On some additive problems with primes and almost-primes
(Number Theory from the Stand Point of Analytic Number Theory)

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On some additive problems with primes and almost-primes

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§1. Introduction and statement of the result.

In 1937 I. M. Vinogradov [11] solved the ternary Goldbach problem proving that for every sufficiently large odd integer \( N \) the equation

\[
p_1 + p_2 + p_3 = N
\]

(1)

has solutions in prime numbers \( p_1, p_2, p_3 \).

Two years later van der Corput [3] used the method of I. M. Vinogradov and established the existence of infinitely many arithmetic progressions of three different primes. A corresponding result for progressions of four or more primes has not been proved yet. In 1981, however, D. R. Heath-Brown [4] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is \( P_2 \) (as usual \( P_r \) denotes an integer with no more than \( r \) prime factors, counted according to multiplicity).

Another famous and still unsolved number theory problem is the so-called prime twins conjecture, which asserts that there exist infinitely many primes \( p \), such that \( p + 2 \) is also a prime. The most important achievement in studying this problem is due to Chen [2]. In 1973 he proved that there exist infinitely many primes \( p \), such that \( p + 2 \) is \( P_2 \).

In 1997 D. I. Tolev and the author [8] applied the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem as well as some arguments belonging to H. Mikawa, and proved that there exist infinitely many non-trivial arithmetic progressions of three primes, such that for two of them, \( p_1 \) and \( p_2 \), say, both the numbers \( p_1 + 2, p_2 + 2 \) are almost–primes.

Later D. I. Tolev [9] obtained an extension of the above result by applying the vector sieve developed by Iwaniec [5] and used also by Brüdern and Fouvry [1]. He established that the equation

\[
p_1 + p_2 = 2p_3
\]

has infinitely many solutions in different primes \( p_1, p_2, p_3 \), such that \( p_1 + 2 = P_5 \), \( p_2 + 2 = P'_5 \), \( p_3 + 2 = P_8 \).

Here we study the solvability of the equation (1) in primes \( p_1, p_2, p_3 \), such that \( p_1 + 2, p_2 + 2, p_3 + 2 \) are almost–primes. We follow the approach of [9] putting emphasis on the examining of the main term where we apply some arguments of [1] (for the other details the reader may refer to [9]).

Our main result is the following
Theorem. Suppose that \( N \equiv 3 \pmod{6} \) is a sufficiently large integer. Then there exist infinitely many solutions of the equation (1) in primes \( p_1, p_2, p_3 \), such that \( p_1 + 2 = P_6, p_2 + 2 = P_6' \), and \( p_3 + 2 = P_8 \).

In fact, the proof yields that for some constant \( c_0 > 0 \) there are at least \( c_0N^2(\log N)^{-6} \) triplets of primes \( p_1, p_2, p_3 \), satisfying (1) and such that for any prime factor \( p \) of \( p_1 + 2 \) or \( p_2 + 2 \) we have \( p \geq N^{0.167} \) and for any prime factor \( p \) of \( p_3 + 2 \) we have \( p \geq N^{0.116} \). Notice that if \( N \) is a sufficiently large odd integer, not satisfying the hypothesis of the Theorem, then for any solution of (1) we have \( 3 \mid p_1 p_2 p_3 (p_1 + 2) (p_2 + 2) (p_3 + 2) \). Therefore, by modifying slightly the given proof, we may obtain that for such \( N \) the equation (1) has infinitely many solutions in primes \( p_1, p_2, p_3 \), such that \( p_1 + 2 = P_6, p_2 + 2 = P_6', p_3 + 2 = P_8 \). Here the extra prime factor in \( P_r \) is 3.

Recently H. Mikawa (unpublished result) used the theory of "well-factorable" functions and showed that the power of \( N \) in the quantity \( D_3 \) (for the definition see formulas (2)) can be taken to be equal to 4/9 instead of 1/3. This enables us to prove the Theorem with \( p_3 + 2 = P_8 \).

