Integral transforms for $\mathcal{D}$-modules and homogeneous manifolds

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1 Integral transforms, sheaves, $\mathcal{D}$-modules

Any problem of integral geometry has aspects of geometric nature (e.g. the support of the transform of a datum) and analytic nature (e.g. the differential equations describing the transform of some class of data). The idea of the approach by sheaves and $\mathcal{D}$-modules (see [8], [4], [9]) is to separate these problems in the calculations of the transform of a constructible sheaf (geometry) and of a coherent $\mathcal{D}$-module (analysis).

Complex integral transforms and real submanifolds. Since we use the theory of $\mathcal{D}$-modules, our framework will be complex, and the real transforms will be read by means of $\mathbb{R}$-constructible sheaves associated to real submanifolds (usually, locally constant sheaves of rank one). Let us explain this point a little more. Let $X$ be a complex analytic manifold with structure sheaf $\mathcal{O}_X$ and $X^\mathbb{R}$ the underlying real analytic manifold: then, the functors $\cdot \otimes \mathcal{O}_X$, $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ and $R\mathcal{H}om(\cdot , \mathcal{O}_X)$ (see [8], [9]) associate a $\mathcal{D}_X$-module to any $\mathbb{R}$-constructible sheaf on $X^\mathbb{R}$. In particular, let $M$ be a real analytic submanifold of $X^\mathbb{R}$ such that $X$ is a complexification of $M$; then, denoting by $j : M \rightarrow X$ the closed embedding and by $(\cdot)^* = R\mathcal{H}om(\cdot , \mathbb{C}_X)$ the duality functor for sheaves, one has $\mathbb{C}_M \otimes \mathcal{O}_X \simeq j_! \mathbb{A}_M$ (analytic functions on $M$), $\mathbb{C}_M \mathcal{T}hom(\cdot, \mathcal{O}_X) \simeq j_! \mathbb{C}^\infty_M$ (smooth functions), $\mathcal{T}hom(\mathbb{C}_M^*, \mathcal{O}_X) \simeq j_! \mathcal{D}b_M$ (Schwartz's distributions) and $R\mathcal{H}om(\mathbb{C}_M^*, \mathcal{O}_X) \simeq H^{d_M^\mathbb{R}}_M(\mathcal{O}_X) \otimes \mathcal{O}_M | X \simeq j_! \mathcal{B}_M$ (Sato's hyperfunctions).

The general integral transform. Let $X$ and $Y$ be complex analytic manifolds, $q_j \ (j = 1, 2)$ the projections of $X \times Y$ on $X$ and $Y$. Roughly speaking, the choice of a function (kernel) $k(x, y)$ on $X \times Y$ determines an integral
transform from data (e.g. functions, cohomology classes) on \( X \) to data on \( Y \) by the law \((f \circ k)(y) := \int_{q_2} k(x, y) f(x) \, dx\), where \( dx \) is some volume element on \( X \). Formally, this can be accomplished also in the categories of sheaves or \( D \)-modules, where the pull-back of \( f \) by \( q_1 \) becomes the inverse image by \( q_1 \), the product by \( k \) the tensor product and the integration along \( q_2 \) the proper direct image by \( q_2 \).

More precisely, let \( D^b(C_X) \) (resp. \( D^b(D_X) \)) be the derived category of sheaves of \( C \)-vector spaces (resp. left \( D \)-modules) on \( X \), i.e. the complexes with bounded cohomology modulo quasi-isomorphisms. Any kernels \( K \in D^b(C_{X \times Y}) \) and \( \mathcal{K} \in D^b(D_{X \times Y}) \) define integral transforms by means of the following functors:

\[
\begin{align*}
\circ K : D^b(C_X) & \to D^b(C_Y), \quad F \circ K = Rq_{2!}(K \otimes q_{1}^{-1}F), \\
\circ \mathcal{K} : D^b(D_X) & \to D^b(D_Y), \quad \mathcal{M} \circ \mathcal{K} = q_{2!}(K \otimes \mathcal{O}_{X \times Y} q_{1}^{-1}\mathcal{M}),
\end{align*}
\]

where \( q_{2!} \) and \( q_{1}^{-1} \) are the direct and inverse images in the sense of \( D \)-modules. The functor \( K \circ \cdot : D^b(C_Y) \to D^b(C_X) \) is similarly defined.

