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Lagrangian properties for the diffraction in the complex domain

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1 Introduction

Let $M$ be a real manifold with boundary and $P$ a second order differential operator with smooth coefficients and real principal symbol $p$. We assume that $p$ is of real principal type and not characteristic on the boundary. Let us consider the classical Dirichlet problem

$$Pu = 0 \quad \text{in} \quad M, \quad u|_{\partial M} = 0.$$ 

If the equation of the boundary is $f = 0$ with $f > 0$ in $M$, the diffractive region is defined by

$$\mathcal{G}_+ = \{ \rho \in \tilde{T}^*\partial M : p(\rho) = 0, \quad \{p, f\} = 0, \quad \frac{\{p, \{p, f\}\}}{\{\{p, f\}, f\}} > 0\}$$

and corresponds to rays tangent to the boundary. The propagation of singularities of $C^\infty$, Gevrey and analytic singularities is known in this setting, see [12], [7], [8]. However, very few lagrangian properties are preserved along diffractive rays. In [9], Lebeau proves that, far away from the data, the operator mapping the Dirichlet data to the normal derivative of the solution belongs to a class of lagrangian Gevrey 3 distributions with weight.

We review a result on the lagrangian properties of the solution at the transition from the shadow to the illuminated region in the $C^\infty$ framework. Using the canonical invariance, we prove that the solution belongs to a class of lagrangian distributions associated to a pair of lagrangian submanifolds. As a consequence, we see that, for a conormal data, the second wave front lies in a lagrangian submanifold.

We next investigate the same problem in the analytic category. Here we use the geometry of complex canonical transforms and the $H_p$ spaces of Sjöstrand. We generalize the definition of bilagragian distributions in this framework and describe the FBI transform of the solution of the boundary value problem.

2 Pairs of lagrangian submanifolds

2.1 Microlocal phase

Let $X$ be a $C^\infty$ manifold of real dimension $n$ and with local coordinates $x_1, \ldots, x_n$. On the cotangent bundle $T^*X$, we consider the canonical 2-form

$$\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$$

where $\xi_j$ are the dual coordinates of $x_j$. This 2-form is called the canonical symplectic form.
where the dual coordinates are defined by \( d\xi_j(\partial \xi_k) = \delta_{jk} \). This manifold is conic for the multiplication \( M_\ell : (x, \xi) \mapsto (x, \ell \xi) \). We denote by \( T^* X = T^* X \setminus \{0\} \) the cotangent bundle with the zero section removed.

A submanifold \( \Lambda \) of \( T^* X \) of dimension \( n \) is lagrangian if \( \sigma|_\Lambda = 0 \). It is said conic if it is invariant through \( T_t \) for every \( t > 0 \).

The classical definition of a phase function for a conic lagrangian submanifold is the following, [1]. For simplicity, we restrict ourself to the case of a real non-degenerate phase function.

**Definition 1** Let \( X \) be a \( C^\infty \) manifold and \( \varphi \) be a \( C^\infty \) real valued function in an open conic subset \( \Gamma \) of \( X \times \mathbb{R}^n \setminus \{0\} \) which is homogeneous of degree 1. The function \( \varphi \) is called a local phase function of \( X \) if \( d\varphi \neq 0 \) in \( \Gamma \) and \( \operatorname{rg}(\varphi''_\theta, \varphi''_\eta) = N \) in the set

\[
C_\varphi = \{(x, \theta) \in \Gamma : \varphi'(x, \theta) = 0\}.
\]

If \( \varphi \) is a local phase function then the differential of the map

\[
j_\varphi : C_\varphi \to \dot{T}^* X : (x, \theta) \mapsto (x, \varphi'_x(x, \theta))
\]

is of rank \( n \). If it is an embedding then \( \varphi \) is called a phase function. Since

\[
j^*_\varphi \sigma = j^*_\varphi d(\xi dx) = d(\varphi'_x dx) = d(d\varphi|_{C_\varphi}) = 0,
\]

its image \( \Lambda_\varphi = j_\varphi(C_\varphi) \) is a lagrangian submanifold of \( \dot{T}^* X \).

### 2.2 2-microlocal phase

The second wave front set along a lagrangian submanifold \( \Lambda \) is defined as a subset of the cotangent bundle of \( \Lambda \). To define lagrangian distributions associated to this geometric setting, we introduce new phase functions.

If \( \Lambda \) is a conic lagrangian submanifold of \( \dot{T}^* X \), then we have the identification

\[
T^* \Lambda \sim T_\Lambda \dot{T}^* X
\]

where the right hand side is the normal bundle of \( \Lambda \). Indeed, if \( k \) is a normal to \( \Lambda \) at a point \( \rho \) then \( T_\rho \Lambda \ni h \mapsto \sigma(h, k) \) is a well-defined 1-form.

Moreover this manifold has two homogeneities: one inherited from \( \Lambda \) and another one as a cotangent bundle. A lagrangian submanifold of \( T^* \Lambda \) is said conic bilagrangian if it is conic for both homogeneities. We introduce phase functions that parameterize such a manifold.

