Title
CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS
(Complex Analysis and Microlocal Analysis)

Author(s)
Uchikoshi, Keisuke

Citation
数理解析研究所講究録 (1999), 1090: 46-49

Issue Date
1999-04

URL
http://hdl.handle.net/2433/62879

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

KEISUKE UCHIKOSHI

Department of Mathematics,
National Defense Academy
Hashirimizu 1-10-20 Yokosuka, Japan
e-mail: uchikosh@cc.nda.ac.jp

ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let $P(x, D)$ be a microdifferential operator defined at $x^* = (0; 0, \cdots, 0, \sqrt{-1}) \in \sqrt{-1}T^*\mathbb{R}^n$ of order $m \geq 2$, written in the form

$$P(x, D) = D_1^n + \sum_{0 \leq j \leq m-1} P_j(x, D')D_1^j,$$

(1) \text{ord } P_j \leq m - j.

Here we have written $D' = (D_2, \cdots, D_n)$. We also write as $D'' = (D_1, \cdots, D_{n-1})$, $D''' = (D_2, \cdots, D_{n-1})$. Let $\sigma_m(P)(x, \xi)$ be the principal symbol of $P(x, D)$. We assume that

$$\sigma_m(P) = \xi_1^m;$$

if $x_1 = 0$, then

$$\sigma_m(P) = \xi_1^m,$$

if $x_1 \neq 0$, then the equation $\sigma_m(P) = 0$ has $m$ distinct roots

$$\xi_1 = \varphi_1(x, \xi'), \cdots, \varphi_m(x, \xi').$$

We denote by $\mathcal{O}$ (resp. $\mathcal{O}_{(j)}$) the sheaf of holomorphic functions (resp. the sheaf of functions $f(x_1^{1/j}, x')$ such that $f(x)$ are holomorphic). Without loss of generality, we may assume that $\varphi_j(x, \xi') \in \mathcal{O}_{(m')}, x_1$ for some $m' \in \mathbb{N}$, that they are homogeneous in $\xi'$ of degree 1, and that they vanish when $x_1 = 0$. From now on, we denote $\mathcal{O} = \mathcal{O}_{(m')}$. It follows that

$$\varphi_j(x, \xi') = x_1^{q_j}a_j(x', \xi'), a_j(x^*) \neq 0 (1 \leq j \leq m).$$

We also assume that

$$i \neq j \quad \Rightarrow \quad (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).$$

We denote by $\mathcal{C}$ (resp. $\mathcal{E}$) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

\[(4) \begin{cases} Pu = 0, \\ D_1^{-1}u(0, x') = v_j(x'), \quad 1 \leq j \leq m, \end{cases}\]

where $u \in C_{R^n, x^*}$ and $v_j \in C_{R^{n-1}, x^*}$ ($x' = (0; 0, \cdots, 0, \sqrt{-1}) \in \sqrt{-1}T^*R^{n-1}$). If $P(x, D)$ is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for $v_1 = \cdots = v_m = 0$), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let $A(x', D')$ be an both-side invertible $m \times m$ matrix whose components $A(\mu, \nu)(x', D') \in \mathcal{E}^R$ are independent of $(x_1, D_1)$. Here we denote by $\mathcal{E}^R$ the sheaf of holomorphic microlocal operators (c.f. [1,6]). We choose $r$ rows of this matrix in an arbitrary way. To be clear, let $1 \leq j_1 < j_2 < \cdots < j_r \leq m$ and choose the $j_1, \cdots, j_r$-th rows of $A$. Then we obtain an $r \times m$ matrix $A'(x', D')$ of holomorphic microlocal operators. We say that $v_1(x'), \cdots, v_m(x') \in C_{R^{n-1}, x^*}$ satisfy an $r$-relation if choosing some $r$ rows of some $A(x', D')$ we have $A'(x', D') \dot{v}(x') = \hat{0}$. Here $\dot{v}$ denotes $\dot{v}(v_1, \cdots, v_m)$. Note that even if $v_1(x'), \cdots, v_m(x')$ satisfy an $r$-relation and another $s$-relation, it does not necessarily mean an $(r + s)$-relation.

We next define a classification of the characteristic roots. Let $\theta \in \{0, \pi\}$. Let

\[(5) (x, \xi') \in R^n \times R^{n-1}, \quad x_1 \neq 0, \quad \arg x_1 = \theta.\]

We define

\[M = \{1, 2, \cdots, m\},\]

\[M_{0, \theta} = \{\lambda \in M; \Re(x_1 \varphi_{\lambda}(x, \xi')) = 0, \text{ if } (x, \xi') \text{ satisfies (5)}\},\]

\[M_{\pm, \theta} = \{\lambda \in M; \pm \Re(x_1 \varphi_{\lambda}(x, \xi')) > 0, \text{ if } (x, \xi') \text{ satisfies (5)}\},\]

\[M' = M \setminus M_{0, \theta} \setminus M_{+, \theta} \setminus M_{-, \theta}.\]

It is easy to see that $M_{0, \theta} \cup M_{+, \theta} \cup M_{-, \theta} \cup M' = M$ is a disjoint union.

