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A dependence domain for a class of microdifferential operators with involutive double characteristics

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In the study of the mathematical models for conical refraction, the theory of 2-microlocal analysis plays an important role. We give, in this note, a sharp microlocal dependence domain for a class of microdifferential operators, which is also an application of 2-microlocal analysis.

1 Statement of the Main Theorem

Let $M$ be a real analytic manifold with complexification $X$, and $P$ a microdifferential operator defined in a neighborhood $U$ in $T^*X$ of a point $\dot{q} \in T^*_MX \setminus M$. We assume that the characteristic variety of $P$ satisfies

$$\text{Char}(P) \subset \{q \in U; p_1(q) = p_2(q) = 0\}$$

with homogeneous holomorphic functions $p_1$ and $p_2$ on $U$ with the following properties:

- $p_1$ and $p_2$ are real valued on $T^*_M X$,
- $dp_1 \wedge dp_2 \wedge \omega_X(q) \neq 0$ if $p_1(q) = p_2(q) = 0$,
- $\{p_1, p_2\}(q) = 0$ if $p_1(q) = p_2(q) = 0$.

Here $\omega_X$ is the canonical 1-form of $T^*X$, and $\{\cdot, \cdot\}$ the Poisson bracket on $T^*X$. 
In this situation, we can define regular involutive submanifolds $V^C \subseteq T^*X$ and $V \subseteq T^*_MX$ by

$$V^C = \{q \in U; p_1(q) = p_2(q) = 0\},$$

$$V = V^C \cap T^*_MX,$$

and we assume, for the simplicity, that $\dot{q} \in V$. Moreover $\Gamma$ denotes the canonical leaf of $V$ passing through $\dot{q}$.

A set $K \subset \Gamma$ is called a $\Gamma$-rectangle if there exists an injective real analytic map

$$\Phi : [0,1] \times [0,1] \longrightarrow \Gamma$$

with the following three properties:

- $\Phi([0,1] \times [0,1]) = K$
- $\Phi(\cdot, t)$ is an integral curve of the Hamiltonian vector field $H_{p_1}$ for any fixed $t \in [0,1]$.
- $\Phi(s, \cdot)$ is an integral curve of the Hamiltonian vector field $H_{p_2}$ for any fixed $s \in [0,1]$.

We give, in this situation,

**Theorem 1.1.** There exists an open neighborhood $U_0$ of $\dot{q}$ in $\Gamma$ with the property that for any $\Gamma$-rectangle $K$ contained in $U_0$ with the four vertices $q_0$, $q_1$, $q_2$, and $q_3$ and for any microfuntion solution $u$ to $Pu = 0$ on $K$,

$$q_1, q_2, q_3 \notin \text{supp}(u) \Rightarrow q_0 \notin \text{supp}(u).$$

This theorem can be deduced from the model case given in the next section.

**2 Theorem in the model case**

Let $M$ be an open subset of $\mathbb{R}^n$ with a complex neighborhood $X$ in $\mathbb{C}^n$ ($n \geq 3$). We take a coordinate system of $M$ (resp. $X$) as $x = (x_1, \cdots, x_n)$ (resp. $z = (z_1, \cdots, z_n)$). Then $(x; \sqrt{-1}\xi \cdot dx)$ (resp. $(z; \zeta \cdot dz)$) denotes a point in $T^*_MX$ (resp. $T^*X$) with $\xi = (\xi_1, \cdots, \xi_n)$ (resp. $\zeta = (\zeta_1, \cdots, \zeta_n)$).
We take a point $q_0 = (0; \sqrt{-1}dX_n) \in T^*_M X$. Let $P$ be a microdifferential operator defined in a neighborhood of $q_0$ whose principal symbol is of the form

$$\zeta_1^{m_1} \zeta_2^{m_2}$$

with $m_1, m_2 \geq 1$. We define an involutive manifold $V$ of $T^*_M X$ by

$$V = \{(x; \sqrt{-1} \xi \cdot dx); \xi_1 = \xi_2 = 0\}$$

and denote by $\Gamma$ the leaf of $V$ passing through the point $q_0$. We take a rectangle $K$ on $\Gamma$ defined by

$$K = \{(x_1, x_2, x'' = 0; \sqrt{-1}dx_n); 0 \leq x_1 \leq t_1, \ 0 \leq x_2 \leq t_2\}.$$ 