A typical situation is when \( \mathcal{K} \) is a regular holonomic \( D_{X \times Y} \)-module and \( K = R\text{Hom}_{D_{X \times Y}}(\mathcal{K}, \mathcal{O}_{X \times Y}) \) (i.e. the complex of holomorphic solutions of \( \mathcal{K} \)) by the Riemann–Hilbert correspondence in Kashiwara's formulation, \( K \) is a perverse sheaf and \( \mathcal{K} \simeq \text{Thom}(K, \mathcal{O}_{X \times Y}) \). For example, we have the geometric correspondences (see [4]): let \( S \) be a smooth complex submanifold of \( X \times Y \) and let \( \mathcal{K} = \mathcal{B}_S \) (the holomorphic hyperfunctions along \( S \)). The Penrose transform (see [6]) is an example. In this case, one has \( K \simeq \mathcal{C}_S[-\text{cod}_{X \times Y}S] \).

If one considers the double fibration (where \( f \) and \( g \) are the projections)

\[
X \leftarrow S \xrightarrow{g} Y,
\]

then it is easy to verify that \( \partial \circ \mathcal{C}_S = Rg_*f^{-1}(\cdot) \) and \( \partial \circ \mathcal{B}_S = g_*f^{-1}(\cdot) \).

**Adjunction formulas.** The arriving point are the adjunction formulas, where a problem of integral geometry is divided into the problems of calculating the transforms of a sheaf on \( Y \) and a \( D \)-module on \( X \). For simplicity, we suppose the manifolds to be compact.

**Proposition 1.** ([4], [9]) Let \( X \) and \( Y \) be compact complex analytic manifolds, \( \mathcal{K} \) a regular holonomic \( D_{X \times Y} \)-module and \( K = R\text{Hom}_{D_{X \times Y}}(\mathcal{K}, \mathcal{O}_{X \times Y}) \). Assume that \( \text{char}(\mathcal{K}) \cap (T^*X \times T^*_Y Y) \subset T^*_{X \times Y}(X \times Y) \). Then, for any \( \mathcal{M} \in D^b(D_X) \) and \( H \in D^b(C_Y) \) one has

\[
\begin{align*}
R\text{Hom}_{D_X}(\mathcal{M}, (K \circ H) \otimes \mathcal{O}_X) & \simeq R\text{Hom}_{D_Y}(\mathcal{M} \circ \mathcal{K}, H \otimes \mathcal{O}_Y)[-d_X^C], \\
R\text{Hom}_{D_X}(\mathcal{M}, R\text{Hom}((K \circ H)^*, \mathcal{O}_X)) & \simeq R\text{Hom}_{D_Y}(\mathcal{M} \circ \mathcal{K}, R\text{Hom}(H^*, \mathcal{O}_Y))[-d_X^C].
\end{align*}
\]
Moreover, similar formulas hold when $H$ has $\mathbb{R}$-constructible cohomology if one replaces $\otimes$ by $\otimes_\mathbb{R}$ and $R\text{Hom}$ by $\mathcal{T}\text{hom}$.

In particular, we are interested in the following case (see [4]). Let $\mathcal{F}$ a holomorphic line bundle on $X$ and $\mathcal{F}^\ast$. Taking $\mathcal{M} = \mathcal{D}\mathcal{F}^\ast = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}^\ast$, we get

\[
\begin{align*}
R\Gamma(X, (K \circ H) \otimes \mathcal{F}) & \cong R\text{Hom}_{\mathcal{D}_Y} (\mathcal{D}\mathcal{F}^\ast \otimes H \otimes (1 - \mathcal{F}^\ast)[-d_X^Y]), \\
R\text{Hom}((K \circ H)^\ast, \mathcal{F}) & \cong R\text{Hom}_{\mathcal{D}_Y} (\mathcal{D}\mathcal{F}^\ast \otimes H \otimes \mathcal{F}^\ast)[-d_X^Y].
\end{align*}
\]

(1)

Hence, (a) we shall compute the $\mathcal{D}$-module transform $\mathcal{D}\mathcal{F}^\ast \otimes \mathcal{K}$, and then (b) we shall make different choices of $H$ in order to obtain various applications.