We should also mention that by applying the method of [9], D. I. Tolev [10] proved that if \( N \) is a sufficiently large integer satisfying the congruent condition \( N \equiv 5 \pmod{24} \) then the equation

\[
p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N
\]

has infinitely many solutions in prime numbers \( p_1, p_2, p_3, p_4, p_5 \) such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) and \( p_4 + 2 \) has at most 6 prime factors and \( p_5 + 2 \) has at most 7 prime factors.

§2. Notations.

Let \( N \) be a sufficiently large integer, such that \( N \equiv 3 \pmod{6} \) and \( \alpha_1, \alpha_2, \alpha_3 \) constants satisfying \( 0 < \alpha_1, \alpha_2 < 1/4, 0 < \alpha_3 < 1/6 \), which we shall specify later.

We put

\[
\begin{align*}
z_1 &= N^{\alpha_1}, \quad i = 1, 2, 3, \quad z_0 = (\log N)^{1000}, \quad D_0 = \exp((\log N)^{0.6}), \\
D_1 &= D_2 = N^{1/2} \exp(-2(\log N)^{0.6}), \quad D_3 = N^{1/3} \exp(-2(\log N)^{0.6}), \\
P(z_0) &= \prod_{2 < p < z_0} p, \quad P(z_0, z_i) = \prod_{z_0 \leq p < z_i} p, \quad i = 1, 2, 3.
\end{align*}
\]

Letters \( m, n, d, l, k, h, \delta, \nu, t, \rho \) denote integers; \( p, p_1, p_2, \ldots \) prime numbers. As usual \( \mu(n) \), \( \varphi(n) \) and \( \tau(n) \) denote Möbius' function, Euler's function and the number of positive divisors of \( n \), respectively; \( (m_1, \ldots, m_k) \) and \( [m_1, \ldots, m_k] \) denote the greatest common divisor and the least common multiple of \( m_1, \ldots, m_k \). Instead of \( m \equiv n \pmod{k} \) we write for simplicity \( m \equiv n(k) \). The notation \( p^r \mid n \) means that \( p^r \mid n \) and \( p^{r+1} \mid n \). For positive \( A \) and \( B \) we write \( A \asymp B \) instead of \( A \ll B \ll A \).
For squarefree odd integers \( k_1, k_2, k_3 \) and prime \( p \) we denote

\[
I_{k_1,k_2,k_3}(N) = \sum_{p_1+p_2+p_3=N, p_i+2=0(k_i), i=1,2,3} \log p_1 \log p_2 \log p_3 ,
\]

(3)

\[
h_{k_1,k_2,k_3}(p) = \begin{cases} 
\frac{1}{(p-1)^3} & \text{if } p \nmid k_1k_2k_3, \ p \nmid N; \\
-\frac{1}{(p-1)^2} & \text{if } p \mid k_1k_2k_3, \ p \mid N; \\
-\frac{1}{(p-1)^2} & \text{if } p \mid k_1k_2k_3, \ p \mid N + 2; \\
\frac{1}{(p-1)} & \text{if } p \mid k_1k_2k_3, \ p \mid N + 2; \\
\frac{1}{(p-1)} & \text{if } p^2 \mid k_1k_2k_3, \ p \mid N + 4; \\
-1 & \text{if } p^2 \mid k_1k_2k_3, \ p \mid N + 4; \\
-1 & \text{if } p^3 \mid k_1k_2k_3, \ p \mid N + 6; \\
p - 1 & \text{if } p^3 \mid k_1k_2k_3, \ p \mid N + 6;
\end{cases}
\]

(4)

\[
\omega(k_1,k_2,k_3) = \prod_{p|k_1k_2k_3} \frac{1 + h_{k_1,k_2,k_3}(p)}{1 + h_{1,1,1}(p)} ,
\]

\[
\Omega(k_1,k_2,k_3) = \frac{\omega(k_1,k_2,k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)} ,
\]

(5)

\[
\varepsilon(N) = \prod_{p|N} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right) .
\]

(6)

§3. Outline of the proof.

