On some additive problems with primes and almost-primes

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§1. Introduction and statement of the result.

In 1937 I. M. Vinogradov [11] solved the ternary Goldbach problem proving that for every sufficiently large odd integer $N$ the equation

$$p_1 + p_2 + p_3 = N \quad (1)$$

has solutions in prime numbers $p_1, p_2, p_3$.

Two years later van der Corput [3] used the method of I. M. Vinogradov and established the existence of infinitely many arithmetic progressions of three different primes. A corresponding result for progressions of four or more primes has not been proved yet. In 1981, however, D. R. Heath-Brown [4] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is $P_2$ (as usual $P_r$ denotes an integer with no more than $r$ prime factors, counted according to multiplicity).

Another famous and still unsolved number theory problem is the so-called prime twins conjecture, which asserts that there exist infinitely many primes $p$, such that $p+2$ is also a prime. The most important achievement in studying this problem is due to Chen [2]. In 1973 he proved that there exist infinitely many primes $p$, such that $p+2$ is $P_2$.

In 1997 D. I. Tolev and the author [8] applied the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem as well as some arguments belonging to H. Mikawa, and proved that there exist infinitely many non-trivial arithmetic progressions of three primes, such that for two of them, $p_1$ and $p_2$, say, both the numbers $p_1 + 2, p_2 + 2$ are almost–primes.

Later D. I. Tolev [9] obtained an extension of the above result by applying the vector sieve developed by Iwaniec [5] and used also by Brüdern and Fouvry [1]. He established that the equation

$$p_1 + p_2 = 2p_3$$

has infinitely many solutions in different primes $p_1, p_2, p_3$, such that $p_1 + 2 = P_5, p_2 + 2 = P_5', p_3 + 2 = P_8$.

Here we study the solvability of the equation (1) in primes $p_1, p_2, p_3$, such that $p_1 + 2, p_2 + 2, p_3 + 2$ are almost–primes. We follow the approach of [9] putting emphasis on the examining of the main term where we apply some arguments of [1] (for the other details the reader may refer to [9]).

Our main result is the following
Theorem. Suppose that \( N \equiv 3 \pmod{6} \) is a sufficiently large integer. Then there exist infinitely many solutions of the equation (1) in primes \( p_1, p_2, p_3 \), such that \( p_1 + 2 = P_5 \), \( p_2 + 2 = P_6' \), \( p_3 + 2 = P_8 \).

In fact, the proof yields that for some constant \( c_0 > 0 \) there are at least \( c_0 N^2 (\log N)^{-6} \) triplets of primes \( p_1, p_2, p_3 \), satisfying (1) and such that for any prime factor \( p \) of \( p_1 + 2 \) or \( p_2 + 2 \) we have \( p \geq N^{0.167} \) and for any prime factor \( p \) of \( p_3 + 2 \) we have \( p \geq N^{0.116} \). Notice that if \( N \) is a sufficiently large odd integer, not satisfying the hypothesis of the Theorem, then for any solution of (1) we have \( 3 \mid p_1 p_2 p_3 (p_1 + 2) (p_2 + 2) (p_3 + 2) \). Therefore, by modifying slightly the given proof, we may obtain that for such \( N \) the equation (1) has infinitely many solutions in primes \( p_1, p_2, p_3 \), such that \( p_1 + 2 = P_6, p_2 + 2 = P_6', p_3 + 2 = P_8 \). Here the extra prime factor in \( P_r \) is 3.

Recently H. Mikawa (unpublished result) used the theory of "well-factorable" functions and showed that the power of \( N \) in the quantity \( D_3 \) (for the definition see formulas (2)) can be taken to be equal to \( 4/9 \) instead of \( 1/3 \). This enables us to prove the Theorem with \( p_3 + 2 = P_6 \).

