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Kyoto University
Generation of $k$-permutations in $O(1)$ time per permutation by reversing sublists

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Abstract

We discuss the problem of generating all $k$-permutations of $n$ objects. Several papers have introduced a technique to alternatingly reverse sublists of a listing for some combinatorial Gray codes in an efficient manner. Our approach is to apply the technique to a listing of all $k$-permutations of $n$ objects constructed recursively by reversing sublists. We show that the list contains $n!/(n-k)!$ permutations so that each string differs from its predecessor by the transposition of two elements. It is easy to convert the construction to a recursive algorithm and then we develop the algorithm that produces successive permutations in a constant amortized time per permutation.

1 Introduction

Many algorithms have been published for generating all permutations of $n$ objects and then there is a number of listings of successive permutations. One of the listings is the transposition order that is introduced independently by Johnson [3] and Trotter [11]. It is well-known that each permutation differs by the transposition of adjacent elements.

Recently several interesting papers have been achieved for generating some combinatorial Gray codes in a constant or constant amortized time per object [1, 9, 5, 6, 8, 10, 2]. However, it is not trivial to generate a listing of combinatorial Gray codes in a unique manner. Most of those papers managed to give a simple recurrence relation for combinatorial Gray codes. Ruskey generalized a close relationship between some combinatorial Gray codes constructed recursively by reversing sublists [9]. To reverse certain sublists seems to contribute a reduction of differences between successive objects.
A $k$-permutation of $n$ objects is an arrangement of the first $k$ objects out of $n$ objects. First, we give a few modified definition for $k$-permutations which are extended into $n$ length strings such that the set of all permutations of $n$ objects contains the smaller set of the extended $k$-permutations. Our approach is to apply the reversing technique to such $k$-permutations. Then a listing of all $k$-permutations is obtained such that successive strings differ by the transposition of two elements. This paper presents a recursive algorithm for generating them in a constant amortized time per string. It is obtained directly from the recursively defined construction for $k$-permutations. Note that we do not count the time for printing permutations.

2 Definitions and properties

To begin with, we extend $k$-permutations of $n$ distinct objects to strings of length $n$: a $k$-permutation of length $n$ consists of $n$ elements which the first $k$ elements are arranged in its original order and the rests are arranged in a lexical order. For example, if a string 52 is a 2-permutation of the set $\{1, 2, 3, 4, 5\}$, then its extension is 52134. When it will not lead to confusion, we call them simply $k$-permutations.

The following useful notations are defined in [9]. If $L$ is a list of strings and $x$ is a symbol, then $x \cdot L$ denotes the list of strings obtained by appending an $x$ to each string of $L$. For example, if $L = 12, 21$, then $3 \cdot L = 312, 321$. If $L$ and $L'$ are lists then $L \circ L'$ denotes the concatenation of the two lists. For example, if $L = 12, 21$ and $L' = 34, 43$, then $L \circ L' = 12, 21, 34, 43$.

For a list $L$, let $\text{first}(L)$ denote the first element on the list and let $\text{last}(L)$ denote the last element on the list. If $L$ is a list $l_1, l_2, \ldots, l_n$, then $\overline{L}$ denotes the list obtained by listing the elements of $L$ in reverse order; i.e., $\overline{L} = l_n, l_{n-1}, l_2, l_1$. Note the obvious equations $\text{first}(\overline{L}) = \text{last}(L)$ and $\text{last}(\overline{L}) = \text{first}(L)$.

Let $T_k(n)$ be a listing of all $k$-permutations of the set $\{p_1, p_2, \ldots, p_n\}$. The construction for the lists consists of two parts, one of which generates $k$ length permutations in their original order and the other of which generates $n - k$ length permutations in a lexical order. The following construction is the case for the original part. The list involves $n$ recursively defined sublists which are alternatingly reversed.

$$T_k(n) = \begin{cases} 
\pi_1 \cdot T_{k-1}(n-1) \circ \pi_2 \cdot T_{k-1}(n-1) \circ \cdots \\
\pi_1 \cdot \pi_n \cdot T_{k-1}(n-1) \circ \pi_2 \cdot T_{k-1}(n-1) \circ \cdots \\
\pi_1 \cdot T_{k-1}(n-1) \circ \pi_2 \cdot T_{k-1}(n-1) \circ \cdots \\
\pi_1 \cdot \pi_n \cdot T_{k-1}(n-1) \circ \pi_2 \cdot T_{k-1}(n-1) \circ \cdots 
\end{cases}$$

if odd $n$, if even $n$,
and the case for the lexical part,

\[ T_k(n) = \pi_1 \cdot T_{k-1}(n-1). \]

