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CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS
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CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

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ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let $P(x, D)$ be a microdifferential operator defined at $x^* = (0; 0, \cdots , 0, \sqrt{-1})$ $\in \sqrt{-1}T^*\mathbb{R}^n$ of order $m \geq 2$, written in the form

$$P(x, D) = D_1^m + \sum_{0 \leq j \leq m} P_j(x, D')D_1^j,$$

$$\text{ord } P_j \leq m - j.$$ (1)

Here we have written $D' = (D_2, \cdots, D_n)$. We also write as $D'' = (D_1, \cdots, D_{n-1}), D''' = (D_2, \cdots, D_{n-1})$. Let $\sigma_m(P)(x, \xi)$ be the principal symbol of $P(x, D)$. We assume that

$$\begin{cases}
\text{if } x_1 = 0, \text{ then } \sigma_m(P) = \xi^n_m, \\
\text{if } x_1 \neq 0, \text{ then the equation } \sigma_m(P) = 0 \text{ has } m \text{ distinct roots } \\
\xi_1 = \varphi_1(x, \xi'), \cdots, \varphi_m(x, \xi').
\end{cases}$$ (2)

We denote by $\mathcal{O}$ (resp. $\mathcal{O}_{(j)}$) the sheaf of holomorphic functions (resp. the sheaf of functions $f(x^{1/j}, x')$ such that $f(x)$ are holomorphic). Without loss of generality, we may assume that $\varphi_j(x, \xi') \in \mathcal{O}_{(m')}^{x'*}$ for some $m' \in \mathbb{N}$, that they are homogeneous in $\xi'$ of degree 1, and that they vanish when $x_1 = 0$. From now on, we denote $\mathcal{O} = \mathcal{O}_{(m')}$. It follows that

$$\begin{cases}
\text{for some } q_j \in \mathbb{N}/m' \text{ and some } a_j(x, \xi') \in \mathcal{O}_{x'*} \text{ we have} \\
\varphi_j(x, \xi') = x_1^{q_j}a_j(x, \xi'), \ a_j(x^*) \neq 0 (1 \leq j \leq m).
\end{cases}$$

We also assume that

$$i \neq j \implies (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).$$ (3)
We denote by \( C \) (resp. \( \mathcal{E} \)) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

\[
\begin{align*}
P u &= 0, \\
D_{1}^{-1} u(0, x') &= v_{j}(x'), \quad 1 \leq j \leq m,
\end{align*}
\]

where \( u \in C_{\mathbb{R}^{n}, x'} \) and \( v_{j} \in C_{\mathbb{R}^{n-1}, x'} \) \( (x' = (0; 0, \ldots, 0, \sqrt{-1}) \in \sqrt{-1}T^{*}\mathbb{R}^{n-1}). \) If \( P(x, D) \) is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for \( v_{1} = \cdots = v_{m} = 0 \)), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let \( A(x', D') \) be an both-side invertible \( m \times m \) matrix whose components \( A_{(\mu, \nu)}(x', D') \in \mathcal{E}_{\mathbb{R}^{n}}^{\mathbb{R}} \) are independent of \( (x_{1}, D_{1}). \) Here we denote by \( \mathcal{E}_{\mathbb{R}}^{\mathbb{R}} \) the sheaf of holomorphic microlocal operators (c.f. [1,6]). We choose \( r \) rows of this matrix in an arbitrary way. To be clear, let \( 1 \leq j_{1} < j_{2} < \cdots < j_{r} \leq m \) and choose the \( j_{1}, \ldots, j_{r} \)-th rows of \( A. \) Then we obtain an \( r \times m \) matrix \( A'(x', D') \) of holomorphic microlocal operators. We say that \( v_{1}(x'), \ldots, v_{m}(x') \in C_{\mathbb{R}^{n-1}, x'} \) satisfy an \( r \)-relation if choosing some \( r \) rows of some \( A(x', D') \) we have \( A'(x', D') \vec{v}(x') = 0. \) Here \( \vec{v} \) denotes \( \{v_{1}, \ldots, v_{m}\}. \) Note that even if \( v_{1}(x'), \ldots, v_{m}(x') \) satisfy an \( r \)-relation and another \( s \)-relation, it does not necessarily mean an \((r + s)\)-relation.

We next define a classification of the characteristic roots. Let \( \theta \in \{0, \pi\}. \) Let

\[
(x, \xi') \in \mathbb{R}^{n} \times \mathbb{R}^{n-1}, \quad x_{1} \neq 0, \quad \arg x_{1} = \theta.
\]

We define

\[
\begin{align*}
M &= \{1, 2, \ldots, m\}, \\
M_{0,0} &= \{\lambda \in M; \Re(x_{1}\varphi_{\lambda}(x, \xi')) = 0, \text{ if } (x, \xi') \text{satisfies (5)}\}, \\
M_{\pm,0} &= \{\lambda \in M; \pm \Re(x_{1}\varphi_{\lambda}(x, \xi')) > 0, \text{ if } (x, \xi') \text{satisfies (5)}\}, \\
M'_{0} &= M \setminus M_{0,0} \setminus M_{+,0} \setminus M_{-,0}.
\end{align*}
\]

It is easy to see that \( M_{0,0} \cup M_{+,0} \cup M_{-,0} \cup M'_{0} = M \) is a disjoint union.

