$
$$g(p^\alpha) = 0 \qquad \text{for } \alpha \geq 3$$

<u>then for</u> $|q| < 1$:

$$\chi(q) = \frac{1}{24} \sum{}' g(n) \left(2\vartheta_3^4(q^n) - \vartheta_3^4(-q^n) - 1\right).$$

This theorem shows that $\chi(q)$ has in some sens a modular character. The multiplicative function g has the following properties [MS]:

a) $|g(k)| \leq k^2$, for k integer

b) if μ is the Möbius function, then (the prime-number theorem) $\sum_{n \geq 1} \frac{\mu(n)}{n} = 0$, hence $\sum_{k \geq 1}' \frac{g(k)}{k^2} = 0$.

To give an idea about the proof of the theorem, we observe that if ζ is the Riemann Zeta-function, then:

$$\zeta(s)\,\zeta(s-1) = \sum_{n\geq 1} \frac{\sigma_1(n)}{n^s} \qquad ; \quad \sigma_1(n) = \sum_{d|n} d$$

So that:

$$\sum_{n\geq 1}{}' \frac{1}{n^s} = (1-2^{-s})\,\zeta(s)$$

and for Re s > 2:

$$(1-2^{-s})(1-2^{-s+1})\,\zeta(s)\,\zeta(s-1) = \sum{}' \frac{\sigma_1(n)}{n}$$

and from the infinite product of ζ, $\zeta(s) = \prod_{p \text{ prime}} (1-1/p)^{-1}$, we obtain

$$\sum_{n\geq 1}{}' \frac{\sigma_1(n)}{n^s} = \prod_{\substack{p\geq 3 \\ p \text{ prime}}} \left(1 - \frac{1+p}{p^s} + \frac{p}{p^{2s}}\right)^{-1}$$

The function g of the theorem, defined on all of the integers by the equality $g(mn) = g(m)\,g(n)$ if m and n have non common divisors, it satisfies:

$$\sum_{n\geq 1}{}' \frac{g(n)}{n^s} = \prod_{\substack{p\geq 3 \\ p \text{ prime}}} \left(1 - \frac{1+p}{p^s} + \frac{p}{p^{2s}}\right).$$

But this means that if we have a relation:

$$\sum{}' \sigma_1(n)\, f(q^n) = F(q)$$

then:

$$\sum{}' g(n)\, F(q^n) = f(q).$$

The theorem follows.

With the identity of the main theorem, the initial problem of the determination of the functional associated to the function X is now posed for the function ϑ_3^4. We explain breefly how to solve this new problem. The details are in [MS] and depend on the Circle Method and modular forms theory.

Let $r_4(n)$ be the number of diffeuent representations of n as a sum of 4 squares. Then:

$$\vartheta_3^4(q) = \left(1+2\sum_{n\geq 1} q^{n^2}\right)^4 = 1+\sum_{n\geq 1} r_4(n) q^n$$

There are so many nice properties of the arithmetical function $r_4(n)$ based on elliptic thêta functions. $\vartheta_3^4(q)$ is a modular form of weight 2 for the subgroup $\Gamma_0(4)$ of $SL(2,\mathbb{Z})$:

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),\ 4|c \right\}$$

if $li_4(z)$ is the function

$$li_4(z) = \frac{z}{(1-z)^2} = -\frac{1}{1-z} + \frac{1}{(1-z)^2}, \quad z \neq 1$$

then

$$li_4(z) = \sum_{n\geq 1} n z^n = li_4(1/z) \qquad \text{for } |z|<1$$

and we have the following theorem:

Theorem. <u>Let S_{hk} be the Gauss sum</u> $S_{hk} = \sum_{1\leq j\leq k} e^{2i\pi j^2 h/k}$

<u>then</u>:
$$\vartheta_3^4(q) = 1+\pi^2 \sum_{\substack{k\geq 1 \\ (h,k)=1}} \left(\frac{S_{hk}}{k}\right)^4 li_4\left(q\, e^{-2i\pi h/k}\right).$$

As application of this theorem, if h is any holomorphic function in the neighbourhood of the closed unit disc, and if γ any closed path around the closed unit disc on which h is holomorphic and positively oriented, then:

$$\frac{1}{2i\pi}\int_\gamma h(z)\, li_4(z)\, dz = h(1) + h'(1)$$

this means that $li_4(z) = \sum_{n\geq 1} n z^n$, $|z|<1$ represents the hyperfunction $\delta_1 + \delta_1'$ on the circle (The point 1 is the point on the circle corresponding to the angle $\theta = 0$).
In consequence $\vartheta_3^4(q) = 1 + \pi^2 \sum_{\substack{k\geq 1 \\ (h,k)=1}} \left(\frac{S_{hk}}{k}\right)^4 li_4(q e^{-2i\pi h/k})$

induces the hyperfunction :

$$T = \pi^2 \sum_{\substack{k\geq 1 \\ (h,k)=1}} \left(\frac{S_{hk}}{k}\right)^4 \left(e^{2i\pi h/k} \delta_{e^{2i\pi h/k}} + e^{4i\pi h/k} \delta'_{e^{2i\pi h/k}}\right)$$

our final result is the answer to the first question of the determination of the functional associated to $\mathcal{X}$ on the unit circle. This is given by the following:

Corollary : <u>Le fonction</u> $\mathcal{X}(z) = \sum_{n\geq 0}' z^{2^n}$, $|z|<1$ <u>induces on the unit circle the hyperfonction:</u>

$$T = \frac{\pi^2}{24} \sum_{p\geq 1}' \sum_{\substack{k\geq 1 \\ (h,k)=1}}' g(p) \left(\frac{S_{hk}}{k}\right)^4 T_{ph,k}$$

<u>Where</u>

$$T_{hk} = 2\left(e^{2i\pi h/k} \delta_{e^{2i\pi h/k}} + e^{4i\pi h/k} \delta'_{e^{2i\pi h/k}}\right) + e^{2i\pi h/k} \delta_{-e^{2i\pi h/k}} - e^{4i\pi h/k} \delta'_{-e^{2i\pi h/k}}{}^{-1}.$$

Remark. Similar results for the function $\mathcal{X}_-(z) = \sum_{n\geq 0} (-1)^n z^{2^n}$ can be obtained but the methods are more complicated.

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A. SEBBAR
Université Bordeaux I
351, Cours de la libération
33405 Talence, FRANCE.