<table>
<thead>
<tr>
<th>Title</th>
<th>The generalized Whittaker functions for the discrete series representations of $SU(3,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ishikawa, Yoshihiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1094: 97-109</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62985">http://hdl.handle.net/2433/62985</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
The generalized Whittaker functions
for the discrete series representations of $SU(3,1)$

Yoshi-hiro Ishikawa

**Problem.** Let $G$ be a semi-simple Lie group and $\pi_\Lambda$ its discrete series representation. What kind of models does $\pi_\Lambda$ have? Exactly when the models exist, with how many multiplicity? What explicit form do functions corresponding to the model have?

More precisely, let $R$ be a closed subgroup of $G$. For $\pi_\Lambda \in \hat{G}_d$ and a representation $\eta$ of $R$, evaluate the upper bound of

$$\dim_c \text{Hom}_{(sc,K)}(\pi_\Lambda^*, \text{Ind}_R^G \eta),$$

where $\pi_\Lambda^*$ is a contragredient of $\pi_\Lambda$. When the dimension does not equal to zero, write down explicitly the functions describing the intertwiners.

Let $G = NAK$ be the Iwasawa decomposition of $G$. When $R$ is the maximal unipotent subgroup $N$ of $G$ and $\eta$ a non-degenerate character of $N$, $\text{Ind}_R^G \eta$ is the Gelfand-Graev representation, hence the problem above is a traditional problem on Whittaker model, considered from various point of view. Here we treat the special unitary group of isometry for the Hermitian form of signature $(3+,1-)$ realized by

$$G = SU(3,1) := \{ g \in SL(4,\mathbb{C}) |^{t} \overline{g} I_{3,1} = I_{3,1} \}.$$ 

Whittaker model for discrete series representations of this group $G$ was investigated by Taniguchi [Ta], where he obtained a formula for dimension of the space of the intertwiners and an explicit form of corresponding functions (Whittaker function for $\pi_\Lambda$). His formula tells that the dimension of the Whittaker model not necessarily smaller than one even if the growth condition on the corresponding functions is imposed: The multiplicity one property is not valid for this model. We replace the Gelfand-Graev representation for the reduced generalized Gelfand-Graev representation and consider the generalized Whittaker model. That is, we take an irreducible infinite dimensional unitary representation of $N$, note $N$ is a Heisenberg group, as $\eta$ and a bigger group containing $N$ as $R$. We investigate this model and give an explicit form of generalized Whittaker functions. By fixing coordinate on $G$ and explicit realization of representations, we reduce the problem to solving a certain system of difference-differential equations for the coefficient functions of generalized Whittaker functions.

We put some remarks. In the case of the group $SU(2,1)$, we obtained an explicit form and the multiplicity one result for generalized Whittaker functions for the standard representations previously [I] from a motivation of automorphic forms. And this is just an "étude" for the work on generalized Whittaker functions on $SU(n,1)$, which will come soon. Main difference from the case of $SU(2,1)$ is in troublesome combinatorial calculation of $K$-types $S$-types.
<Groups and algebras>

We fix a coordinate on subgroups of $G$ as follows,

$$K = \{ \begin{pmatrix} k & \det k^{-1} \end{pmatrix} \mid k \in U(3) \}, \quad A = \{ a_r := \begin{pmatrix} 1_2 & c & s \\ c & s & c \end{pmatrix} \mid s = (r - r^{-1})/2, \quad r \in \mathbb{R}_{>0} \},$$

$$N = \exp n = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & -\bar{z}_1 \\ \bar{z}_2 & -\bar{z}_2 & \alpha \\ -z_1 & -z_2 & \alpha \beta \end{pmatrix} \mid \alpha = 1 - \frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + it, \quad \beta = 1 + \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2} - it, \quad z_1, z_2 \in \mathbb{C}, t \in \mathbb{R} \}. $$

Here the Lie algebra $n$ of $N$ is given by

$$n := \bigoplus_{p=1}^{2} (\mathbb{R}X_p + \mathbb{R}Y_p) \oplus \mathbb{R}W,$$

$$X_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} -i & i \\ -i & -i \end{pmatrix},$$

$$X_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -i & i \\ -i & -i \end{pmatrix}, \quad W = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix},$$

where $i$ denotes the complex unity $\sqrt{-1}$. By natural isomorphisms we identify these groups as

$$K \cong U(3), \quad N \cong H(\mathbb{C}^2).$$

Here $H(\mathbb{C}^2)$ denotes the real Heisenberg group of dimension 5. The center $Z(N)$ of $N$ is of the form

$$Z(N) = \{ z_t := \begin{pmatrix} 1_2 & 1 + it & -it \\ 1 + it & -it & 1 - it \end{pmatrix} \mid t \in \mathbb{R} \}.$$ 