We should also mention that by applying the method of [9], D. I. Tolev [10] proved that if \( N \) is a sufficiently large integer satisfying the congruent condition \( N \equiv 5 \pmod{24} \) then the equation

\[
p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N
\]

has infinitely many solutions in prime numbers \( p_1, p_2, p_3, p_4, p_5 \) such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) and \( p_4 + 2 \) has at most 6 prime factors and \( p_5 + 2 \) has at most 7 prime factors.

§2. Notations.

Let \( N \) be a sufficiently large integer, such that \( N \equiv 3 \pmod{6} \) and \( \alpha_1, \alpha_2, \alpha_3 \) — constants satisfying \( 0 < \alpha_1, \alpha_2 < 1/4, 0 < \alpha_3 < 1/6 \), which we shall specify later.

We put

\[
z_i = N^{\alpha_i}, \quad i = 1, 2, 3, \quad z_0 = (\log N)^{1000}, \quad D_0 = \exp ((\log N)^{0.6}),
\]

\[
D_1 = D_2 = N^{1/2} \exp (-2(\log N)^{0.6}), \quad D_3 = N^{1/3} \exp (-2(\log N)^{0.6}),
\]

\[
P(z_0) = \prod_{2 < p < z_0} p, \quad P(z_0, z_i) = \prod_{z_0 < p < z_i} p, \quad i = 1, 2, 3.
\]

Letters \( m, n, d, l, k, h, \delta, \nu, t, \rho \) denote integers; \( p, p_1, p_2, \ldots \) — prime numbers. As usual \( \mu(n) \), \( \varphi(n) \) and \( \tau(n) \) denote Möbius' function, Euler's function and the number of positive divisors of \( n \), respectively; \( (m_1, \ldots, m_k) \) and \( [m_1, \ldots, m_k] \) denote the greatest common divisor and the least common multiple of \( m_1, \ldots, m_k \). Instead of \( m \equiv n \pmod{m} \) we write for simplicity \( m \equiv n(k) \). The notation \( p^r \mid n \) means that \( p^r \mid n \) and \( p^{r+1} \mid n \). For positive \( A \) and \( B \) we write \( A \approx B \) instead of \( A \ll B \ll A \).
For squarefree odd integers \( k_1, k_2, k_3 \) and prime \( p \) we denote
\[
I_{k_1,k_2,k_3}(N) = \sum_{p_1+p_2+p_3=N \atop p_1+2 \equiv 0(k_i), \ i=1,2,3} \log p_1 \log p_2 \log p_3 ,
\]
(3)
\[
h_{k_1,k_2,k_3}(p) = \begin{cases}
1/(p-1)^3 & \text{if } p \nmid k_1k_2k_3, \ p \nmid N ; \\
-1/(p-1)^2 & \text{if } p \nmid k_1k_2k_3, \ p \mid N ; \\
-1/(p-1)^2 & \text{if } p \mid k_1k_2k_3, \ p \mid N + 2 ; \\
1/(p-1) & \text{if } p \mid k_1k_2k_3, \ p \mid N + 2 ; \\
1/(p-1) & \text{if } p^2 \mid k_1k_2k_3, \ p \mid N + 4 ; \\
-1 & \text{if } p^2 \mid k_1k_2k_3, \ p \mid N + 4 ; \\
-1 & \text{if } p^3 \mid k_1k_2k_3, \ p \mid N + 6 ; \\
p-1 & \text{if } p^3 \mid k_1k_2k_3, \ p \mid N + 6 ;
\end{cases}
\]
(4)
\[
\omega(k_1, k_2, k_3) = \prod_{p|k_1k_2k_3} \frac{1 + h_{k_1,k_2,k_3}(p)}{1 + h_{1,1,1}(p)} ,
\]
\[
\Omega(k_1, k_2, k_3) = \frac{\omega(k_1, k_2, k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)} ,
\]
(5)
\[
\varepsilon(N) = \prod_{p|N} \left( 1 + \frac{1}{(p-1)^3} \right) \prod_{p|N} \left( 1 - \frac{1}{(p-1)^2} \right) .
\]
(6)

§3. Outline of the proof.