Let \( \Gamma_0 \) be an open subset of \( X \times \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^m \setminus \{0\} \) such that \( (x, \theta, \eta) \in \Gamma_0 \) and \( s, t > 0 \) imply \( (x, t\theta, s\eta) \in \Gamma_0 \). Such an open set is called a profile. An open subset \( \Gamma \) of \( X \times \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^m \setminus \{0\} \) is said biconic with profile \( \Gamma_0 \) if

- \( (x, \theta, \eta) \in \Gamma \) and \( t > 0 \) imply \( (x, t\theta, t\eta) \in \Gamma \),

- for each compact subset \( K \) of \( \Gamma_0 \), there is \( \epsilon > 0 \) such that \( (x, \theta, s\eta) \in \Gamma \) if \( (x, \theta, \eta) \in K \) and \( 0 < s < \epsilon \).
If $\Gamma$ is biconic with respect to a family of profiles, it is also biconic with respect to their union. The profile of $\Gamma$ is the largest profile $\Gamma_0$ such that the last condition is satisfied. We also introduce

$$\Gamma_1 = \{(x, \theta, \eta) : \exists \eta \text{ such that } (x, \theta, \eta) \in \Gamma\}.$$

This is an open conic subset of $X \times \mathbb{R}^N \setminus \{0\}$.

Let $p, q \in \mathbb{R}$ and $r \in \mathbb{N}_0$. A $C^\infty$ function $f : \Gamma \to \mathbb{R}^m$ is said bihomogeneous of degree $(p, q; r)$ if

- $f(x, t \theta, t \eta) = t^p f(x, \theta, \eta)$ if $(x, \theta, \eta) \in \Gamma$, $t > 0$,
- for every $(x_0, \theta_0, \eta_0) \in \Gamma_0$, there is a neighborhood $V$ of $(x_0, \theta_0, \eta_0)$ and a $C^\infty$ function $F$ in $V \times \mathbb{R}^d - \epsilon, \epsilon$ satisfying

$$f(x, \theta, s \eta) = s^q F(x, \theta, \eta, s^{1/r})$$

if $(x, \theta, \eta, s) \in V \times \mathbb{R}^d - \epsilon, \epsilon$.

The integer $r$ is inserted here essentially for technical reasons. In the application, it does not affect the 2-microlocal geometry but has some effects on the microlocal lagrangian submanifolds involved. We say that $f$ has the regularity $r$.

**Definition 2** Let

- $\Lambda$ be a conic lagrangian submanifold of $T^* X$,
- $\varphi$ be a $C^\infty$ real valued function which is homogeneous of degree 1 in $\Gamma_1$,
- $\psi$ be a $C^\infty$ real valued function which is bihomogeneous of degree $(1, 1; r)$ in $\Gamma$

and

$$C_{\varphi, \psi} = \{(x, \theta, \eta) \in \Gamma_0 : \varphi(x, \theta) = 0, \psi(x, \theta, \eta) = 0\}.$$ 

The pair $(\varphi, \psi)$ is a local 2-phase function of $\Lambda$ (with regularity $r$) if

- $\varphi$ is a local phase function that parameterizes $\Lambda$,
- at each point of $C_{\varphi, \psi}$, the vector $(\psi'_1, \psi'_\theta)$ is different from 0 and

$$\text{rk} \begin{pmatrix} \psi''_1,_{\eta x} & \psi''_1,_{\eta \theta} & \psi''_1,_{\eta} \\ \varphi''_{\theta x} & \varphi''_{\theta \theta} & 0 \end{pmatrix} = N + M.$$ 

If $\varphi$ is a phase function, the last condition means that the map $(\rho, \eta) \mapsto \psi_1(j_{\varphi^{-1}}(\rho), \eta)$ is a local phase function of $\Lambda$. This definition has the following consequences.

a) The map

$$j_{\varphi, \psi} : C_{\varphi, \psi} \to T^* \Lambda : (x, \theta, \eta) \mapsto ((x, \varphi'), j_\varphi((\psi'_1,_{\eta x}, \psi'_1,_{\theta})_{TC_\varphi})).$$

is a lagrangian immersion.
Following the identification $\tilde{T}^*\Lambda \sim T_\Lambda \tilde{T}^*X$, the map $j_{\varphi,\psi}$ can be identified with

$$C_{\varphi,\psi} \to \tilde{T}_\Lambda T^*X : (x, \theta, \eta) \mapsto ((x, \varphi_x'), (h, \tilde{\psi}_{1,x} + \varphi''_{xx}.h + \varphi''_{x\theta}.k))$$

where $h, k$ satisfy

$$\varphi''_{x\theta}.h + \varphi''_{\theta\theta}.k + \tilde{\psi}_{1,\theta} = 0.$$

b) Let $(\varphi, \psi)$ be a local 2-phase function (with regularity $r$) in a biconic set $\Gamma$ and $(x_0, \theta_0, \eta_0) \in C_{\varphi,\psi}$. By the definition, $\varphi$ is a local phase function in $\Gamma$ and there is a biconic open subset $\tilde{\Gamma}$ of $\Gamma$ whose profile contains $(x_0, \theta_0, \eta_0)$ such that $(x, (\theta, \eta)) \mapsto \varphi(x, \theta) + \psi(x, \theta, \eta)$ is a local phase function in $\tilde{\Gamma}$. A local 2-phase function $(\varphi, \psi)$ is called a 2-phase function if $j_{\varphi}$, $j_{\varphi+\psi}$ and $j_{\varphi,\psi}$ are embeddings.