**Remark 1.** Let $p_j$ ($j = 1, 2$) be the projections of $T^\ast(X \times Y)$ on $T^\ast X$ and $T^\ast Y$ respectively, and denote by $p_j$ the composition with the antipodal map. Assuming, as above, the "non-characteristicity condition" $\text{char}(\mathcal{K}) \cap (T^\ast X \times T^\ast Y) \subset T^\ast_{X \times Y}(X \times Y)$, one has $\text{char}(\mathcal{D}\mathcal{F}^\ast \otimes \mathcal{K}) \subset p_j^\ast \text{char}(\mathcal{K})$. Therefore, it is important to study the "microlocal correspondence" $T^\ast X \leftarrow \text{char}(\mathcal{K}) \rightarrow T^\ast Y$ in order to get informations on the transform $\mathcal{D}\mathcal{F}^\ast \otimes \mathcal{K}$.

## 2 Generalized flag manifolds and relations to representation theory

We specialize the preceding discussion to the case of compact homogeneous manifolds. Let $G$ be a complex semisimple Lie group, $P$ and $Q$ two parabolic subgroups containing a same Borel subgroup. Let $X = G/P$ and $Y = G/Q$ be the corresponding compact homogeneous manifolds. The diagonal $G$-action on $X \times Y$ has a finite number of orbits, and the only closed one is $S = G/(P \cap Q)$, which is again a compact homogeneous manifold of $G$. Let $\mathcal{K}$ be a $G$-equivariant regular holonomic $\mathcal{D}_{X \times Y}$-module (e.g. the one associated to one of these orbits) and $\mathcal{F}$ be a $G$-equivariant holomorphic line bundle on $X$: then $\mathcal{D}\mathcal{F}^\ast$ (resp. $\mathcal{D}\mathcal{F}^\ast \otimes \mathcal{K}$) is a quasi $G$-equivariant $\mathcal{D}_X$- (resp. $\mathcal{D}_Y$-) module (we refer e.g. to [10] for all these notions).

Let $G_0$ be a real form of $G$, and let $G_0$ act on $X$ and $Y$ by restricting the $G$-action. Then, if $H$ is a $G_0$-equivariant sheaf (e.g. we shall consider locally constant sheaves of rank one on the closed $G_0$-orbit in $Y$), so are $K \circ H$ and the duals, and the formulas (1) and (2) may be interpreted as isomorphisms in the derived category of representations of $G_0$. 
3 The case of Grassmannians

Let $W \simeq \mathbb{C}^n$ and $G = SL_n(\mathbb{C})$. For $1 \leq p \leq n - 1$, the subgroup $P_p$ of matrices in $G$ with the left bottom $(n - p) \times p$ block equal to zero is the “standard $p$th” maximal parabolic subgroup of $G$, and the quotient $X = G/P_p$ is naturally identified to the Grassmann manifold of $p$-dimensional subspaces of $W$. Recall that $X$ is a compact manifold of complex dimension $p(n - p)$. The homogeneous action of $G$ on $X$ yields the following natural identification:

$$T^* X \simeq \{(x; \alpha) : x \in X, \alpha \in \text{Hom}_{\mathbb{C}}(\frac{W}{x}, x)\}.$$

Let $1 \leq p \neq q \leq n - 1$, $X = G/P_p$ and $Y = G/P_q$; assume for simplicity $p < q \leq n - p$. The diagonal $G$-action on $X \times Y$ has orbits

$$S_j = \{(x, y) \in X \times Y : \dim_{\mathbb{C}}(X \cap y) = j\} \quad (j = 0, \ldots, p).$$