Let \( m_{0,0}, \ m_{\pm,0} \) be the number of the elements belonging to \( M_{0,0}, \ M_{\pm,0}, \) respectively. We assume that

\[
(6) \quad M'_{0} = \emptyset, \quad \forall \theta \in \{0, \pi\}.
\]

We also need a condition for the microfunctions. Let

\[
\begin{align*}
\omega(r) &= \{(x, \xi) \in \sqrt{-1}T^{*}\mathbb{R}^{n}; \ |x| < r, \ |\xi''| < r \Im \xi_{n}\}, \\
\omega'(r) &= \{(x', \xi') \in \sqrt{-1}T^{*}\mathbb{R}^{n-1}; \ |x'| < r, \ |\xi''| < r \Im \xi_{n}\},
\end{align*}
\]

and

\[
\begin{align*}
\omega_{0}(r) &= \{(x, \xi) \in \omega(r); \ |x'| \leq r^{-1}|x_{1}|, \ |\xi''| \leq r^{-1}|x_{1}| \Im \xi_{n}\}, \\
\omega_{0}(r) &= \{tx^{*}; t > 0\}.
\end{align*}
\]
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We define

\[ C_0 = \lim_{r \to 0} \Gamma(\omega_0(r), C_{R^n}, \omega_0(r)), \]
\[ C'_0 = \lim_{r \to 0} \Gamma(\omega'_0(r), C_{R^{n-1}}, \omega'_0(r)). \]

Then we have the following

**Theorem.** We assume (1) – (3) and (6). Let \( v_1(x'), \ldots, v_m(x') \in C'_0 \). Then there exists an \( m_{+0} \)-relation and an \( m_{+\pi} \)-relation such that the Cauchy problem (4) has a solution \( u \in C_0 \) if, and only if, \( v_1(x'), \ldots, v_m(x') \) satisfy these relations.

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

**Example 0 (Lewy-Mizohata operators).** If \( P_\pm = D_1 \pm \sqrt{-1} x_1 D_n \), then we have \( M_{+,\theta} = \{1\} (= M) \), \( M_{+,\theta} = 0 \). The above theorem means that \( P_- u = 0 \), \( u(0, x') = v(x') \) is solvable for any \( v \in C'_0 \) without any relations. In fact using the defining function we only need to let \( u(x) = v(x'', x + \sqrt{-1} x_1^2/2) \). On the other hand, \( P_+ u = 0 \), \( u(0, x') = v(x') \) is solvable only for the case when \( v(x') \) satisfies a one-relation. This means \( v = 0 \), and \( u = 0 \). It follows that \( P_+ u = 0 \Rightarrow u = 0 \), i.e., \( P_+ \) is hypo-elliptic in \( C_0 \) (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to \( M_{+\theta} \) cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case \( m = 2 \).

**Example 1 (microhyperbolic operators).** Let \( P(x, D) = D_1^2 - x_1^2 D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1 \). Without loss of generality, we may assume that \( P' \) is a polynomial in \( D_1 \) of degree 1. Since \( \varphi_1(x, \xi') = x_1 \xi_n, \varphi_2(x, \xi') = -x_1 \xi_n \), and \( \arg \xi_n = \pi/2 \), it is easy to see that \( M_{0, \theta} = \{1, 2\}, M_{+, \theta} = 0 \) for \( \theta \in \{0, \pi\} \). It follows that that (4) is solvable for arbitrary \( v_1(x'), v_2(x') \in C'_0 \) without any relations (See [3]).

**Example 2 (Tricomi operators).** Let \( P(x, D) = D_1^2 - x_1 D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1 \). We have \( \varphi_1(x, \xi') = \sqrt{x_1} \xi_n, \varphi_2(x, \xi') = -\sqrt{x_1} \xi_n \). It follows that \( M_{0, 0} = \{1, 2\}, m_{+, 0} = 0 \), and that \( M_{+, \pi} = \{1\}, M_{-, \pi} = \{2\}, m_{+, 0} = 1 \). It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let \( \omega \subset \sqrt{-1} T^* R^n \) be a small neighborhood of \( x^* \), and let \( \omega^\theta = \{x, \xi \in \omega; x_1 \neq 0, \arg x_1 = \theta\}, \theta \in \{0, \pi\} \).

At first we consider an elliptic boundary value problem in \( \omega_{\pi} \), giving one boundary datum on \( \{x_1 = 0\} \). Then we can always extend this solution to the hyperbolic region \( \omega_0 \). This case was considered also by [4].

**Example 3 (hypoelliptic operators).** Let \( P(x, D) = D_1^2 + x_1^2 D_n^2 + P'(x, D), \) \( \text{ord} P' \leq 1 \). Since \( \varphi_1(x, \xi') = \sqrt{-1} x_1 \xi_n, \varphi_2(x, \xi') = -\sqrt{-1} x_1 \xi_n \), it is easy to see that \( M_{0, \theta} = \{1\}, M_{+, \theta} = \{2\}, m_{+, \theta} = 1 \) for \( \theta \in \{0, \pi\} \). There exist an \( m_{+, 0} \)-relation and an \( m_{+, \pi} \)-relation such that the Cauchy problem (4) uniquely has a solution \( u \in C_0 \) if, and only if, \( v_1(x'), \ldots, v_m(x') \in C'_0 \) satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case \( v_1 = v_2 = 0 \), and \( u = 0 \). In other words, \( Pu = 0 \) does not have any
non-trivial solutions. It is well-known that this is true if the principal symbol $\sigma_1(P')$ of the lower order term satisfies $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1}, \sqrt{-13}, \sqrt{-15}, \cdots\}$ (See [2,5]). Of course our result applies for higher order operators, too.

REFERENCES