Consider the sum

\[
\Gamma = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 .
\]

Any non-trivial estimate from below of \( \Gamma \) implies the solvability of (1) in primes, such that \( p_i + 2 = P_{h_i} \), \( h_i = [\alpha_i^{-1}] \), \( i = 1, 2, 3 \). We see that

\[
\Gamma = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_6 ,
\]

where

\[
\Lambda_i = \begin{cases} 
\sum_{d|(p_i+2,F(z_0,a_i))} \mu(d) & \text{for } i = 1, 2, 3, \\
\sum_{d|(p_i-3+2,F(z_0))} \mu(d) & \text{for } i = 4, 5, 6 .
\end{cases}
\]

Denote

\[
\Lambda_i^\pm = \begin{cases} 
\sum_{d|(p_i+2,F(z_0,a_i))} \lambda_i^\pm(d) & \text{for } i = 1, 2, 3, \\
\sum_{d|(p_i-3+2,F(z_0))} \lambda_i^\pm(d) & \text{for } i = 4, 5, 6 .
\end{cases}
\]

(7)
where $\lambda^\pm_i(d)$ are the Rosser’s weights of order $D_i$, $0 \leq i \leq 3$ (see Iwaniec [6], [7]). In particular, we have

$$|\lambda^\pm_i(d)| \leq 1, \quad \lambda^\pm_i(d) = 0 \text{ for } d \geq D_i, \quad 0 \leq i \leq 3. \quad (8)$$

We find that

$$\Lambda^-_i \leq \Lambda_i \leq \Lambda^+_i, \quad 1 \leq i \leq 6$$

(for the proof see [7]). Consequently we may apply Lemma 3 of [9], which is the analogue of Lemma 13 of [1], and we get

$$\Gamma \geq \Gamma_0, \quad (9)$$

where

$$\Gamma_0 = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \left( \Lambda^-_1 \Lambda^+_2 \Lambda^+_3 \Lambda^+_4 \Lambda^+_5 \Lambda^+_6 + \Lambda^+_1 \Lambda^-_2 \Lambda^+_3 \Lambda^+_4 \Lambda^+_5 + \cdots + \Lambda^+_1 \Lambda^+_2 \Lambda^+_3 \Lambda^+_4 \Lambda^-_5 \Lambda^-_6 - 5 \Lambda^+_1 \Lambda^+_2 \Lambda^+_3 \Lambda^+_4 \Lambda^-_5 \Lambda^-_6 \right). \quad (10)$$

We use (3), (7) and change the order of summation to obtain

$$\Gamma_0 = \sum_{d_i \mid P(z_0, z_1), \delta_i \mid P(z_0), i=1,2,3} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) I(d_1 \delta_1, d_2 \delta_2, d_3 \delta_3) (N), \quad (11)$$

where

$$\kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) = \lambda^-_1(d_1) \lambda^+_2(d_2) \lambda^+_3(d_3) \lambda^-_0(\delta_1) \lambda^-_0(\delta_2) \lambda^-_0(\delta_3) + \cdots + \lambda^+_1(d_1) \lambda^-_2(d_2) \lambda^-_3(d_3) \lambda^-_0(\delta_1) \lambda^-_0(\delta_2) \lambda^-_0(\delta_3) - 5 \lambda^+_1(d_1) \lambda^+_2(d_2) \lambda^+_3(d_3) \lambda^-_0(\delta_1) \lambda^-_0(\delta_2) \lambda^-_0(\delta_3).$$

By applying the Hardy–Littlewood circle method we find an asymptotic formula for the sum $I_{k_1, k_2, k_3}(N)$ which we substitute in (10). Proceeding in the same way as in Lemma 11 and Lemma 13 of [9] we derive

$$\Gamma_0 = \frac{1}{2} N^2 \mathcal{S}(N) W + \mathcal{O}\left( \frac{N^2}{\log^4 N} \right), \quad (12)$$

where $\mathcal{S}(N)$ is defined by (6) and

$$W = \sum_{d_i \mid P(z_0, z_1), \delta_i \mid P(z_0), i=1,2,3} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) \Omega(d_1 \delta_1, d_2 \delta_2, d_3 \delta_3). \quad (13)$$