The closed orbit is $S_p \simeq G/(P_p \cap P_q)$ (the flag manifold of type $(p, q)$ in $W$), $S_0$ is the open generic orbit and the other $S_j$’s are smooth locally closed submanifolds. Again, for $1 \leq j \leq p$ one has the following useful identifications:

$$T^*_S (X \times Y) \simeq \{(x, y; \gamma) : (x, y) \in X \times Y, \gamma \in \text{Hom}_{\mathbb{C}}(\frac{W}{x+y}, x \cap y)\},$$

$$p_1(x, y; \gamma) = (x; \frac{W}{x} \xrightarrow{\pi} \frac{W}{x+y} \gamma \xrightarrow{i} x),$$

$$p_2(x, y; \gamma) = (y; \frac{W}{y} \xrightarrow{\pi} \frac{W}{x+y} \gamma \xrightarrow{i} y).$$

where $\pi$ and $i$ are the natural maps.

The holomorphic line bundles on $X$ are parametrized (up to isomorphisms) by $\lambda \in \mathbb{Z}$, and we shall denote by $\mathcal{O}_X(\lambda)$ the $-\lambda$th holomorphic tensor power of the determinant of the tautological vector bundle on $X$. In other words, let $F_p(W) = \{v = (v_1, \ldots, v_p) \in W^p : v_1 \wedge \cdots \wedge v_p \neq 0\}$ (the manifold of $p$-frames in $W$, an open dense subset of $W^p$) and $\pi : F_p(W) \to X$ the natural $GL_p(\mathbb{C})$-bundle assigning to any $v = (v_1, \ldots, v_p) \in F_p(W)$ the $p$-subspace of $W$ spanned by the $v_j$’s: then, for any open subset $U \subset X$ one has

$$\Gamma(U; \mathcal{O}_X(\lambda)) = \{\phi \in \Gamma(\pi^{-1}(U); \mathcal{O}_{F_p(W)}) : \phi(vA) = (\det A)^\lambda \phi(v) \forall A \in GL_p(\mathbb{C})\}.$$

We will write $\mathcal{D}_X(\lambda) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lambda)$ for short.
4 Applications

We announce results in two different applications.

4.1 The Grassmann duality ([11])

In the above notations, let $W \simeq \mathbb{C}^n$, $G = SL_n(C)$, $X = G/P_p$, $Y = G/P_{n-p}$ (we assume $p \leq n/2$), $\Omega = S_0$ and $S = (X \times Y) \setminus \Omega$. We consider the integral transform from $X$ to $Y$ given by $K = C_\Omega$ and $K = B_\Omega = T hom(C, \mathcal{O}_{X \times Y})$, i.e. the sheaf of meromorphic functions on $X \times Y$ with poles on $S$. (This choice generalizes the projective duality (see [5]), which is obtained for $p = 1$.) The nice geometric properties of the correspondence (e.g. for any $y \in Y$ the “slices” $\Omega_y = \{x \in X : (x, y) \in \Omega\}$ are affine charts of $X$) allow us to prove that:

**Theorem 1a.** The functor $\cdot \circ C_\Omega : D^b(C_X) \rightarrow D^b(C_Y)$ is an equivalence of categories preserving the objects with $R$- or $C$-constructible cohomologies; similarly, the functor $\cdot \circ B_\Omega : D^b(D_X) \rightarrow D^b(D_Y)$ is an equivalence of categories preserving the objects with good coherent or regular holonomic cohomologies.

The closed singular manifold $S$ is a non-smooth (if $p > 1$) hypersurface of $X \times Y$, Whitney-stratified by $S = \bigcup_{j=1}^p S_j$. The group $G$ acts prehomogeneously on $X \times Y$ with singular locus $S$, and this action is locally isomorphic to that of $GL_p(C)$ on $M_p(C)$ whose semi-invariant is $f : M_p(C) \rightarrow C$, $f(a) = \det(a)$ with $b$-function $b(s) = (s + 1) \cdots (s + p)$. This is a regular prehomogeneous vector space, and hence we get char($B_\Omega$) = $T_{X \times Y}^*(X \times Y) \cup \bigcup_{j=1}^p T_{S_j}^*(X \times Y)$. From the above identifications, it is then easy to check that the microlocal correspondence $T^*X \leftarrow \text{char}(B_\Omega) \rightarrow T^*Y$ induces a contact transformation between two open dense subsets $U_X \subset T^*X$ and $U_Y \subset T^*Y$, whose graph $\Lambda$ is contained in $T_{S_p}^*(X \times Y)$, and moreover $p_1^{-1}(U_X) = p_2^{-1}(U_Y) = \Lambda$. Using this fact and Theorem 1a, we obtain the following result:

**Theorem 1b.** Let $\lambda^* = -n - \lambda$: then $\mathcal{D}_X(-\lambda) \circ B_\Omega \simeq \mathcal{D}_Y(-\lambda^*)$ if $b(\lambda^* - \nu) \neq 0$ for any $\nu = 1, 2, \ldots$, i.e. if $\lambda \geq -n + p$.

Applying Theorem 1b to (1) and (2) we get the following isomorphisms
for any \(-n + p \leq \lambda \leq -p\) and any \(H \in D^b(C_X)\):

\[
\text{RG}(X; H \otimes \mathcal{O}_X(\lambda)) \simeq \text{RG}(Y; (H \circ C_\Omega) \otimes \mathcal{O}_Y(\lambda^*))[N],
\]

\[
\text{RG}(X; R\text{Hom}(H, \mathcal{O}_X(\lambda))) \simeq \text{RG}(Y; R\text{Hom}(H \circ C_\Omega, \mathcal{O}_Y(\lambda^*)))[-N],
\]

(where \(N = p(n - p)\)) and similarly for \(\otimes\) and \(R\text{Hom}\) replaced by \(\otimes\overline{\mathcal{X}}\) and \(\mathcal{T}\text{hom}\) when \(H\) has \(\mathbf{R}\)-constructible cohomology. Hence, we are left with the choice of \(H\) and the calculation of \(H \circ C_\Omega\). (Using the symmtry of the transform, here we have written the formulas with \(H\) a sheaf on \(X\) rather than on \(Y\).)

**Example 1.** Let \(Q\) be a hermitian form of signature \((p, n - p)\) on \(W \simeq \mathbb{C}^n\), and let \(G_0 = SU_{p,n-p}(Q)\) be the corresponding real form of \(G\). The \(G_0\)-orbits in \(X\) are \(U'_{i,j} = \{x \in X : Q_x\mid_y \text{ has signature } (i, j)\}\) for \(0 \leq i + j \leq p\) (the only closed orbit is \(U'_{0,0}\), i.e. the \(Q\)-isotropic \(p\)-subspaces, and the open orbits are \(U'_{i,j}\) with \(i + j = p\)). Similarly, the \(G_0\)-orbits in \(Y\) are \(U''_{i,j} = \{y \in Y : Q_y\mid_x \text{ has signature } (i, j)\}\) for \(0 \leq i \leq p, j \geq n - 2p\) and \(i + j \leq n - p\). Let \(y_0 \in U'' = U''_{0,n-p}\), and let \(E'_0 = \{x \in X : x \cap y_0 = 0\} \simeq \mathbb{C}^n\): then \(U' = U'_{p,0}\) is a relatively compact open subset of \(E_0\); similarly, fixed \(x_0 \in U',\ U''\) is a relatively compact open subset of the affine chart \(E''_0 = \{y \in Y : x_0 \cap y = 0\} \simeq \mathbb{C}^N\). Let us consider the closure \(\overline{U'} = \bigcup_{j=0}^{p} U'_{j,0}\), and choose \(H = C_{\overline{U'}}\): then it is possible to prove that \(C_{\overline{U'}} \circ C_\Omega \simeq C_U\) and then from the above adjunction formulas we get

\[
\text{RG}(\overline{U'}, \mathcal{O}_{E'_0}) \simeq \text{RG}_c(U''; \mathcal{O}_{E''_0})[N], \quad \text{RG}(\overline{U''}; \mathcal{O}_{E'_0}) \simeq \text{RG}(U''; \mathcal{O}_{E''_0})[-N]
\]

where all complexes are concentrated in degree zero.