Consider the sum
\[
\Gamma = \sum_{p_1+p_2+p_3=N \atop (p_i+2,F(z_i))=1 , \ i=1,2,3} \log p_1 \log p_2 \log p_3.
\]
Any non-trivial estimate from below of \( \Gamma \) implies the solvability of (1) in primes, such that \( p_i + 2 = P_{h_i} \), \( h_i = [\alpha_i^{-1}] \), \( i = 1, 2, 3 \). We see that
\[
\Gamma = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6 ,
\]
where
\[
\Lambda_i = \left\{ \begin{array}{ll}
\sum_{d|(p_i+2,F(z_i))} \mu(d) & \text{for } i = 1, 2, 3, \\
\sum_{d|(p_i-3+2,F(z_0))} \mu(d) & \text{for } i = 4, 5, 6.
\end{array} \right.
\]
Denote
\[
\Lambda_i^\pm = \left\{ \begin{array}{ll}
\sum_{d|(p_i+2,F(z_0))} \lambda_i^\pm(d) & \text{for } i = 1, 2, 3, \\
\sum_{d|(p_i-3+2,F(z_0))} \lambda_i^\pm(d) & \text{for } i = 4, 5, 6.
\end{array} \right.
\]
(7)
where $\lambda_i^\pm(d)$ are the Rosser's weights of order $D_i$, $0 \leq i \leq 3$ (see Iwaniec [6], [7]). In particular, we have

$$|\lambda_i^\pm(d)| \leq 1, \quad \lambda_i^\pm(d) = 0 \text{ for } d \geq D_i, \quad 0 \leq i \leq 3. \quad (8)$$

We find that

$$\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+ , \quad 1 \leq i \leq 6$$

(see Iwaniec [6], [7]). Consequently we may apply Lemma 3 of [9], which is the analogue of Lemma 13 of [1], and we get

$$\Gamma \geq \Gamma_0 , \quad (9)$$

where

$$\Gamma_0 = \sum_{p_1 + p_2 + p_3 = N} \log p_1 \log p_2 \log p_3 \left( \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_5^+ \Lambda_6^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^+ \Lambda_6^+ + \cdots + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^- - 5 \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+ \right). \quad (10)$$

We use (3), (7) and change the order of summation to obtain

$$\Gamma_0 = \sum_{d_i \mid P(z_0,z_i), \delta_i \mid P(z_0) \mid d | P(z_0), \delta | d}, \quad \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) I_{d_1, \delta_1, d_2, \delta_2, d_3, \delta_3} (N), \quad (10)$$

where

$$\kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) = \lambda_1^-(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \lambda_5^+(\delta_1) \lambda_6^+(\delta_2) \lambda_6^+(\delta_3) + \cdots + \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \lambda_5^+(\delta_1) \lambda_6^+(\delta_2) \lambda_6^+(\delta_3) - 5 \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \lambda_5^+(\delta_1) \lambda_6^+(\delta_2) \lambda_6^+(\delta_3). \quad (11)$$

By applying the Hardy–Littlewood circle method we find an asymptotic formula for the sum $I_{k_1,k_2,k_3}(N)$ which we substitute in (10). Proceeding in the same way as in Lemma 11 and Lemma 13 of [9] we derive

$$\Gamma_0 = \frac{1}{2} N^2 \mathcal{S}(N) W + \mathcal{O} \left( \frac{N^2}{\log^4 N} \right), \quad (12)$$

where $\mathcal{S}(N)$ is defined by (6) and

$$W = \sum_{d_i \mid P(z_0,z_i), \delta_i \mid P(z_0) \mid d | P(z_0), \delta | d}, \quad \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) \Omega(d_1, d_2, d_3) \Omega(\delta_1, d_2, d_3) \Omega(\delta_1, \delta_2, \delta_3). \quad (13)$$