The Cartan decomposition of $\mathfrak{g} = \text{Lie} G$ is given by $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, with $\mathfrak{k} = \text{Lie} K$,

$$\mathfrak{p} = \{ \begin{pmatrix} O_2 & X \\ i\bar{X} & 0 \end{pmatrix} \mid X \in \mathbb{C}^3 \}.$$ 

The action of the Levi subgroup $L$ of $P := \exp \mathfrak{p}$ on $N$ is naturally extended to that on $\overline{N}$. By Ston-von Neumann theorem, the unitary dual $\overline{N}$ of $N$ is exhausted by unitary characters and infinite dimensional irreducible unitary representations. And the infinite dimensional ones $\rho$ are determined by their central characters $\psi$. Hence the stabilizer $S$ of $\rho$ in $L$ is the centralizer of $Z(N)$ and of the following form

$$S = \{ \text{diag}(m, d, d) \in G \mid m \in U(2), \quad d = (\det m)^{-1/2} \},$$

which coincides with the Levi part of $P$. Using this $S$, we define the group $R$ as

$$R := S \ltimes N \cong U(2) \ltimes H(\mathbb{C}^2).$$
Let \( t := \{ \text{diag}(ih_1, ih_2, ih_3, ih_4) | h_j \in \mathbb{R}, h_1 + \cdots + h_4 = 0 \} \) be a Cartan subalgebra of \( \mathfrak{t} \) and define roots \( \beta_{ij} : t_c \ni \text{diag}(ih_1, ih_2, ih_3, ih_4) \mapsto t_i - t_j \in \mathbb{C} \). We denote \( \Sigma_c \) and \( \Sigma_n \) the sets of compact and noncompact roots, respectively. In our choice of coordinate,

\[
\Sigma_c = \{ \beta_{12}, \beta_{13}, \beta_{23}, \beta_{21}, \beta_{31}, \beta_{32} \}, \quad \Sigma_n = \{ \beta_{14}, \beta_{24}, \beta_{34}, \beta_{41}, \beta_{42}, \beta_{43} \},
\]

and matrix element \( E_{ij} (1 \leq i, j \leq 3) \) generates the root space \( g_{\beta_{ij}} \). We put

\[
X_{\beta_{ij}} = \begin{cases} -E_{ij} & \text{when } (i, j) = (2, 1), (3, 1), (3, 2) \\ E_{ij} & \text{otherwise}, \end{cases}
\]

and take it as a root vector in \( g_{\beta_{ij}} \). These root vectors decompose with respect to the Iwasawa decomposition as

\[
X_{\beta_{34}} = \frac{1}{2} H_{34}' + \frac{1}{2} H + \frac{i}{2} W; \quad X_{\beta_{34}} = -\frac{1}{2} H_{34}' + \frac{1}{2} H - \frac{i}{2} W;
\]

\[
X_{\beta_{14}} = X_{\beta_{13}} - \frac{1}{2} X_1 - \frac{i}{2} Y_1; \quad X_{\beta_{14}} = X_{\beta_{31}} - \frac{1}{2} X_1 + \frac{i}{2} Y_1;
\]

\[
X_{\beta_{24}} = X_{\beta_{23}} - \frac{1}{2} X_2 - \frac{i}{2} Y_2; \quad X_{\beta_{24}} = X_{\beta_{32}} - \frac{1}{2} X_2 + \frac{i}{2} Y_2,
\]

where \( H_{34}' \) is a generator \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) of \( \alpha := \log \Lambda \). This is used to calculate the action of Schmid operators.

<Representations>
We fix realization of representations of groups.

Parameterization of irreducible \( K \)-modules
In this subsection, we recall the Gel'fand-Zetlin basis, which gives a nice realization of irreducible representation of \( K \). The set \( L^+_T \) of \( \Sigma_c^+ \)-dominant \( T \)-integral weights is given by \( L^+_T = \{(l, m, n) \in \mathbb{Z}^3 | l \geq m \geq n \} \). For a given \( \Sigma_c^+ \)-dominant \( T \)-integral weight \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \in L^+_T \), let \( V_{\lambda} \) be a complex vector space spanned by \( v(Q) \)'s.

\[
V_{\lambda} := \bigoplus_{Q \in GZ(\lambda)} \mathbb{C} v(Q).
\]

Here the index set \( GZ(\lambda) \) is the set of the Gel'fand-Zetlin schemes with top raw \( \lambda \):

\[
GZ(\lambda) := \left\{ Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \lambda_3 \\ k & \mu_2 & \lambda_3 \end{pmatrix} \middle| \begin{array}{c} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3, \\
\mu_1 \geq k \geq \mu_2, \lambda_i, \mu_j, k \in \mathbb{Z} \end{array} \right\}.
\]