**4.2 The generalized Radon-Penrose transform ([3])**

Let \(W \simeq \mathbb{C}^{n+1}\), \(G = SL_{n+1}(\mathbb{C})\), \(X = G/P_1\), \(Y = G/P_{k+1}\) (with \(1 \leq k \leq n-2\)) and \(S = S_1\). Note that \(X\) is a \(n\)-dimensional complex projective space and \(S\) is the flag manifold of type \((1, k+1)\) in \(W\); one has \(\dim_{\mathbb{C}} X = n\), \(\dim_{\mathbb{C}} Y = (k+1)(n-k)\) and \(\dim_{\mathbb{C}} S = n + k(n-k)\). We consider the integral transform from \(X\) to \(Y\) given by \(K = C_S[-(n-k)]\) and \(K = B_S\). (This is a natural generalization of Penrose's twistors correspondence (see [6]), which is obtained for \(n = 3\) and \(k = 1\).) We have \(\text{char}(B_S) = \Lambda = T_\Sigma^*(X \times Y)\), and thus let us consider the microlocal correspondence \(T^*X \leftrightarrow \Lambda \rightarrow T^*Y\): it is easy to check that \(p_1|_{\Lambda}\) is smooth and surjective and \(p_2|_{\Lambda}\) is a closed embedding identifying \(\Lambda\) to a smooth regular involutive submanifold \(V \subset \dot{T}^*Y\) (in fact,
it is \( V \simeq \{(y; \beta) : y \in Y, \ \beta \in \text{Hom}_C(W, y), \ \text{rank}(\beta) = 1\} \), which implies that the correspondence induces microlocally a contact transformation with holomorphic parameters. Using the theory of [4], we prove that:

**Theorem 2a.** \( \mathcal{D}_X(-\lambda) \otimes \mathcal{B}_S \) is concentrated in degree zero if and only if \( \lambda < 0 \), and \( H^0(\mathcal{D}_X(-\lambda) \otimes \mathcal{B}_S) \) is a \( \mathcal{D}_Y \)-module with simple characteristic along \( V \).

For any \( \lambda \in \mathbb{Z} \) we introduce a pair of \( G \)-equivariant holomorphic vector bundles \( \mathcal{H}_\lambda \) and \( \tilde{\mathcal{H}}_\lambda \) on \( Y \), and a \( G \)-invariant differential operator (the *ultrahyperbolic system*) \( P_\lambda \) acting between them. The description of these objects, that will be given in detail in [3], depends upon the sign of \( \lambda^* = -k - 1 - \lambda \) (positive, null and negative helicity cases in Penrose’s terminology [6]): it can be partially found e.g. in [2, Ex. 9.7.1] and, in a real version, in [7].

Let \( \mathcal{N}_{P_\lambda} \) be the coherent \( \mathcal{D}_Y \)-module associated to the differential operator \( P_\lambda \), i.e. \( \mathcal{N}_{P_\lambda} \) is defined by the exact sequence of \( \mathcal{D}_Y \)-modules (where \( \mathcal{D}\mathcal{H}_\lambda^* := \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}_\lambda^* \) and \( P_\lambda^* \) is the transpose to \( P_\lambda \)):

\[
\mathcal{D}\tilde{\mathcal{H}}_\lambda^* \xrightarrow{P_\lambda^*} \mathcal{D}\mathcal{H}_\lambda^* \longrightarrow \mathcal{N}_{P_\lambda} \longrightarrow 0.
\]

The \( \mathcal{D}_Y \)-module \( \mathcal{N}_{P_\lambda} \) has simple characteristic along \( V \), and we prove that:

**Theorem 2b.** For any \( \lambda < 0 \), \( \mathcal{D}_X(-\lambda) \otimes \mathcal{B}_S \) is isomorphic to \( \mathcal{N}_{P_\lambda} \).

