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<th>Gevrey Cohomology Groups for Confluent Hypergeometric Systems (Complex Analysis and Microlocal Analysis)</th>
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1. CONTIGUITY RELATIONS

A "hypergeometric system" is a system of partial differential equations containing a parameter and admits a contiguity relation with respect to the parameter. In the language of $D$-modules, this empirical fact is formulated as follows: Let $\mathcal{M}(c)$ be a left $\mathcal{D}_X$-module containing a parameter $c$, that is, a left $\mathcal{D}_X[c]$-module, where $\mathcal{D}_X[c] = \mathcal{D}_X \otimes \mathbb{C}[c]$. A contiguity relation for $\mathcal{M}(c)$ with respect to $c$ is a commutative diagram:

$$
\cdots \xrightarrow{Q^2(c)} \mathcal{D}_X[c]^{m_2} \xrightarrow{Q^1(c)} \mathcal{D}_X[c]^{m_1} \xrightarrow{Q^0(c)} \mathcal{D}_X[c]^{m_0} \\
\downarrow P^2(c) \quad \downarrow P^1(c) \quad \downarrow P^0(c)
$$

$\cdots \xrightarrow{Q^2(c+1)} \mathcal{D}_X[c]^{m_2} \xrightarrow{Q^1(c+1)} \mathcal{D}_X[c]^{m_1} \xrightarrow{Q^0(c+1)} \mathcal{D}_X[c]^{m_0}$

of left $\mathcal{D}_X[c]$-modules such that the following sequence is a free resolution of $\mathcal{M}(c)$:

$$
\cdots \longrightarrow \mathcal{D}_X[c]^{m_2} \xrightarrow{Q^1(c)} \mathcal{D}_X[c]^{m_1} \xrightarrow{Q^0(c)} \mathcal{D}_X[c]^{m_0} \longrightarrow \mathcal{M}(c) \longrightarrow 0
$$

Here $Q^i(c)$ is an $m_{i+1} \times m_i$ matrix of holomorphic partial differential operators depending polynomially on $c$ and acting on $\mathcal{D}_X[c]^{m_{i+1}}$ by right multiplication, where each element of $\mathcal{D}_X[c]^{m_{i+1}}$ is regarded as a row vector. As for the operators $P^i(c)$, we require that each $P^i(c)$ should be an $m_i \times m_i$ matrix of holomorphic functions (not of partial differential operators) depending polynomially on $c$ and acting on $\mathcal{D}_X[c]^{m_i}$ by right multiplication.

**Example 1.1.** Consider Humbert's confluent hypergeometric system $\Phi_2(b_1, b_2; c)$:

$$
\begin{align*}
L_1(c)f &= \{x\partial_x^2 + y\partial_y\partial_x + (c-x)\partial_x - b_1\}f = 0, \\
L_2(c)f &= \{y\partial_y^2 + x\partial_x\partial_y + (c-y)\partial_y - b_2\}f = 0,
\end{align*}
$$

on $X = \mathbb{P}^1 \times \mathbb{P}^1$ with parameters $b_1, b_2$ and $c$, (see [2]). Let $\mathcal{M}(c)$ be the $\mathcal{D}_X[c]$-module associated to the system $\Phi_2(b_1, b_2; c)$, where $b_1, b_2$ are regarded as fixed.
Then $\mathcal{M}(c)$ has a contiguity relation:

$$
0 \xrightarrow{\mathcal{M}(c)} D_X^3 \xrightarrow{Q^1(c)} D_X^9 \xrightarrow{Q^6(c)} D_X^6
$$

where

$$
P^0(c) = 
\begin{pmatrix}
    c & x & y & 0 & 0 & 0 \\
    b_1 & x & 0 & 0 & 0 & 0 \\
    b_2 & 0 & y & 0 & 0 & 0 \\
    0 & 1 + b_1 & 0 & x & 0 & 0 \\
    0 & b_2 & b_1 & 0 & \frac{1}{2}(x + y) & 0 \\
    0 & 0 & 1 + b_2 & 0 & 0 & y
\end{pmatrix}
$$