One can verify that if $(\varphi, \psi)$ is a local 2-phase function in $\Gamma$ and $(x_0, \theta_0, \eta_0) \in C_{\varphi,\psi}$ then there is a biconic open set $\tilde{\Gamma}$ whose profile contains $(x_0, \theta_0, \eta_0)$ such that $(\varphi, \psi)$ is a 2-phase function in $\tilde{\Gamma}$.

Hence, if $(\varphi, \psi)$ is a 2-phase function then

$$\{(x, \varphi_x'), (h, \tilde{\psi}_{1,x} + \varphi''_{xx}.h + \varphi''_{x\theta}.k) : (x, \theta) \in C_{\varphi,\psi}, \tilde{\psi}_{1,\theta} + \varphi''_{\theta\theta}.h + \varphi''_{\theta\theta}.k = 0\}$$

is a conic bilagrangian submanifold of $\tilde{T}^*\Lambda_\varphi$. It is denoted $\Lambda_{\varphi,\psi}$.

c) If $(\varphi, \psi)$ is a 2-phase function, then

$$n - \text{rg}(\pi_{\Lambda_\varphi, X}) = N - \text{rg}(\varphi''_{\theta\theta}), \quad n - \text{rg}(\pi_{\Lambda_{\varphi,\psi},X}) = M - \text{rg}(\psi''_{1,\eta\eta}),$$

and

$$n - \text{rg}(\pi_{\Lambda_{\varphi,\psi}, X}) = N + M - \text{rk} \begin{pmatrix} \psi''_{1,\eta\eta} & \psi''_{1,\eta\theta} \\ 0 & \varphi''_{\theta\theta} \end{pmatrix}.$$

### 2.3 Pairs of lagrangian submanifolds

We now describe the geometric setting associated to a 2-phase. If $Y$ is a submanifold of a $C^\infty$ manifold $X$, the blowup of $X$ along $Y$ is

$$\tilde{X}_Y = (X \setminus Y) \cup \tilde{T}_Y X.$$

The sets

$$\bigcap_{1 \leq j \leq p} \left\{ x \in \omega : f_j(x) > 0 \right\} \cup \left\{ (x, h) \in \tilde{T}_Y X : x \in \omega, df_j(x).h > 0 \right\}$$

where $\omega$ is an open subset of $X$ and $f_j \in C^\infty(\omega)$, $f_j|_{Y \cap \omega} = 0$ for all $j$, form a basis of topology of $\tilde{X}_Y$. For this topology, the projection $\pi : \tilde{X}_Y \to X$ is continuous.

**Definition 3** A pair $(\Lambda_0, \Lambda_1)$ is a 2-microlocal pair of lagrangian submanifolds of $\tilde{T}^*X$ if

- $\Lambda_0$ is a conic lagrangian submanifolds of $\tilde{T}^*X$, $\Lambda_1 \subset (\tilde{T}^*X)^\wedge_{\Lambda_0}$,
- $\Lambda_1 \cap (\tilde{T}^*X \setminus \Lambda_0)$ is a conic lagrangian submanifold of $\tilde{T}^*X$,
for each \((\rho, h) \in \Lambda_1 \cap \tilde{T}_\Lambda^* T^* X\), there is an open neighborhood \(V\) of \((\rho, h)\) in \((\tilde{T}^* X)_{\Lambda_0}\) and a 2-phase function \((\varphi, \psi)\) such that

\[
\Lambda_0 \cap \pi(V) = \Lambda_\varphi \quad \text{and} \quad \Lambda_1 \cap V = \Lambda_{\varphi+\psi} \cup \Lambda_{\varphi,\psi}.
\]

In this situation, we say that the 2-phase function \((\varphi, \psi)\) defines \((\Lambda_0, \Lambda_1)\). Let \(T_{\Lambda_0} \Lambda_1 = \Lambda_1 \cap \tilde{T}_\Lambda^* (T^* X)\). This is a conic bilagrangian submanifold of \(\tilde{T}^* \Lambda_0\).