Again, the application of Theorem 2b to (1) and (2) yields the following isomorphisms for any \( \lambda < 0 \) and any \( H \in \mathcal{D}^b(C_Y) \):

\[
\text{R}\Gamma(X, (C_S \circ H) \otimes \mathcal{O}_X(\lambda)) \simeq \text{RHom}_{\mathcal{D}_Y}(\mathcal{N}_{P_\lambda}, H \otimes \mathcal{O}_Y)[-k],
\]

\[
\text{RHom}_{\mathcal{D}_Y}((C_S \circ H)^*, \mathcal{O}_X(\lambda)) \simeq \text{RHom}_{\mathcal{D}_Y}(\mathcal{N}_{P_\lambda}, \text{RHom}(H^*, \mathcal{O}_Y))[-k]
\]

and similarly for \( \otimes \) and \( \text{RHom} \) replaced by \( \check{\otimes} \) and \( \check{\text{Thom}} \) when \( H \) has \( \mathbb{R} \)-constructible cohomology.

If we choose \( H \) to be a locally constant sheaf of rank one on the closed orbit of some real form \( G_0 \) of \( G \) in \( Y \), we can recover and improve many known results of real integral geometry. We give two hints in this direction (these results will appear in [3]).

**Example 2.** Let \( W_\mathbb{R} \) be a \((n+1)\)-dimensional real subspace of \( W \) such that \( W \simeq \mathbb{C} \otimes_\mathbb{R} W_\mathbb{R} \), and let \( G_0 = SL_{n+1}(\mathbb{R}) \) be the corresponding real form of \( G \). Assuming for simplicity that \( k+1 \leq (n+1)/2 \), the \( G_0 \)-orbits in \( Y \) are \( \mathcal{N}_j = \{ y \in Y : \dim_\mathbb{R}(y \cap W_\mathbb{R}) = j \} \) \((j = 0, \ldots, k+1)\), and \( N = \mathcal{N}_{k+1} \) is
naturally identified to the real Grassmann manifold of \((k+1)\)-subspaces of \(W_{\mathbb{R}}\). Similarly, the \(G_{0}\)-orbits in \(X\) are \(M_{i} = \{x \in X : \dim_{\mathbb{R}}(x \cap W_{\mathbb{R}}) = i\}\) \((i = 0, 1)\), and \(M = M_{1}\) is naturally identified to the real projective space of \(W_{\mathbb{R}}\). It is known that \(N\) (in particular, \(M\)) is not simply connected: namely, one has \(\pi_{1}(N) \cong \mathbb{Z}/2\mathbb{Z}\). We denote by \(C_{N}(\epsilon)\) \((\epsilon = 0, 1)\) the two distinct locally constant sheaves on \(N\), with the convention that \(C_{N}(0) = C_{N}\). For example, for \(\epsilon = 1\) we recover and improve the results of [7], whereas for \(\epsilon = 0\) the results should be new.

Example 3. Let \(1 \leq k \leq q \leq n-1\), \(Q\) a hermitian form on \(W\) of signature \((q+1, n-q)\), and let \(G_{0} = SU_{q+1,n-q}(Q)\) be the associated real form of \(G\). Assuming for simplicity that \(q + 1 \leq (n + 1)/2\), the \(G_{0}\)-orbits in \(Y\) are \(N_{i,j} = \{y \in Y : Q|_{y}\) has signature \((i,j)\}\) for \(0 \leq i + j \leq k + 1\). The closed orbit is \(N = N_{0,0}\), the \(Q\)-isotropic \((k+1)\)-subspaces of \(W\): one can prove that \(N\) is a generic real submanifold of \(Y\) of dimension \((k+1)(2n-3k-1)\), simply connected if \(k + q + 1 < n\) and affine if \(k = q\). Similarly, the \(G_{0}\)-orbits in \(X\) are \(M_{0,0}, M_{1,0}\) and \(M_{0,1}\); the closed orbit \(M = M_{0,0}\) is a simply connected real hypersurface of \(X\), and \(M_{1,0}\) and \(M_{0,1}\) are the two connected components of \(X \setminus M\). Here, we can extend some results known only in the case of the Penrose transform (see e.g. [1]) by calculating \(C_{S} \circ C_{N}\).

References


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