$$
P^1(c) = 
\begin{pmatrix}
    c & 0 & x & 0 & y & 0 & 1 & 0 & 0 \\
    0 & c & 0 & x & 0 & y & 0 & 1 & 0 \\
    b_1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & b_1 & 0 & x & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
    b_2 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 \\
    0 & b_2 & 0 & 0 & 0 & 0 & x & 0 & -\frac{1}{2} \\
    0 & 0 & 0 & 0 & 0 & 0 & y & \frac{1}{2} & \frac{1}{2} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(x + y) & 0
\end{pmatrix}
$$

$$
P^2(c) = 
\begin{pmatrix}
    c & 1 & 0 \\
    b_1 x + b_2 y & x + y & -1 \\
    (b_1 + b_2) x y & x y & 0
\end{pmatrix}
$$

$$
Q^0(c) = 
\begin{pmatrix}
    \partial_x & -1 & 0 & 0 & 0 & 0 \\
    \partial_y & 0 & -1 & 0 & 0 & 0 \\
    0 & \partial_x & 0 & -1 & 0 & 0 \\
    0 & \partial_y & 0 & 0 & -1 & 0 \\
    0 & 0 & \partial_x & 0 & -1 & 0 \\
    0 & 0 & \partial_y & 0 & 0 & -1 \\
    -b_1 & c - x & 0 & x & y & 0 \\
    -b_2 & 0 & c - y & 0 & x & y \\
    0 & -b_2 & b_1 & 0 & x - y & 0
\end{pmatrix}
$$

$$
Q^1(c) = 
\begin{pmatrix}
    \partial_y & -b_2 & -b_2 x \\
    -\partial_x & b_1 & b_1 y \\
    0 & x \partial_y & x(\delta_y + b_2) \\
    1 & -(\delta_x - x + c) & -y(\delta_x - x + c - b_2) \\
    -1 & \delta_y - y + c & x(\delta_y - y + c - b_1) \\
    0 & -y \partial_x & -y(\delta_x + b_1) \\
    0 & \delta_y & \delta_y + b_2 \\
    0 & -\partial_x & -(\delta_x + b_1) \\
    0 & 1 & \delta_x + \delta_y + c
\end{pmatrix}^T
$$

Here $\partial_y = \partial/\partial y$, $\delta_y = y\partial_y$, and $T$ stands for the transpose of a matrix.
2. MAPPING CONES

From the contiguity relation (1.1), one obtains a $\mathcal{D}_{X \times \mathbb{P}^1}$-module $\mathcal{N}(c)$ containing a parameter as follows: Let $y$ be an inhomogeneous coordinate of $\mathbb{P}^1$ and set $\partial_y = \partial / \partial y$, $\delta_z = y \partial_y$. Given a nonzero polynomial $\phi(c) \in \mathbb{C}[c]$ independent of $i$, set

$$f^i(c) = \phi(\delta_y) - P^i(\delta_y + c) \partial_y.$$  

Then the contiguity relation (1.1) induces a commutative diagram

$$
\cdots \xrightarrow{Q^2(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_2} \xrightarrow{Q^1(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_1} \xrightarrow{Q^0(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_0} \\
\downarrow f^2(c) \quad \downarrow f^1(c) \quad \downarrow f^0(c) \\
\cdots \xrightarrow{Q^2(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_2} \xrightarrow{Q^1(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_1} \xrightarrow{Q^0(\delta_y+c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_0},
$$  