Example 4 In \(\tilde{T}^* \mathbb{R}^n\), consider

\[
\varphi(x, \xi) = x \cdot \xi, \quad \psi(x, \xi, \eta') = \frac{\eta' \cdot \xi}{\xi_n} - H(\eta', \xi_n).
\]

where \(\xi = (\xi', \xi_n)\) and \(H\) is bihomogeneous of degree \((1, 1; r)\). We have

\[
\Lambda_\varphi = \{(0, \xi) : \xi_n \neq 0\}
\]

and

\[
\Lambda_{\varphi+\psi} = \{(-\frac{\eta''}{\xi_n}, \frac{\eta'' \cdot H'}{\xi_n}, (\xi_n H', \xi_n)) : \xi_n \neq 0\}.
\]

If \(H(\eta', \xi_n) = \eta_1^2/\eta_2^2\) in \(\mathbb{R}^3\), the projection of \(T_{\Lambda_\varphi} \Lambda_{\varphi+\psi}\) on \(\Lambda_\varphi\) is the cusp

\[
\{(0, \xi) : (\xi_1)^3 = (\xi_2)^2 \xi_3 : \xi_3 \neq 0\}.
\]

It can be shown, see [4], that the property of being a microlocal pair of lagrangian submanifolds is preserved by an homogeneous canonical transformation.

Let us describe the equivalence of 2-phase functions.

Two 2-phase functions \((\varphi, \psi)\) and \((\tilde{\varphi}, \tilde{\psi})\) defined in biconic open subsets \(\Gamma\) and \(\tilde{\Gamma}\) of \(X \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}\) are said equivalent if there is a \(C^\infty\) diffeomorphism \(\Gamma \to \tilde{\Gamma} : (x, \theta, \eta) \mapsto (f(x, \theta, \eta), g(x, \theta, \eta))\) such that

- \(\varphi(x, f(x, \theta, \eta)) + \psi(x, f(x, \theta, \eta), g(x, \theta, \eta)) = \tilde{\varphi}(x, \theta) + \tilde{\psi}(x, \theta, \eta)\),
- \(f\) is strictly bihomogeneous of degree \((1, 0; r)\) and \(g\) is bihomogeneous of degree \((1, 1; r)\),
- \(D_\theta f_0\) and \(D_\eta g_1\) are invertible in \(\Gamma_0\).

These two pairs define the same 2-microlocal pair.

If \(\Delta\) is a diagonal real invertible matrix, the pair of phases

\[
\varphi(x, \theta) = \tilde{\varphi}(x, \theta') + \frac{\langle \Delta \theta', \theta' \rangle}{2|\theta'|}, \quad \psi(x, \theta', \eta) = \tilde{\psi}(x, \theta', \eta)
\]

defines the same lagrangian submanifolds as \(\tilde{\varphi}\) and \(\tilde{\psi}\). In the same way,

\[
\varphi(x, \theta) = \tilde{\varphi}(x, \theta) \quad \psi(x, \theta, \eta) = \tilde{\psi}(x, \theta, \eta') + \frac{\langle \Delta \eta', \eta' \rangle}{2|\eta'|}
\]

defines the same lagrangian submanifolds as \(\tilde{\varphi}\) and \(\tilde{\psi}\).

It can be shown that the transition between two 2-phase functions defining the same 2-microlocal pair of lagrangian submanifolds can be obtained by a composition of the previous reductions.
3 Bilagrangian distributions

3.1 Symbols

We use only classical symbols. This is enough for the applications that we consider here.

Definition 5 If $m, p \in \mathbb{R}$ and $X$ is an open subset of $\mathbb{R}^n$, we denote by $S^{m,p}(X, \mathbb{R}^N, \mathbb{R}^M)$ the set of all $a \in C^\infty(X \times \mathbb{R}^N \times \mathbb{R}^M)$ such that for every compact subset $K$ of $X$ and all multiorders $\alpha, \beta, \gamma$ there is a $C > 0$ satisfying

$$|D_x^\alpha D_\theta^\beta D_\eta^\gamma a(x, \theta, \eta)| \leq C(1 + |\theta| + |\eta|)^{m-|\beta|}(1 + |\eta|)^{p-|\gamma|}$$

for all $(x, \theta, \eta) \in K \times \mathbb{R}^N \times \mathbb{R}^M$.

Write

$$S_2^\infty = \bigcup_{m, p \in \mathbb{R}} S^{m,p}, \quad S^{m,-\infty} = \bigcap_{p \in \mathbb{R}} S^{m,p}.$$  

It is clear that $S^{m,p}$ is a Fréchet space with semi-norms given by the smallest constants which can be used in the definition.

Oscillatory integrals can be defined using symbols in $S^{m,p}$ and 2-phase functions.

Theorem 6 Let $(\varphi, \psi)$ be a 2-phase function in an open biconic set $\Gamma$ and let $F$ be a closed conic subset of $\Gamma$ such that $F \ll \Gamma$. For every $u \in C_0^\infty(X)$, the linear form

$$a \mapsto \iiint e^{i(\varphi(x, \theta) + \psi(x, \theta, \eta))} a(x, \theta, \eta) u(x) \, dx \, d\theta \, d\eta$$

defined in the set of all $a \in S^{-\infty}(X; \mathbb{R}^N \times \mathbb{R}^M)$ satisfying $\text{supp}(a) \subset F$, can be extended on $S_2^\infty$ in a unique way such that it is continuous on the set of $a \in S^{m,p}(X, \mathbb{R}^N, \mathbb{R}^M)$ satisfying $\text{supp}(a) \subset F$ for every $m, p$.