Let $m_{0, \theta}$, $m_{\pm, \theta}$ be the number of the elements belonging to $M_{0, \theta}$, $M_{\pm, \theta}$, respectively. We assume that

\[(6) M' = \emptyset, \quad \forall \theta \in \{0, \pi\}.\]

We also need a condition for the microfunctions. Let

\[\omega(r) = \{(x, \xi) \in \sqrt{-1}T^*R^n; |x| < r, \quad |\xi''| < r \Im \xi_n\},\]

\[\omega'(r) = \{(x', \xi') \in \sqrt{-1}T^*R^{n-1}; |x'| < r, \quad |\xi''| < r \Im \xi_n\},\]

and

\[\omega_0(r) = \{(x, \xi) \in \omega(r); |x'| \leq r^{-1}|x_1|, \quad |\xi''| \leq r^{-1}|x_1| \Im \xi_n\},\]

\[\omega_0(r) = \{tx'^*; \quad t > 0\}.\]
CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

We define

\[ C_0 = \lim_{r \to 0} \Gamma_{\omega_0(r)}(C_{\mathbb{R}^n}, \omega_0(r)), \]
\[ C'_0 = \lim_{r \to 0} \Gamma_{\omega'_0(r)}(C_{\mathbb{R}^{n-1}}, \omega'_0(r)). \]

Then we have the following

**Theorem.** We assume (1) – (3) and (6). Let \( v_1(x'), \ldots, v_m(x') \in C'_0. \) Then there exists an \( m_{+,0} \)-relation and an \( m_{+,\pi} \)-relation such that the Cauchy problem (4) has a solution \( u \in C_0 \) if, and only if, \( v_1(x'), \ldots, v_m(x') \) satisfy these relations.

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

**Example 0 (Lewy-Mizohata operators).** If \( P = D_1 \pm \sqrt{-1}x_1D_n \), then we have \( M_{+,\theta} = \{1\} (= M), \ M_{-,\theta} = \emptyset. \) The above theorem means that \( P_-u = 0, u(0, x') = v(x') \) is solvable for any \( v \in C'_0 \) without any relations. In fact using the defining function we only need to let \( u(x) = v(x'', x_n + \sqrt{-1}x_1^2/2) \). On the other hand, \( P_+u = 0, u(0, x') = v(x') \) is solvable only for the case when \( v(x') \) satisfies an one-relation. This means \( v = 0 \) and \( u = 0 \). It follows that \( P_+u = 0 \Rightarrow u = 0 \), i.e., \( P_+ \) is hypo-elliptic in \( C_0 \) (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to \( M_{+,\theta} \) cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case \( m = 2 \).

**Example 1 (microhyperbolic operators).** Let \( P(x, D) = D_1^2 - x_1^2 D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1. \) Without loss of generality, we may assume that \( P' \) is a polynomial in \( D_1 \) of degree 1. Since \( \varphi_1(x, \xi') = x_1\xi_n, \varphi_2(x, \xi') = -x_1\xi_n, \arg \xi_n = \pi/2, \) it is easy to see that \( M_{0,0} = \{1, 2\}, \ M_{+,\theta} = \emptyset \) for \( \theta \in \{0, \pi\}. \) It follows that (4) is solvable for arbitrary \( v_1(x'), v_2(x') \in C'_0 \) without any relations (See [3]).

**Example 2 (Tricomi operators).** Let \( P(x, D) = D_1^2 - x_1D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1. \) We have \( \varphi_1(x, \xi') = \sqrt{x_1}\xi_n, \varphi_2(x, \xi') = -\sqrt{x_1}\xi_n. \) It follows that \( M_{0,0} = \{1, 2\}, \ m_{+,0} = 0, \) and that \( M_{+,\pi} = \{1\}, \ M_{-,\pi} = \{2\}, \ m_{+,\theta} = 1. \) It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let \( \omega \subset \sqrt{-1}\mathbb{T}^*\mathbb{R}^n \) be a small neighborhood of \( x^* \), and let \( \omega^\theta = \{ (x, \xi) \in \omega; \ x_1 \neq 0, \ arg x_1 = \theta \}, \ \theta \in \{0, \pi\}. \) At first we consider an elliptic boundary value problem in \( \omega_\pi \), giving one boundary datum on \( \{x_1 = 0\}. \) Then we can always extend this solution to the hyperbolic region \( \omega_0. \) This case was considered also by [4].

**Example 3 (hypoelliptic operators).** Let \( P(x, D) = D_1^2 + x_1^2 D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1. \) Since \( \varphi_1(x, \xi') = \sqrt{-1}x_1\xi_n, \varphi_2(x, \xi') = -\sqrt{-1}x_1\xi_n, \) it is easy to see that \( M_{-,\theta} = \{1\}, \ M_{+,\theta} = \{2\}, \ m_{+,\theta} = 1 \) for \( \theta \in \{0, \pi\}. \) There exist an \( m_{+,0} \)-relation and an \( m_{+,\pi} \)-relation such that the Cauchy problem (4) uniquely has a solution \( u \in C_0 \) if, and only if, \( v_1(x'), \ldots, v_m(x') \in C'_0 \) satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case \( v_1 = v_2 = 0, \) and \( u = 0. \) In other words, \( Pu = 0 \) does not have any
non-trivial solutions. It is well-known that this is true if the principal symbol \( \sigma_1(P') \) of the lower order term satisfies \( \xi_n^{-1} \sigma_1(P') \not\in \{\sqrt{-1}, \sqrt{-13}, \sqrt{-15}, \ldots\} \) (See [2,5]). Of course our result applies for higher order operators, too.

REFERENCES