(2.1)

where the horizontal lines are exact. Let $\mathcal{N}(c)$ be the $\mathcal{D}_{X \times \mathbb{P}^1}[c]$-module having $M(f(c))[-1]$ as its free resolution, where $M(f(c))$ is the mapping cone of the morphism (1.2). Namely, $\mathcal{N}(c)$ is the $\mathcal{D}_{X \times \mathbb{P}^1}[c]$-module such that

$$
\cdots \xrightarrow{D^3(c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_2+m_1} \xrightarrow{D^2(c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_2+m_1} \xrightarrow{D^1(c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_1+m_0} \xrightarrow{D^0(c)} \mathcal{D}_{X \times \mathbb{P}^1}[c]^{m_0} \xrightarrow{D^0(c)} \mathcal{N}(c) \xrightarrow{0} 
$$

is a free resolution of $\mathcal{N}(c)$, where the operator $D^i(c)$ is given by

$$D^i(c) = \begin{pmatrix}
Q^i(\delta_y + c) & 0 \\
\phi(\delta_y) - P^i(\delta_y + c) \partial_y & -Q^{i-1}(\delta_y + c)
\end{pmatrix}.$$  

In this situation, we say that $\mathcal{N}(c)$ is obtained as the mapping cone of a contiguity relation for $\mathcal{M}(c)$. We observe that $\mathcal{N}(c)$ is a system of partial differential equations on $X \times \mathbb{P}^1$ having singularities along the hypersurface $X \times \{\infty\}$. It is an empirical fact that a confluent hypergeometric system $\mathcal{N}(c)$ often appears as the mapping cone of a contiguity relation for another hypergeometric system $\mathcal{M}(c)$, at least locally around an irregular singular point.

**Example 2.1.** Let $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$ denote Humbert's confluent hypergeometric system on $X = (\mathbb{P}^1)^n$ with parameters $b_1, \ldots, b_n$ and $c$, (see [1]). Note that $\Phi_2^{(1)}(b_1; c)$ is Kummer's equation and $\Phi_2(b_1, b_2; c) = \Phi_2^{(2)}(b_1, b_2; c)$ is considered in Example 1.1. If $\mathcal{M}(c)$ is the system $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$ and $\phi(c) = c - b_{n+1}$, then $\mathcal{N}(c)$ is the system $\Phi_2^{(n+1)}(b_1, \ldots, b_{n+1}; c)$.

3. GEVREY COHOMOLOGY GROUPS

Let $\mathcal{N}(c)$ be a $\mathcal{D}_{X \times \mathbb{P}^1}[c]$-module obtained as the mapping cone of a contiguity relation for a $\mathcal{D}_X[c]$-module $\mathcal{M}(c)$. We are interested in computing the extension groups $\operatorname{Ext}^i_{\mathcal{D}_{X \times \mathbb{P}^1}}(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$ for generic values of $c \in \mathbb{C}$. Here $\mathcal{O}_X[[1/y]]_{s,a}$
is the sheaf of (formal) Gevrey functions, that is, $\mathcal{O}_X[[1/y]]_{s,a}$ consists of the functions $f = \sum_{n=0}^{\infty} u_n(x)y^{-n}$ with $u_n(x) \in \mathcal{O}_X$ such that for any $n$,

$$||u_n|| \leq C(f, b) b^n (n!)^{s-1} \quad (\forall b > a),$$

where $C(f, b)$ is a constant depending only on $f$ and $b$. It can easily been seen that all the formal extension groups $\text{Ext}^i(N(c), \mathcal{O}_X[[1/y]])$ are trivial, but the Gevrey extension groups $\text{Ext}^i(N(c), \mathcal{O}_X[[1/y]]_{s,a})$ are, in general, nontrivial.