By (4) it is obvious that for squarefree odd integers $k_1, k_2, k_3$ we have

$$h_{k_1, k_2, k_3}(p) = \begin{cases} h_{1,1,1}(p) & \text{if } p \nmid k_1 k_2 k_3; \\ h_{p,1,1}(p) & \text{if } p \mid k_1 k_2 k_3; \\ h_{p,p,1}(p) & \text{if } p^2 \mid k_1 k_2 k_3; \\ h_{p,p,p}(p) & \text{if } p^3 \mid k_1 k_2 k_3. \end{cases} \quad (14)$$
Define

\[ \omega_1(p) = \omega(p, 1, 1), \quad \omega_2(p) = \omega(p, p, 1), \quad \omega_3(p) = \omega(p, p, p). \]  \tag{15} 

It is clear that if \( p > 2 \) then

\[
\omega_1(p) = \begin{cases} 
1 & \text{if } p \mid N; \\
\frac{(p-1)^2}{p^2-3p+3} & \text{if } p \not| N+2; \\
\frac{(p-1)(p-2)}{p^2-3p+3} & \text{if } p \mid N+2; \\
0 & \text{if } p \mid N+4; \\
\frac{(p-1)^2}{p^2-3p+3} & \text{if } p \not| N+4; 
\end{cases} 
\]

\[
\omega_2(p) = \begin{cases} 
\frac{(p-1)^2}{p^2-3p+3} & \text{if } p \mid N+6, \quad p > 3; \\
0 & \text{if } p \not| N+6. 
\end{cases} 
\]

\[
\omega_3(p) = \begin{cases} 
4 & \text{if } p = 3; \\
\frac{(p-1)^3}{p^2-3p+3} & \text{if } p \mid N+6, \quad p > 3; \\
0 & \text{if } p \not| N+6. 
\end{cases} 
\]  \tag{16} 

By (4), (5), (14), (15) we get

\[ \omega(k_1, k_2, k_3) = \prod_{p \mid k_1 k_2 k_3} \omega(p). \]  \tag{17} 

The next statement is the analogue of Lemma 12 of [1]. The Lemma follows easily from (16), (17).

**Lemma 1.** For squarefree odd \( k \), let

\[ \omega^*(k) = \prod_{p \mid k} \omega_1(p). \]

If \( k_1, k_2, k_3 \) is a triplet of integers, we put \( k_{1,2} = (k_1, k_2), \ k_{1,3} = (k_1, k_3), \ k_{2,3} = (k_2, k_3) \). Then

(i) there exists a function \( g \) of the three variables \( k_{1,2} \), such that for any squarefree odd \( k_1, k_2, k_3 \) we have

\[ \omega(k_{1,2}, k_{1,3}) \leq 10 \omega^*(k_1) \omega^*(k_2) \omega^*(k_3) g(k_{1,2}, k_{1,3}, k_{2,3}) \]

and

\[ g(k_{1,2}, k_{1,3}, k_{2,3}) \leq 10 (\max_{k_{1,2}})^{10}; \]

(ii) for any squarefree odd \( k_1, k_2, k_3 \) we have the inequality

\[ \omega(k_1, k_2, k_3) \leq 10 \omega(k_1) \omega(k_2) \omega(k_3). \]
where $\tilde{\omega}(m)$ is the multiplicative function defined on squarefree odd $m$ by

$$\tilde{\omega}(p) = \begin{cases} 2 & \text{if } p \mid N + 6; \\ 2p^{1/3} & \text{if } p \mid N + 6. \end{cases}$$

Suppose that the integers $d_1, d_2, d_3, \delta_1, \delta_2, \delta_3$ satisfy the conditions imposed in (13). Using (5) and (17) we easily get

$$\Omega(d_1 \delta_1, d_2 \delta_2, d_3 \delta_3) = \Omega(d_1, d_2, d_3) \Omega(\delta_1, \delta_2, \delta_3).$$