3.2 Distribution class

Let $X$ be a $C^\infty$ manifold of dimension $n$ and let $(\Lambda_0, \Lambda_1)$ be a 2-microlocal pair of lagrangian submanifolds of $T^*X$.

Definition 7 The space $I^{m,p}(X, \Lambda_0, \Lambda_1)$ is the set of all locally finite sums of an element of $I^m(X, \Lambda_0)$, an element of $I^{m+p}(X, \Lambda_1 \cap T^*X)$ and distributions of the form

$$I_{\varphi, \psi, a}(u) = (2\pi)^{-((n+2(N+M))/4)} \iiint e^{i(\varphi(x, \theta) + \psi(x, \theta, \eta))} a(x, \theta, \eta) u(x) \, dx \, d\theta \, d\eta$$

where $(U, \chi)$ is a chart of $X$, $u \in C_0^\infty(X)$, $(\varphi, \psi)$ is a 2-phase function of $(\Lambda_0, \Lambda_1)$ defined in an open biconic subset $\Gamma$ of $\chi(U) \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}$ and

$$a \in S^{m+(n-2N)/4, p-M/2}(\chi(U), \mathbb{R}^N, \mathbb{R}^M)$$

satisfies $\text{supp}(a) \ll \Gamma$. 
It can be shown that this space is invariant by composition with a Fourier integral operators. Moreover, any 2-phase function defining the pair \((\Lambda_0, \Lambda_1)\) near a point \(\rho_0 \in \Lambda_0\) can be used to define any element of \(I^{m,p}(X, \Lambda_0, \Lambda_1)\) near \(\rho_0\).

The singularities of an element of \(I^{m,p}(X, \Lambda_0, \Lambda_1)\) are included in the lagrangian submanifolds involved, [4].

**Theorem 8** If \(u \in I^{m,p}(X, \Lambda_0, \Lambda_1)\) then

\[
WF(u) \subset \Lambda_0 \cup \Lambda_1, \quad WF_{\Lambda_0}^{(2)}(u) \subset T_{\Lambda_0} \Lambda_1.
\]

**4 Application to diffraction**

Let us consider the boundary value problem

\[
\begin{cases}
(-\Delta + (1 + x_n)\partial_t^2)u = 0 \\
u|_{x_n=0} = \delta_0, \quad u|_{t<0} = 0
\end{cases}
\]

where we use the decomposition \((t, x', x_n) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}_+\). This is a model for the strictly diffractive problems in the \(C^\infty\) category, see [11].

Let

\[p(x_n, \tau, \xi) = |\xi|^2 - (1 + x_n)\tau^2\]

be the principal symbol of the operator and \(\tau(\sigma, \xi') = |\xi'|^2 - \tau^2\) be the boundary hamiltonian. Two lagrangian submanifolds are involved here. On one hand, we consider the flowout \(\Lambda_0 = \Lambda_{0,+} \cup \Lambda_{0,-}\) of

\[\{((0,0), (\tau, \xi)) : \tau = \pm|\xi'| \neq 0, \xi_n = 0\}\]

through \(H_\tau\) on the boundary and followed by \(H_p\) intersected with \(t > 0\) and \(x_n > 0\). On the other hand, the flowout \(\Lambda_1 = \Lambda_{1,+} \cup \Lambda_{1,-}\) of

\[\{((0,0), (\tau, \xi)) : \tau = \pm|\xi|, \xi_n \neq 0\}\]

through \(H_p\) intersected with \(t > 0\) and \(x_n > 0\). These two manifolds are smooth but are tangent at their intersection.

It can be checked that \((\Lambda_{0,\pm}, \Lambda_{1,\pm})\) is a 2-microlocal pair of lagrangian submanifolds with

\[
T_{\Lambda_{0,\pm}} \Lambda_{1,\pm} = \{((2/3)x_n^{3/2} + 2\sqrt{x_n}, x', x_n), (\pm|\xi'|, \xi', \mp|\xi'\sqrt{x_n})\}
\]

\[((0,0,0), (\pm|\xi|, \xi_n, \mp|\xi|/\sqrt{x_n})) : \sigma, x_n > 0, \xi' \neq 0\}.
\]

A 2-phase function \((\varphi_{\pm}, \psi_{\pm})\) of \((\Lambda_{0,\pm}, \Lambda_{1,\pm})\) is given by

\[\varphi_{\pm}(t, x, \xi') = x'.\xi' \pm |\xi'|(t - \frac{2}{3}x_n^{3/2})\]

and

\[\varphi_{\pm}(t, x, \sigma, \xi') = x'.\xi' \pm |\xi'|(1 - \frac{\sigma}{|\xi'|})^{-1/2}(t - \frac{2}{3}((x_n + \frac{\sigma}{|\xi'|})^{3/2} - (\frac{\sigma}{|\xi'|})^{3/2})).\]
This 2-phase function has the regularity 2.