Here $x'' = (x_3, \ldots, x_n)$. The vertices of $K$ are denoted by

$$q_0, q_1 = (t_1, 0, 0; \sqrt{-1}dx_n), q_2 = (0, t_2, 0; \sqrt{-1}dx_n), q_3 = (t_1, t_2, 0; \sqrt{-1}dx_n).$$

Then we have

**Theorem 2.1.** Let $u$ be a microfunction defined in a neighborhood of $K$. We assume that $u$ satisfies

$$Pu = 0$$

and that the three points $q_1, q_2, q_3$ are not in $\text{supp}(u)$:

$$q_1, q_2, q_3 \notin \text{supp}(u).$$

Then

$$q_0 \notin \text{supp}(u).$$

**Remark 2.2.** The phenomenon in Theorem 2.1 was first observed by Y. Okada [O] for $C^\infty$ wavefront set of microdistribution solutions. His result concerns with the case $m_1 = m_2 = 1$ under a Levi condition on the lower order term of $P$. He employed a microlocal version of Goursat problem in the complex domain.

To give an implication of Theorem 2.1, we recall a result obtained by N. Tose [T2].
Theorem 2.3. Let $u$ be a microfunction solution to $Pu = 0$ on an open subset $U$ of $\Gamma$. Then there exist a family $\{b_{\lambda}^{(1)}\}_{\lambda \in \Lambda_1}$ of integral curves on $\Gamma$ of $\partial/\partial x_1$ and another family $\{b_{\lambda}^{(2)}\}_{\lambda \in \Lambda_2}$ of integral curves on $\Gamma$ of $\partial/\partial x_2$ which satisfy the properties that the set

$$
\bigcup_{\lambda \in \Lambda_1} b_{\lambda}^{(1)} \cup \bigcup_{\lambda \in \Lambda_2} b_{\lambda}^{(2)}
$$

is included in $\text{supp}(u)$ and that $\text{supp}(u)$ has unique continuation property on the set

$$
\Omega = U \setminus \left( \bigcup_{\lambda \in \Lambda_1} b_{\lambda}^{(1)} \cup \bigcup_{\lambda \in \Lambda_2} b_{\lambda}^{(2)} \right).
$$

More precisely, if a point $q \in \Omega$ is not in $\text{supp}(u)$, then the connected component of $\Omega$ containing $q$ is disjoint with $\text{supp}(u)$.

In the situation of Theorem 2.3, we take a point

$$
\dot{q} = (s_1, s_2, x'' = 0; \sqrt{-1}dx_n) \in \Gamma.
$$

We assume that, for a neighborhood $U_1$ of $\dot{q}$, the only one integral curve $b_{\lambda_1}^{(1)}$ of $\partial/\partial x_1$ and the only one $b_{\lambda_2}^{(2)}$ of $\partial/\partial x_2$ pass $U_1$. We assume, for simplicity, that the both two curves pass $\dot{q}$:

$$
\dot{q} \in b_{\lambda_j}^{(j)} \quad (j = 1, 2).
$$

We assume that

$$
\text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 > s_2\} = \emptyset
$$

and that

$$
\text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 < s_2\} = \emptyset.
$$

In this situation, if a point

$$
\dot{q}' \in \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 < s_2\}
$$

does not belong to $\text{supp}(u)$, then it follows from Theorem 2.1 that

$$
\text{supp}(u) \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 > s_2\} = \emptyset.
$$

We also give the following decomposition proposition.
Proposition 2.4. Let \( u \in C_{M}(K) \) be a solution to \( Pu = 0 \). Then there exist two microfunctions \( u_1 \) and \( u_2 \) in \( C_{M}(K) \) with the properties

\[
\begin{align*}
• & \quad u = u_1 + u_2, \\
• & \quad Pu_j = 0 \quad (j = 1, 2), \\
• & \quad SS_{V}^{2}(u_j) \setminus V \subset \{(x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}(x_1\cdot dx_1 + x_2\cdot dx_2)); x_j^* = 0\}.
\end{align*}
\]

Here \( SS_{V}^{2}(\cdot) \) is the second singular spectrum along \( V \). Using this decomposition, we can prove the Theorem 2.1, but we omit the details.

References