The main idea for tackling the problem is to introduce an auxiliary complex $C$ of $\mathcal{D}_X$-modules (called the harmonic complex), quasi-isomorphic to the solution complex $\mathbb{R}\text{Hom}(N(c), \mathcal{O}_X[[1/y]]_{s,a})$, in such a manner that computing the cohomology groups $H^i(C)$ is more accessible than computing $\text{Ext}^i(N(c), \mathcal{O}_X[[1/y]]_{s,a})$ directly. In the next section we construct such a complex $C$ by expressing it combinatorially in terms of the contiguity operators $P^i(c)$ as well as the differential operators $Q^i(c)$. The construction of $C$ is formal, that is, it does not require analysis. However, determing admissible indices $(s, a)$ for which $C$ is quasi-isomorphic to $\mathbb{R}\text{Hom}(N(c), \mathcal{O}_X[[1/y]]_{s,a})$ depends strongly upon hard analysis, that is, upon Gevrey estimates of solutions to certain finite difference equations arising from the contiguity relation. The necessary analysis is developed in [3]. In this report we restrict our attention to the algebraic aspect of the theory, leaving the analytic aspect to the above-mentioned paper.

4. HARMONIC COMPLEX

To construct the harmonic complex $C$, we first set

$$P^i_n = \frac{P^i(n-c)}{(n-c)^{\deg P^i(c)}}, \quad Q^i_n = Q^i(n-c) \quad (n = 0, 1, 2, \ldots),$$

where $\deg P^i(c)$ is the degree of $P^i(c)$ as a polynomial of $c$. Then $P^i_n$ and $Q^i_n$ define the operators $P^i_n : \mathcal{O}_X^m \rightarrow \mathcal{O}_X^m$ and $Q^i_n : \mathcal{O}_X^m \rightarrow \mathcal{O}_X^{m+1}$. The following assumption is very natural for the operators $P^i_n$ and $Q^i_n$ arising from hypergeometric systems.

Assumption 4.1. Assume that $P^i_n$ and $Q^i_n$ admit factorial asymptotic expansions:

$$P^i_n \sim \sum_{j=0}^{\infty} P^{i,j}(n-c)_j \quad (n \rightarrow +\infty),$$

$$Q^i_n = \sum_{j=0}^{N^i_n} Q^{i,j}(n-c)_j + o(1) \quad (n \rightarrow +\infty),$$

where $(x)_j$ and $\langle x \rangle_j$ are defined by

$$(x)_j = \frac{(-1)^j j!}{x(x+1)\cdots(x+j-1)}, \quad \langle x \rangle_j = \frac{x(x-1)\cdots(x-j+1)}{(-1)^j j!},$$

and that there exists a direct sum decomposition $\mathcal{O}_X^m = U_0^i \oplus U_1^i$ with the associated projections $X^i : \mathcal{O}_X^m \rightarrow U_0^i$ and $Y^i : \mathcal{O}_X^m \rightarrow U_1^i$ such that

$$X^i P^{i,0} X^i = X^i, \quad X^i P^{i,0} Y^i = O,$$

$$Y^i P^{i,0} X^i = O, \quad X^i P^{i,1} X^i = O,$$

$$I_1 - Z^i : U_1^i \rightarrow U_1^i$$

is invertible,

where $I_1$ is the identity operator on $U_1^i$ and $Z^i := Y^i P^{i,0} Y^i$. 

**Definition 4.2.** Under Assumption 4.1 we define $C^i$ and $d^i : C^i \rightarrow C^{i+1}$ by

$$
\begin{align*}
C^i &= U^i_0, \\
d^i &= Q^i_{0,0} + \sum_{j=1}^{N^i-1} Q^i_{j-1,j} \sum_{J \in S_j} A^i_{j-1,0},
\end{align*}
$$

where $S_j$ is the set of all nonempty subsets of $\{1,2,\ldots,j\}$. The operators $A^i_{j} : U^i \rightarrow U^i (J \in S_j)$ are defined as follows. We first set

$$
P^i_{jk} = \sum_{m=1}^{j-k} \frac{(k-1)_+!(m-1)!}{(k+m-1)!} \left[ m-1 \right] P^i_{j-k,m},$$

for $0 \leq k < j$, where $a_+ = \max\{a,0\}$ and

$$
\begin{bmatrix} a \\ j \end{bmatrix} = \begin{cases} 1 & (j = 0), \\ \frac{1}{j!}a(a+1)(a+2) \cdots (a+j-1) & (j = 1,2,3,\ldots). \end{cases}
$$

