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Author(s)
Okada, Yasunori; Tose, Nobuyuki

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A dependence domain for a class of microdifferential operators with involutive double characteristics

Yasunori Okada (Chiba University) (岡田靖則)
Nobuyuki Tose (Keio University) (戸瀬信之)

In the study of the mathematical models for conical refraction, the theory of 2-microlocal analysis plays an important role. We give, in this note, a sharp microlocal dependence domain for a class of microdifferential operators, which is also an application of 2-microlocal analysis.

1 Statement of the Main Theorem

Let $M$ be a real analytic manifold with complexification $X$, and $P$ a microdifferential operator defined in a neighborhood $U$ in $T^*X$ of a point $\mathcal{q} \in T^*_M X \setminus M$. We assume that the characteristic variety of $P$ satisfies

$$\text{Char}(P) \subset \{q \in U; p_1(q) = p_2(q) = 0\}$$

with homogeneous holomorphic functions $p_1$ and $p_2$ on $U$ with the following properties:

- $p_1$ and $p_2$ are real valued on $T^*_M X$,
- $dp_1 \wedge dp_2 \wedge \omega_X(q) \neq 0$ if $p_1(q) = p_2(q) = 0$,
- $\{p_1, p_2\}(q) = 0$ if $p_1(q) = p_2(q) = 0$.

Here $\omega_X$ is the canonical 1-form of $T^*X$, and $\{\cdot, \cdot\}$ the Poisson bracket on $T^*X$. 

In this situation, we can define regular involutive submanifolds $V^\mathbb{C} \subset T^*X$ and $V \subset T^*_MX$ by

$$V^\mathbb{C} = \{ q \in U; p_1(q) = p_2(q) = 0 \},$$

$$V = V^\mathbb{C} \cap T^*_MX,$$

and we assume, for the simplicity, that $\dot{q} \in V$. Moreover $\Gamma$ denotes the canonical leaf of $V$ passing through $\dot{q}$.

A set $K \subset \Gamma$ is called a $\Gamma$-rectangle if there exists an injective real analytic map

$$\Phi: [0,1] \times [0,1] \longrightarrow \Gamma$$

with the following three properties:

- $\Phi([0,1] \times [0,1]) = K$
- $\Phi(\cdot,t)$ is an integral curve of the Hamiltonian vector field $H_{p_1}$ for any fixed $t \in [0,1]$.
- $\Phi(s,\cdot)$ is an integral curve of the Hamiltonian vector field $H_{p_2}$ for any fixed $s \in [0,1]$.

We give, in this situation,

**Theorem 1.1.** There exists an open neighborhood $U_0$ of $\dot{q}$ in $\Gamma$ with the property that for any $\Gamma$-rectangle $K$ contained in $U_0$ with the four vertices $q_0$, $q_1$, $q_2$, and $q_3$ and for any microfuntion solution $u$ to $Pu = 0$ on $K$,

$$q_1, q_2, q_3 \notin \text{supp}(u) \implies q_0 \notin \text{supp}(u).$$

This theorem can be deduced from the model case given in the next section.

## 2 Theorem in the model case

Let $M$ be an open subset of $\mathbb{R}^n$ with a complex neighborhood $X$ in $\mathbb{C}^n$ ($n \geq 3$). We take a coordinate system of $M$ (resp. $X$) as $x = (x_1, \cdots, x_n)$ (resp. $z = (z_1, \cdots, z_n)$). Then $(x; \sqrt{-1} \xi \cdot dx)$ (resp. $(z; \zeta \cdot dz)$) denotes a point in $T^*_MX$ (resp. $T^*X$) with $\xi = (\xi_1, \cdots, \xi_n)$ (resp. $\zeta = (\zeta_1, \cdots, \zeta_n)$).
We take a point $q_0 = (0; \sqrt{-1}dX_n) \in T_M^*X$. Let $P$ be a microdifferential operator defined in a neighborhood of $q_0$ whose principal symbol is of the form

$$\zeta_1^{m_1} \zeta_2^{m_2}$$

with $m_1, m_2 \geq 1$. We define an involutive manifold $V$ of $T_M^*X$ by

$$V = \{(x; \sqrt{-1}\xi \cdot dx; \xi_1 = \xi_2 = 0)\}$$

and denote by $\Gamma$ the leaf of $V$ passing through the point $q_0$. We take a rectangle $K$ on $\Gamma$ defined by

$$K = \{(x_1, x_2, x'' = 0; \sqrt{-1}dx_n); 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2\}.$$ 

Here $x'' = (x_3, \cdots, x_n)$. The vertices of $K$ are denoted by

$q_0, q_1 = (t_1, 0, 0; \sqrt{-1}dx_n), q_2 = (0, t_2, 0; \sqrt{-1}dx_n), q_3 = (t_1, t_2, 0; \sqrt{-1}dx_n)$. 