These are subject to the terminal condition that \( T_k(0) = \emptyset \). The construction appends \( \pi_i \)'s \( \in \{p_1, p_2, \ldots, p_n\} \) to sublists in a lexical order from left to right and each sublist is reconstructed with the set obtained by deleting a given element and renumbering the rests from \( \pi_1 \) to \( \pi_{n-1} \). This constraint requires that permutations contain all distinct elements.

**Lemma 2.1** The list \( T_k(n) \) satisfies the following properties:

1. Successive \( k \)-permutations in \( T_k(n) \) differ in exactly two elements.
2. \( \text{first}(T_k(n)) = p_1 p_2 \cdots p_n \).
3. \( \text{last}(T_k(n)) = \begin{cases} p_n p_{n-1} p_1 p_2 \cdots p_{n-2} & \text{if odd } n \text{ and } k \geq 2, \\ p_n p_1 p_2 \cdots p_{n-1} & \text{otherwise}. \end{cases} \)

**Proof.** The proof is by induction on \( n \). The list obviously has the stated properties for \( 1 \leq k \leq n \leq 2 \). Suppose that the lemma is true for \( n \geq 3 \). We must show it to be correct for \( n+1 \). For convenience, we assume the \( i \)th element in a permutation to be placed in the position \( n-i \), that is, the last element is placed in the position 0.

Obviously the list \( T_1(n+1) \) contains \( n+1 \) permutations in which the \( i \)th permutation is \( p_i p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n+1} \) and the permutation differs from its predecessor by two elements in positions \( n \) and \( n-i \). Otherwise, for \( k \geq 2 \), the list contains \( n+1 \) sublists and we need to inspect the transposition of successive permutations at the interface between the \( i \)th sublist and the \( (i+1) \)st sublist. The transposition behaves in different ways according to the parities \( n \) and \( i \).

The first case is for even \( i \). The \( i \)th sublist is reverse and the \( (i+1) \)st sublist is natural. The contiguous permutations between the \( i \)th sublist and the \( (i+1) \)st sublist differ by two elements, since the last permutation of the \( i \)th sublist is the lexically first one, as shown below.

\[
\begin{align*}
p_i \cdot T_{k-1}(n) & \begin{cases} \vdots \\
p_1 \ p_1 \cdots p_{i-1} \ p_{i+1} \ p_{i+2} \cdots p_{n+1} \\
p_{i+1} \cdot T_{k-1}(n) & \begin{cases} \vdots \\
p_{i+1} \ p_1 \cdots p_{i-1} \ p_i \ p_{i+2} \cdots p_{n+1} \end{cases} \\
\end{cases}
\end{align*}
\]

The underlined elements that are swapped appear in positions \( n \) and \( n-i \). When odd \( n+1 \), this case occurs on the last interface. The third property
holds, since the last permutation of $T_k(n+1)$ is $p_{n+1} \cdot \text{last}(T_{k-1}(n))$, that is, $p_{n+1}p_n p_1 \cdots p_{n-1}$.

The second case is for odd $i$. The $i$th sublist is natural and the $(i+1)$st sublist is reverse. (1) When odd $n+1$, the contiguous permutations between the $i$th sublist and the $(i+1)$st sublist are shown below.

$$p_i \cdot T_{k-1}(n) = \begin{cases} p_i & p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n+1} \\ \\ \vdots \\ p_i & p_{n+1} p_1 \cdots p_{i-1} p_{i+1} \cdots p_n \\ \vdots \\ p_{i+1} & p_{n+1} p_1 \cdots p_{i-1} p_i p_{i+2} \cdots p_n \\ \vdots \\ p_{i+1} & p_1 \cdots p_{i} p_{i+2} \cdots p_{n+1} \end{cases}$$

$$p_{i+1} \cdot \overline{T_{k-1}(n)} = \begin{cases} p_{i+1} & p_{n+1} p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n-1} \\ p_{i+1} & p_{n+1} p_1 \cdots p_{i-1} p_i p_{i+2} \cdots p_{n-1} \\ \vdots \\ p_{i+1} & p_1 \cdots p_{i} p_{i+2} \cdots p_{n+1} \end{cases}$$