Note that $\Omega(\delta_1, \delta_2, \delta_3)$ is a symmetrical function with respect to $\delta_1$, $\delta_2$, $\delta_3$. Hence, we obtain by (11), (13)

$$W = \sum_{i=1}^{6} L_i H_i - 5L_7 H_7,$$

where

$$L_1 = \sum_{d_i \mid P(z_0, z), i=1,2,3} \lambda_i^-(d_1) \lambda_i^+(d_2) \lambda_i^+(d_3) \Omega(d_1, d_2, d_3),$$

$$L_2 = \sum_{d_i \mid P(z_0, z), i=1,2,3} \lambda_i^+(d_1) \lambda_i^+(d_2) \lambda_i^+(d_3) \Omega(d_1, d_2, d_3),$$

$$L_3 = \sum_{d_i \mid P(z_0, z), i=1,2,3} \lambda_i^+(d_1) \lambda_i^+(d_2) \lambda_i^-(d_3) \Omega(d_1, d_2, d_3),$$

$$L_4 = L_5 = L_6 = L_7 = \sum_{d_i \mid P(z_0, z), i=1,2,3} \lambda_i^+(d_1) \lambda_i^+(d_2) \lambda_i^-(d_3) \Omega(d_1, d_2, d_3),$$

$$H_1 = H_2 = H_3 = H_7 = \sum_{\delta_i \mid P(z_0), i=1,2,3} \lambda_i^-(\delta_1) \lambda_i^-(\delta_2) \lambda_i^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3),$$

$$H_4 = H_5 = H_6 = \sum_{\delta_i \mid P(z_0), i=1,2,3} \lambda_i^-(\delta_1) \lambda_i^+(\delta_2) \lambda_i^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3).$$

It is easy to prove the following

**Lemma 2.** Suppose that $\phi(n_1, n_2, n_3)$ is a function defined on the set of integers and such that for any two triplets $n_1, n_2, n_3$ and $l_1, l_2, l_3$, satisfying $(n_1 n_2 n_3, l_1 l_2 l_3) = 1$, we have $\phi(n_1 l_1, n_2 l_2, n_3 l_3) = \phi(n_1, n_2, n_3) \phi(l_1, l_2, l_3)$. Then the function

$$\Phi(n) = \sum_{d_1, d_2, d_3 \mid n} \phi(d_1, d_2, d_3)$$

is multiplicative.

Applying Lemma 1 and Lemma 2 we find asymptotic formulas for the sums $H_i$. Define

$$H^{(\mu)} = \sum_{\delta_i \mid P(z_0), i=1,2,3} \mu(\delta_1) \mu(\delta_2) \mu(\delta_3) \Omega(\delta_1, \delta_2, \delta_3).$$

(19)
Lemma 3. We have
\[ H_i = H^{(\mu)} + \mathcal{O}((\log N)^{-10}), \quad 1 \leq i \leq 7, \]
and
\[ H^{(\mu)} \asymp (\log z_0)^{-3}. \tag{20} \]

Now we are able to estimate from below the quantity $W$, defined by (18). We put
\[ \mathcal{F}(z_0, z_i) = \prod_{z_0 \leq p < z_i} \left(1 - \frac{\omega_1(p)}{p - 1}\right), \quad s_i = \frac{\log D_i}{\log z_i}, \quad i = 1, 2, 3, \tag{21} \]
where $\omega_1(p)$ is defined by (16). Suppose that $c^* > 0$ is an absolute constant and let $\theta_i, s_i, i = 1, 2, 3$ satisfy
\[ \theta_1 + \theta_2 + \theta_3 = 1, \quad \theta_i > 0, \quad f(s_i) - 2\theta_i F(s_i) > c^*, \quad i = 1, 2, 3, \]
where $f$ and $F$ are the functions of the linear sieve. Following the arguments in the proof of Lemma 15 of [9] it is easy to establish that
\[ W \geq H^{(\mu)} \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \left(\sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}\left((\log N)^{-1/3}\right)\right). \tag{22} \]

Finally, we choose
\[ \alpha_1 = \alpha_2 = 0.167, \quad \alpha_3 = 0.116, \quad \theta_1 = \theta_2 = 0.345, \quad \theta_3 = 0.31 \]
and compute that for sufficiently large $N$ we have
\[ f(s_i) - 2\theta_i F(s_i) > 10^{-5}, \quad i = 1, 2, 3. \tag{23} \]

Therefore, using (2), (20)–(23) we get
\[ W \gg \frac{1}{\log^3 N}. \]

The last estimate and (9), (12) imply
\[ \Gamma \gg \frac{N^2}{\log^3 N}, \]
which suffices to complete the proof of the Theorem.
References


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