We denote by $I^m_{p}(X, \Lambda_0)$ the set of all lagrangian distributions on $\Lambda_0$ with symbol in $S^m_{\rho}$. This means that the symbol satisfies the following inequalities

$$|D_x^\alpha D_\theta^\beta a(x, \theta)| \leq C_{\alpha, \beta}(1 + |\theta|)^{m-|\beta|+(1-\rho)(|\alpha|+|\beta|)}.$$

An analysis of the solution of the initial boundary value problem given in [2] leads to the following result.

**Theorem 9** The solution $u$ of the previous boundary value problem belongs to

$$I^{\frac{n}{4}-1,\frac{3}{4}}(\mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}_{-1}^n \times \Lambda_0, \Lambda_1 \cup \tau_{\Lambda_0} \Lambda_1) + I^{\frac{n}{2m}-\frac{1}{2}}(\mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}_{-1}^n \times \Lambda_0).$$

5 **The geometry in the complex domain**

Our purpose is to define the phase functions used to characterize the bilagrangian distributions in the formalism of the Fourier-Bros-Iagolnitzer transform. In the microlocal case, we closely follow [6] and collect some material from [9], see also [13].

As usual, we identify

- $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ and write $z = x + iy$,
- $\zeta \in T^*_z \mathbb{C}^n$ with $(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ using $\zeta(h) = \sum_j \zeta_j h_j$,
- $T^*_z \mathbb{C}^n$ with $T^*_{x,y} \mathbb{R}^{2n}$ by mapping the $\mathbb{C}$-linear form $\zeta \in T^*_z \mathbb{C}^n$ to the $\mathbb{R}$-linear form $h \mapsto -\Im \zeta(h)$.

This map is symplectic if $T^*\mathbb{R}^{2n}$ is endowed with the usual canonical 2-form and $T^*\mathbb{C}^n$ with the 2-form $-\Im \sigma$ defined below.

It follows that if $f$ is a holomorphic function, $\partial f \in T^*_z \mathbb{C}^n$ is identified with $d(-\Im f) \in T^*_{x,y} \mathbb{R}^{2n}$ since $d(-\Im f) = -\Im(df) = -\Im(\partial f)$.

In the same way, if $\varphi$ is a real function then $d \varphi \in T^*_{x,y} \mathbb{R}^{2n}$ is identified with $\frac{i}{2} D_z \varphi \in \mathbb{C}^n$.

All the constructions described in this section are local even this is not stated explicitly.

5.1 **FBI transform**

Writing $z = x + iy$ and $\zeta = \xi + i\eta$, the canonical 2-form on $T^*\mathbb{C}^n$ is

$$\sigma = \sum_j d\zeta_j \wedge dz_j.$$ 

Its real and imaginary parts

$$\Re \sigma = \sum_j (d\xi_j \wedge dx_j - d\eta_j \wedge dy_j), \quad \Im \sigma = \sum_j (d\eta_j \wedge dx_j + d\xi_j \wedge dy_j).$$
are symplectic forms on $\mathbb{R}^{2n}$.

Let $\varphi$ be a real $C_1$ function defined in a neighborhood of $z_0 \in \mathbb{C}^n$ and

$$\Lambda_\varphi = \{(z, \frac{2}{i} D_2 \varphi(z)) : z \in \mathbb{C}^n\}.$$ 

This manifold is $\mathfrak{S}$-lagrangian since it is identified with

$$\{(z, d\varphi(z)) : z \in \mathbb{C}^n\} \subset T^* \mathbb{R}^{2n}.$$ 

If $j_\varphi$ denotes the \text{iff}_{1\mathfrak{S}i_{o}z\mapsto\backslash z/_{z,\frac{2}{i}D\varphi}(Z)_{\mathit{1}}/\}|\) then

$$j_\varphi^*(\Re_{\sigma}) = j_\varphi^*(\sigma) = j_\varphi^*(d(\frac{2}{i} \partial\varphi) = \frac{2}{i} \overline{\partial}\partial\varphi.$$ 

It follows that, if $\overline{\partial}\partial\varphi$ is non degenerate, $j_\varphi$ is a symplectic map from $(\mathbb{C}^n, \frac{2}{i} \overline{\partial}\partial\varphi)$ onto $(\Lambda_\varphi, \Re_{\sigma})$. Its inverse is the projection.

The following result is proven in [7], see also [3].

**Theorem 10** Let $\varphi$ be a strictly plurisubharmonic function near $z_0 \in \mathbb{C}^n$ and $\chi : T^* \mathbb{R}^n \rightarrow \Lambda_\varphi$ a canonical transform defined near $(y_0, \eta_0)$ such that $\chi(y_0, \eta_0) = (z_0, \frac{2}{i} D_2 \varphi(z_0))$. Here $\Lambda_\varphi$ is endowed with the 2-form $\Re_{\sigma}$. There is a unique holomorphic function $g(z, y)$ near $(z_0, y_0)$, such that

- the complexification of $\chi$ is
  $$\chi^\mathbb{C} : T^* \mathbb{C}^n \rightarrow T^* \mathbb{C}^n : (y, -D_y g(z, y)) \mapsto (z, D_z g(z, y)), $$

- $i g(z_0, y_0) = \varphi(z_0), -D_y g(z_0, y_0) = \eta_0$,

- the function $y \mapsto -\Re g(z, y)$ has a non degenerate critical point $y(z)$ with signature $(0, n)$ and critical value $\varphi(z)$. Moreover, we have
  $$(y(z), -D_y g(z, y(z))) = \chi^{-1}(z, \frac{2}{i} D_2 \varphi(z)).$$

For example, if $\chi : (x, \xi) \mapsto (x - i\xi, \xi)$ and $\varphi(z) = \frac{1}{2} |\Im z|^2$, then $g(z, y) = \frac{1}{2} (z - y)^2$.