By (4) it is obvious that for squarefree odd integers $k_1, k_2, k_3$ we have

$$h_{k_1,k_2,k_3}(p) = \begin{cases} h_{1,1,1}(p) & \text{if } p \nmid k_1 k_2 k_3; \\ h_{p,1,1}(p) & \text{if } p \nmid k_1 k_2 k_3; \\ h_{p,p,1}(p) & \text{if } p^2 \nmid k_1 k_2 k_3; \\ h_{p,p,p}(p) & \text{if } p^3 \nmid k_1 k_2 k_3. \end{cases} \quad (14)$$
Define
\[ \omega_1(p) = \omega(p, 1, 1), \quad \omega_2(p) = \omega(p, p, 1), \quad \omega_3(p) = \omega(p, p, p). \] (15)

It is clear that if \( p > 2 \) then
\[ \omega_1(p) = \begin{cases} 1 & \text{if } p | N ; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } p \nmid N + 2 ; \\ \frac{(p-1)(p-2)}{p^2-3p+3} & \text{if } p \nmid N \ (N+2) \end{cases} \]
\[ \omega_2(p) = \begin{cases} \frac{p-1}{p-2} & \text{if } p | N ; \\ 0 & \text{if } p | N + 4 ; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } p \nmid N + 4 \end{cases} \]
\[ \omega_3(p) = \begin{cases} 4 & \text{if } p = 3 ; \\ \frac{(p-1)^3}{p^2-3p+3} & \text{if } p | N + 6 , \quad p > 3 ; \\ 0 & \text{if } p \nmid N + 6 \end{cases} \] (16)

By (4), (5), (14), (15) we get
\[ \omega(k_1, k_2, k_3) = \prod_{p^\nu \mid k_1, k_2, k_3} \omega(p). \] (17)

The next statement is the analogue of Lemma 12 of [1]. The Lemma follows easily from (16), (17).

**Lemma 1.** For squarefree odd \( k \), let
\[ \omega^*(k) = \prod_{p \mid k} \omega_1(p). \]

If \( k_1, k_2, k_3 \) is a triplet of integers, we put \( k_{1,2} = (k_1, k_2), \ k_{1,3} = (k_1, k_3), \ k_{2,3} = (k_2, k_3) \). Then
\[ \omega(k_{1,2}, k_{1,3}, k_{2,3}) = \omega^*(k_1) \omega^*(k_2) \omega^*(k_3) g(k_{1,2}, k_{1,3}, k_{2,3}) \]
and
\[ g(k_{1,2}, k_{1,3}, k_{2,3}) \leq 10 (\max k_{i,j})^{10}; \]
\[ (i) \text{ there exists a function } g \text{ of the three variables } k_{i,j}, \text{ such that for any squarefree odd } k_1, k_2, k_3 \text{ we have} \]
\[ \omega(k_1, k_2, k_3) = \omega^*(k_1) \omega^*(k_2) \omega^*(k_3) g(k_{1,2}, k_{1,3}, k_{2,3}) \]
and
\[ g(k_{1,2}, k_{1,3}, k_{2,3}) \leq 10 (\max k_{i,j})^{10}; \]
\[ (ii) \text{ for any squarefree odd } k_1, k_2, k_3 \text{ we have the inequality} \]
\[ \omega(k_1, k_2, k_3) \leq 10 \tilde{\omega}(k_1) \bar{\omega}(k_2) \bar{\omega}(k_3), \]
where \( \tilde{\omega}(m) \) is the multiplicative function defined on squarefree odd \( m \) by

\[
\tilde{\omega}(p) = \begin{cases} 
2 & \text{if } p | N + 6; \\
2p^{1/3} & \text{if } p | N + 6.
\end{cases}
\]

Suppose that the integers \( d_1, d_2, d_3, \delta_1, \delta_2, \delta_3 \) satisfy the conditions imposed in (13). Using (5) and (17) we easily get

\[
\Omega(d_1, d_2, d_3) = \Omega(\delta_1, \delta_2, \delta_3).
\]