The underlined elements that are swapped appear in positions $n$ and $n - i - 1$. (2) When even $n+1$, we can give some formulae for detecting the transposition of successive permutations at each interface. The successive permutations at the last interface differ by the elements in positions $n$ and $n - 1$, as shown below.

$$p_n \cdot T_{k-1}(n) = \begin{cases} p_n & p_1 \cdots p_{n-1} p_{n+1} \\ \vdots \\ p_n & p_{n+1} p_1 \cdots p_{n-1} \\ p_{n+1} & p_n p_1 \cdots p_{n-1} \\ \vdots \\ p_{n+1} & p_1 \cdots p_{n-1} p_n \end{cases}$$

$$p_{n+1} \cdot \overline{T_{k-1}(n)} = \begin{cases} p_{n+1} & p_{n+1} p_n p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n-1} \\ p_{n+1} & p_{n+1} p_n p_1 \cdots p_{i-1} p_i p_{i+2} \cdots p_{n-1} \\ \vdots \\ p_{n+1} & p_1 \cdots p_{i} p_{i+2} \cdots p_{n+1} \end{cases}$$

The last property holds in this case. Otherwise, for $i < n$, the transposition behaves in two different ways depending upon the value of $k$. When $k = 2$, the elements of permutations that appear in positions greater than 2 are arranged in a lexical order. The contiguous permutations between the $i$th sublist and the $(i+1)$st sublist are identical in arrangements as the ones for the case (1), that is, the elements that are swapped appear in positions $n$ and $n - i - 1$. When $k > 2$, they are shown below.

$$p_i \cdot T_{k-1}(n) = \begin{cases} p_i & p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n+1} \\ \vdots \\ p_i & p_{n+1} p_n p_1 \cdots p_{i-1} p_{i+1} \cdots p_{n-1} \\ p_{i+1} & p_{n+1} p_n p_1 \cdots p_{i-1} p_i p_{i+2} \cdots p_{n-1} \\ \vdots \\ p_{i+1} & p_1 \cdots p_{i} p_{i+2} \cdots p_{n+1} \end{cases}$$

The elements that are swapped appear in positions $n$ and $n - i - 2$. The list $T_k(n+1)$ has the stated properties. The proof is complete.
procedure interchange(n,k,i:integer);
begin
  if (k=1) or not(odd(i)) then swap(n-1,n-i-1)
  else if odd(n) then swap(n-1,n-i-2)
  else if i=n-1 then swap(n-1,n-2)
  else if k=2 then swap(n-1,n-i-2) else swap(n-1,n-i-3);
end {of procedure};

Figure 1: The procedure interchange(n,k,i).

procedure gen(n,k:integer);
var i:integer;
begin
  if k>0 then
    for i:=1 to n do begin
      if odd(i) then gen(n-1,k-1) else neg(n-1,k-1);
      if i<n then interchange(n,k,i);
    end
  end {of procedure};

Figure 2: The recursive procedure gen(n,k).

3 Implementation and analysis

To begin with, we summarize the transposition of successive permutations between the $i$th sublist and the $(i+1)$st sublist and show it in a Pascal procedure, in Figure 1. The procedure swap(i,j) swaps the elements in positions $i$ and $j$. The definition of the list $T_k(n)$ leads directly to a recursive algorithm for generating all $k$-permutations of $n$ objects. The Pascal procedure gen(n,k) generates the list $T_k(n)$, shown in Figure 2 and the procedure neg(n,k) is a symmetric procedure of gen(n,k) which generates the reversed list $T_k(n)$.

Let us analyze the running time of gen(n,k). The procedure gen does $n$ recursive calls to either gen or neg in the while statement. We also know that it calls interchange once per loop and the interchange operation takes a constant time to find the two elements that are swapped. Thus the total amount of computations is proportional to the number of recursive calls, which is $O(n!/(n-k)!))$. To summarize above, the procedure gen generates all $k$-permutations in an amortized constant time to go from one string to the next.
References