The FBI transform associated to $\varphi$, $\chi$ near the points $(y_0, \eta_0), z_0$ is

$$T_\chi u(z, \lambda) = \int e^{i\varphi(z, y)} a(z, y, \lambda) u(y) \, dy$$

where $a$ is a classical symbol.

### 5.2 Lagrangian submanifolds

In this setting, lagrangian submanifold can be parameterized by a holomorphic function.
Proposition 11  Let $\Lambda$ be a lagrangian submanifold of $T^*\mathbb{R}^n$, $h$ be a phase function of $\Lambda$ near $\rho_0$ and $\chi$ be a local canonical map from $T^*\mathbb{R}^n$ to $\Lambda_{\rho}$ mapping $\rho_0$ to $z_0$. If $g$ the FBI phase defined in theorem 10 and

$$\phi_\Lambda(z) = cv(z,\theta)(g(z,\theta) + h(x,\theta))$$

then $\varphi_\Lambda = -\Im \phi_\Lambda$. The critical points are given by

$$(x,\theta) = j_\mathbb{C}^{-1} \circ \chi_\mathbb{C}^{-1}(z, Dz\phi_\Lambda(z)).$$

Here $j$ is the immersion $(x,\theta) \mapsto (x, h'_x)$ and $j_\mathbb{C}$ is its complexification.

We have

$$\chi(\Lambda) = \{ (z, Dz\phi_\Lambda(z)): z \in \mathbb{C}^n \}$$

and

$$\varphi_\Lambda(z) \leq \varphi(z).$$

The equality holds if and only if $(z, \frac{\partial}{\partial z}\varphi(z)) \in \chi(\Lambda)$.

In this formalism, the lagrangian distributions are defined in the following way.

Definition 12  Let $u$ be a distribution in an open subset $\Omega$ of $\mathbb{R}^n$, $\Lambda$ a lagrangian submanifold of $T^*\Omega$. With the notations of proposition 11, $u$ is said lagrangian at $\rho_0$ if, in a neighborhood of $z_0$, we have

$$(T_\chi u)(z,\lambda) = e^{i\lambda\phi_\Lambda(z)}b(z,\lambda)$$

where $b$ is a classical analytic symbol.

This is equivalent to the fact that $u$ can be written $u = u_1 + u_2$ with $\rho_0 = j_h(x_0,\theta_0)$ not in the singular spectrum of $u_2$ and

$$u_1(x) = \int_{\Gamma} e^{i h(x,\theta)}a(x,\theta) d\theta$$

where $\Gamma$ is a conic neighborhood of $\theta_0$ and $a$ is a classical analytic symbol near $(x_0,\theta_0)$.

5.3 Pairs of lagrangian submanifolds

Let us consider the FBI transform of a 2-phase function. For simplicity, we restrict ourself to the case of one 2-microlocal parameter.

Proposition 13  Let $(\Lambda_0, \Lambda_1)$ be a 2-microlocal pair of lagrangian submanifolds and $(h, \psi)$ be a 2-phase function for the pair $(\Lambda_0, \Lambda_1)$ near a point $\rho_0 \in \Lambda_0$. We assume that $h$ is analytic and that $\psi$ is an analytic function of $(x,\theta,\sigma^{1/2})$,

$$\psi(x,\theta,\sigma) = \psi_1(x,\theta)\sigma + \psi_{3/2}(x,\theta)\sigma^{3/2} + \psi_2(x,\theta)\sigma^2 + O(\sigma^{5/2}).$$

If $g$ is an FBI phase function associated to a local canonical map $\chi$ such that $\chi(\rho_0) = z_0 \in \mathbb{C}^n$, we have

$$\phi(z,\sigma) = cv(z,\theta)\left( g(z,\theta) + h(x,\theta) + \psi(x,\theta,\sigma) \right)$$

$$= \Phi_0(z) + \Phi_1(z)\sigma + \Phi_{3/2}(z)\sigma^{3/2} + \Phi_2(z)\sigma^2 + O(\sigma^{5/2}).$$

Here $\Phi_1$ and $\Phi_{3/2}$ are real on $\pi \circ \chi(\Lambda_0)$, $\Phi_1(z_0) = 0$, $D_z\Phi_1(z_0) \neq 0$ and $\Im \Phi_2(z_0) > 0$. 
With the notations of the proposition 13, a distribution $u$ is said \textit{analytic bilagrangian} at $\rho_0$ with respect to $(\Lambda_0, \Lambda_1)$ if, in a neighborhood of $z_0$, we have

$$(T_\chi u)(z, \lambda) = \int_0^\delta e^{i\phi(\sigma)}a(z, \sigma, \lambda) d\sigma$$

where $a$ is holomorphic in an open set of the form

$$\{(z, \sigma) \in \mathbb{C}^n \times \mathbb{C} : |z - z_0| < \epsilon, |\Im \sigma| < \epsilon \Re \sigma\}$$

and is bounded by $C\lambda^m$ for $\lambda > 1$.