Then we have

**Theorem 2.1.** Let $u$ be a microfunction defined in a neighborhood of $K$. We assume that $u$ satisfies

$$Pu = 0$$

and that the three points $q_1, q_2, q_3$ are not in $\text{supp}(u)$:

$$q_1, q_2, q_3 \notin \text{supp}(u).$$

Then

$$q_0 \notin \text{supp}(u).$$

**Remark 2.2.** The phenomenon in Theorem 2.1 was first observed by Y. Okada [O] for $C^\infty$ wavefront set of microdistribution solutions. His result concerns with the case $m_1 = m_2 = 1$ under a Levi condition on the lower order term of $P$. He employed a microlocal version of Goursat problem in the complex domain.

To give an implication of Theorem 2.1, we recall a result obtained by N. Tose [T2].
**Theorem 2.3.** Let $u$ be a microfunction solution to $Pu = 0$ on an open subset $U$ of $\Gamma$. Then there exist a family $\{b^{(1)}_{\lambda}\}_{\lambda \in \Lambda_1}$ of integral curves on $\Gamma$ of $\partial/\partial x_1$ and another family $\{b^{(2)}_{\lambda}\}_{\lambda \in \Lambda_2}$ of integral curves on $\Gamma$ of $\partial/\partial x_2$ which satisfy the properties that the set

$$ \bigcup_{\lambda \in \Lambda_1} b^{(1)}_{\lambda} \cup \bigcup_{\lambda \in \Lambda_2} b^{(2)}_{\lambda} $$

is included in $\text{supp}(u)$ and that $\text{supp}(u)$ has unique continuation property on the set

$$ \Omega = U \setminus \left( \bigcup_{\lambda \in \Lambda_1} b^{(1)}_{\lambda} \cup \bigcup_{\lambda \in \Lambda_2} b^{(2)}_{\lambda} \right). $$

More precisely, if a point $q \in \Omega$ is not in $\text{supp}(u)$, then the connected component of $\Omega$ containing $q$ is disjoint with $\text{supp}(u)$.

In the situation of Theorem 2.3, we take a point

$$ \dot{q} = (s_1, s_2, x'' = 0; \sqrt{-1}dx_n) \in \Gamma. $$

We assume that, for a neighborhood $U_1$ of $\dot{q}$, the only one integral curve $b^{(1)}_{\lambda_1}$ of $\partial/\partial x_1$ and the only one $b^{(2)}_{\lambda_2}$ of $\partial/\partial x_2$ pass $U_1$. We assume, for simplicity, that the both two curves pass $\dot{q}$:

$$ \dot{q} \in b^{(j)}_{\lambda_j} \quad (j = 1, 2). $$

We assume that

$$ \text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 > s_2\} = \emptyset $$

and that

$$ \text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 < s_2\} = \emptyset. $$

In this situation, if a point

$$ \dot{q}' \in \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 < s_2\} $$

does not belong to $\text{supp}(u)$, then it follows from Theorem 2.1 that

$$ \text{supp}(u) \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 > s_2\} = \emptyset. $$

We also give the following decomposition proposition.
Proposition 2.4. Let $u \in C_{M}(K)$ be a solution to $Pu = 0$. Then there exist two microfunctions $u_1$ and $u_2$ in $C_{M}(K)$ with the properties

- $u = u_1 + u_2$,
- $Pu_j = 0$ ($j = 1, 2$),
- $SS^2_V(u_j) \setminus V \subset \{(x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}(x_1^* \cdot dx_1 + x_2^* \cdot dx_2)); x_j^* = 0\}$.

Here $SS^2_V(\cdot)$ is the second singular spectrum along $V$. Using this decomposition, we can prove the Theorem 2.1, but we omit the details.

References