Using the operators $P^i_{jk}$ defined above, we next set

$$
A^i_{jk} = X^i P^i_{j+1,k} + (I + \frac{1}{j}X^i P^i_{j+1,j})(I - Z^i)^{-1}(Y^i P^i_{j+1,j} - \delta_{j,k+1}^i Z^i),
$$

for $0 \leq k < j$, where $I$ is the identity operator on $U^i$ and $\delta_{ij}$ is Kronecker’s symbol. Then for each $J = \{j_1,j_2,\ldots,j_k\} \in S_j$ with $j_1 < j_2 < \cdots < j_k$, the operator $A^i_{J}$ is defined by $A^i_{J} = A^i_{j_k,j_{k-1}} A^i_{j_{k-1},j_{k-2}} \cdots A^i_{j_2,j_1} A^i_{j_1,0}$.

**Lemma 4.3.** $C$ so defined is a complex, i.e., $d^i$ maps $C^i$ into $C^{i+1}$ and $d^{i+1}d^i = 0$.

5. **QUASI-ISOMORPHISM**

**Theorem 5.1.** For suitable Gevrey indices $(s,a)$, we have for any $c \in \mathbb{C} \setminus \mathbb{Z}$,

$$
\mathbb{R}\text{Hom}_{\mathcal{D}_{\mathbb{C} \times \mathbb{C}^1}}(N(c), \mathcal{O}_X[[1/y]]_{s,a}) \simeq C.
$$

A Gevrey index $(s,a)$ for which (5.1) holds is said to be admissible. To describe admissible Gevrey indices, we set

$$
\begin{align*}
\bar{s} &= \max_i \{ \deg P^i(c) - s^i \} - \deg \phi(c) + 2, \\
\underline{s} &= \min_i \deg P^i(c) - \deg \phi(c) + 2,
\end{align*}
$$

where

$$
\begin{align*}
p^i &= \min_j \{ j ; X^i P^i_{j} Y^i \neq O \}, \\
q^i &= \min_j \{ j ; Y^i P^i_{j} X^i \neq O \}, \\
r^i &= \min_j \{ j ; Y^i P^i_{j} Y^i \neq O \}, \\
s^i &= \min_i \{ p^i + q^i - 1, r^i \}.
\end{align*}
$$

- Case $\underline{s} < s < \bar{s}$: $(s,a)$ is admissible for any $a \geq 0$.
- Case $s = \underline{s}$ or $\bar{s}$: admissible values of $a$ can be determined explicitly in terms of the coefficients $P^i_{j,j}$ of the asymptotic expansion of $P^i_{n}$, though the description of them are rather complicated (and hence omitted). See [4] for details.
**Example 5.2.** Recall that if $\mathcal{M}(c) = \Phi_2^{(n)}(b_1, \ldots, b_n; c)$ and $\phi(c) = c - b_{n+1}$, then $\mathcal{N}(c) = \Phi_2^{(n)}(b_1, \ldots, b_{n+1}; c)$, (Example 2.1). In this case the harmonic complex $C$ is isomorphic to the de Rham complex $\Omega_{(\mathbb{P}^1)^n}[-1]$ shifted by one, and $\bar{s} = 1, \bar{s} = 2$. Theorem 5.1 implies that

$$\dim \text{Ext}^i(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a}) = \dim H^i(C) = \begin{cases} 1 & (i = 1) \\ 0 & (i \neq 0). \end{cases}$$

where the second equality follows from Poincaré's lemma.

H. Majima [5] also computed the Gevrey extension groups for the Humbert system $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$.

**REFERENCES**

5. Majima, H., article in this volume.