Note that \( \Omega(\delta_1, \delta_2, \delta_3) \) is a symmetrical function with respect to \( \delta_1, \delta_2, \delta_3 \). Hence, we obtain by (11), (13)

\[
W = \sum_{i=1}^{6} L_i H_i - 5L_7 H_7,
\]

where

\[
L_1 = \sum_{\delta_i \mid P(z_0, z_i), i=1,2,3} \lambda_1^-(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3),
\]

\[
L_2 = \sum_{\delta_i \mid P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3),
\]

\[
L_3 = \sum_{\delta_i \mid P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^-(d_3) \Omega(d_1, d_2, d_3),
\]

\[
L_4 = L_5 = L_6 = L_7 = \sum_{\delta_i \mid P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^-(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3),
\]

\[
H_1 = H_2 = H_3 = H_7 = \sum_{\delta_i \mid P(z_0)} \lambda_1^+(\delta_1) \lambda_2^+(\delta_2) \lambda_3^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3),
\]

\[
H_4 = H_5 = H_6 = \sum_{\delta_i \mid P(z_0)} \lambda_1^-(\delta_1) \lambda_2^+(\delta_2) \lambda_3^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3).
\]

It is easy to prove the following

**Lemma 2.** Suppose that \( \phi(n_1, n_2, n_3) \) is a function defined on the set of integers and such that for any two triplets \( n_1, n_2, n_3 \) and \( l_1, l_2, l_3 \), satisfying \( (n_1 n_2 n_3, l_1 l_2 l_3) = 1 \), we have \( \phi(n_1 l_1, n_2 l_2, n_3 l_3) = \phi(n_1, n_2, n_3) \phi(l_1, l_2, l_3) \). Then the function

\[
\Phi(n) = \sum_{d_1, d_2, d_3 \mid n} \phi(d_1, d_2, d_3)
\]

is multiplicative.

Applying Lemma 1 and Lemma 2 we find asymptotic formulas for the sums \( H_i \). Define

\[
H^{(\mu)} = \sum_{\delta_i \mid P(z_0), i=1,2,3} \mu(\delta_1) \mu(\delta_2) \mu(\delta_3) \Omega(\delta_1, \delta_2, \delta_3).
\]
Lemma 3. We have

\[ H_i = H^{(\mu)} + O((\log N)^{-10}) \], \quad 1 \leq i \leq 7, \]

and

\[ H^{(\mu)} \asymp (\log z_0)^{-3}. \] (20)

Now we are able to estimate from below the quantity \( W \), defined by (18). We put

\[ F(z_0, z_i) = \prod_{z_0 \leq p < z_i} \left(1 - \frac{\omega(p)}{p-1}\right), \quad s_i = \frac{\log D_i}{\log z_i}, \quad i = 1, 2, 3, \] (21)

where \( \omega(p) \) is defined by (16). Suppose that \( c^* > 0 \) is an absolute constant and let \( \theta_i, s_i, i = 1, 2, 3 \) satisfy

\[ \theta_1 + \theta_2 + \theta_3 = 1, \quad \theta_i > 0, \quad f(s_i) - 2\theta_i F(s_i) > c^*, \quad i = 1, 2, 3, \]

where \( f \) and \( F \) are the functions of the linear sieve. Following the arguments in the proof of Lemma 15 of [9] it is easy to establish that

\[ W \geq H^{(\mu)} \prod_{j=1}^{3} F(z_0, z_j) \left( \sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + O \left( (\log N)^{-1/3} \right) \right). \] (22)

Finally, we choose

\[ \alpha_1 = \alpha_2 = 0.167, \quad \alpha_3 = 0.116, \quad \theta_1 = \theta_2 = 0.345, \quad \theta_3 = 0.31 \]

and compute that for sufficiently large \( N \) we have

\[ f(s_i) - 2\theta_i F(s_i) > 10^{-5}, \quad i = 1, 2, 3. \] (23)

Therefore, using (2), (20)–(23) we get

\[ W \gg \frac{1}{\log^3 N}. \]

The last estimate and (9), (12) imply

\[ \Gamma \gg \frac{N^2}{\log^3 N}, \]

which suffices to complete the proof of the Theorem.
References


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