Since $\Im \Phi_2(z_0) > 0$ and $\Phi_1(z_0), \Phi_{3/2}(z_0)$ are real, we can choose $\delta > 0$ small such that

$$-\Im \phi(z_0, \delta) < -\Im \phi_{\Lambda_0}(z_0).$$

For example, if

$$\Lambda_0 = \{((0, x_n), (\xi', 0))\}, \quad \Lambda_1 = \{((0, 0), (\xi', \xi_n))\}$$

and $g(z, y) = i(z - y)^2/2$, we have

$$\Phi_{\Lambda_0}(z) = \frac{i}{2}z^2, \quad \Phi_{\Lambda_1}(z) = \frac{i}{2}z^2$$

and

$$\phi(z, \sigma) = \frac{i\sigma^2}{2} + \sigma z_n + \frac{i\sigma^2}{2}.$$

\section{Bilagrangian structure of the parametrix}

Let us show how, at the transition of the shadow and the illuminated region, the parametrix defines a bilagrangian distribution if the boundary data is conormal.

Using [11], we may assume that the operator can be written

$$P(x, D) = D_{x_n}^2 + R(x, D_{x'})$$

in the half space $\{x_n > 0\}$. Its principal symbol is

$$p(x, \xi) = \xi_n^2 + r(x, \xi').$$

Let $r_0(x', \xi') = r(x', 0, \xi')$. We assume that the point $(x_0', \xi_0')$ is diffractive. This means that $r_0(x_0', \xi_0') = 0$ and $dr_0 \neq 0$, $\partial_{x_n} r < 0$.

Following [7], we first perform a complex canonical transform. We choose the weight function $\varphi_0(x') = |x'|^2/2$ and a canonical map

$$\chi_0 : T^* \mathbb{R}^{n-1} \to (\Lambda_{\varphi_0}, \Re \sigma)$$

mapping $(x_0', \xi_0')$ to $(0, 0)$ and the glancing region $\{r_0 = 0\}$ to $\{\Im z_1 = 0\}$. To this canonical map is associated a FBI transform.
After this transform, we obtain a pseudodifferential operator
\[ P(x, \tilde{D}, \lambda) = \tilde{D}_{x_{n}}^{2} + R(x, \tilde{D}_{x^{f}}, \lambda) \]

near \((0, 0)\) on \(\Lambda_{\varphi_{0}}\). Its principal symbol \(p(x, \xi) = \xi_{n}^{2} + r(x, \xi')\) is real on \(\Lambda_{\varphi_{0}}\) and \(p(x, \xi) = 0\) is equivalent to \(x_{n} + q(x', \xi) = 0\) with
\[ q(x', \xi) = \xi_{1} - e(x', \xi')\xi_{n}^{2} + O(\xi_{n}^{4}), \quad e(0, 0) > 0. \]

In the \(H_{\varphi}\) space, the problem is reduced to find an outgoing solution to
\[ P(x, \tilde{D}, \lambda)u(x, \lambda) = 0, \quad u|_{\tilde{x}_{n}=0} = g. \quad (1) \]

Define, as above, \(\Lambda_{0}\) as the flowout of the set of diffractive points through the boundary hamiltonian \(H_{p}\) followed by \(H_{p}\) and \(\Lambda_{1}\) as the flowout of all the characteristic points at \(x = 0\) through \(H_{p}\).

In the boundary value problem (1), we consider the boundary data \(g(x', \lambda) = \exp(i\lambda x'^{2})\) corresponding to a Dirac mass. Using the Lebeau construction of the parametrix, we obtain the following estimation.

**Theorem 14** The function
\[ \varphi(z, \sigma) = c\nu_{(x, \eta')} \left( \frac{i}{2} (z_{n} - x_{n})^{2} + H(z', \sigma, \eta'', \sqrt{x_{n} + \sigma}) \right. \]
\[ \left. - x_{1} \sigma - x''. \eta'' + F(x', \sqrt{\sigma}, \eta'') + \frac{i x' x'^{2}}{2} \right) \]

satisfies the conditions of proposition 13. Moreover, the solution \(u\) of the boundary value problem (1) can be written \(u_{1} + u_{2}\) where \(u_{1}\) is analytic bilagrangian and
\[ |u_{2}(x, \lambda)| \leq C_{\epsilon} e^{\lambda(\varphi_{\Lambda_{0}}(z) + C_{d}(z, \pi \chi(\Lambda_{0}))^{3}) + \epsilon \lambda} \]

near 0 for every \(\epsilon > 0\).

**References**