The \( \mathfrak{t}_c \)-module structure defined by

\[
\tau_{\lambda}(H_{14})v(Q) = kv(Q), \quad \tau_{\lambda}(H_{24}')v(Q) = (|\mu| - k)v(Q), \quad \tau_{\lambda}(H_{34}')v(Q) = (|\lambda| - |\mu|)v(Q),
\]

\[
\tau_{\lambda}(X_{\beta_{34}})v(Q) = a_{1}^{+}(Q)v(Q^{+\epsilon_1}) + a_{2}^{+}(Q)v(Q^{+\epsilon_2}), \quad \tau_{\lambda}(X_{\beta_{34}})v(Q) = a_{1}^{-}(Q)v(Q_{-1}) + a_{2}^{-}(Q)v(Q_{-2}),
\]

\[
\tau_{\lambda}(X_{\beta_{23}})v(Q) = b_{1}^{+}(Q)v(Q^{+\epsilon_1}) + b_{2}^{+}(Q)v(Q^{+\epsilon_2}), \quad \tau_{\lambda}(X_{\beta_{23}})v(Q) = b_{1}^{-}(Q)v(Q_{-1}) + b_{2}^{-}(Q)v(Q_{-2}),
\]

\[
\tau_{\lambda}(X_{\beta_{21}})v(Q) = b_{1}^{+}(Q)v(Q^{+\epsilon_1}) + b_{2}^{+}(Q)v(Q^{+\epsilon_2}), \quad \tau_{\lambda}(X_{\beta_{21}})v(Q) = b_{1}^{-}(Q)v(Q_{-1}) + b_{2}^{-}(Q)v(Q_{-2}).
\]
gives an irreducible $K$-module $(\tau_\lambda, V_\lambda)$ via the highest weight theory. The coefficients appearing above are given as follows.

\[
\begin{align*}
\tau_2^1(Q) &= \frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - 1)(\lambda_3 - \mu_1 - 2)}{(d'_\mu + 1)(d'_\mu + 2)} \sqrt{\mu_1 + 1 - k}, \\
\tau_2^2(Q) &= \frac{-(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)(\lambda_3 - \mu_2 - 1)}{d'_\mu(d'_\mu + 1)} \sqrt{k - \mu_2}, \\
\tau_2^3(Q) &= \frac{a^{-1}(Q) - (\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)(\lambda_3 - \mu_2)}{(d'_\mu + 1)(d'_\mu + 2)} \sqrt{k + 1 - \mu_2}, \\
\tau_2^{-1}(Q) &= -\sqrt{(\mu_1 + 1 - k)(k - \mu_2)}.
\end{align*}
\]

And the indices $Q^{\pm_1}, Q^{\pm_2}, Q_{\pm 1}$ mean

\[
Q^{\pm_1} = \left( \begin{array}{ccc} \lambda_1 & \mu_1 \pm 1 & \lambda_2 \\ \mu_2 & \lambda_3 \end{array} \right), \quad Q^{\pm_2} = \left( \begin{array}{ccc} \lambda_1 & \mu_1 \pm 1 & \lambda_2 \\ \mu_2 \pm 1 & \lambda_3 \end{array} \right),
\]

\[
Q_{\pm 1} = \left( \begin{array}{ccc} \lambda_1 & \mu_1 & \lambda_2 \\ \mu_2 \pm 1 & \lambda_3 \\ k \pm 1 \end{array} \right),
\]

respectively. The basis $\{v(Q) \mid Q \in GZ(\lambda)\}$ prescribed above is called the Gef'and-Zetlin basis of $(\tau_\lambda, V_\lambda)$.

**Tensor products with $\mathfrak{p}_C$**

We regard the 6-dimensional vector space $\mathfrak{p}_C$ as a $\mathfrak{t}_C$-module via the adjoint representation. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are invariant subspaces, and

\[
\mathfrak{p}_+ := \mathbb{C}X_{\beta_4} \oplus \mathbb{C}X_{\beta_2} \oplus \mathbb{C}X_{\beta_3} \cong V_{\beta_4}, \quad \mathfrak{p}_- := \mathbb{C}X_{\beta_4} \oplus \mathbb{C}X_{\beta_2} \oplus \mathbb{C}X_{\beta_3} \cong V_{\beta_4}.
\]

Given an irreducible $K$-module $V_\alpha$, Clebsch-Gordan's theorem tells us the following decomposition of $V_\alpha \otimes \mathfrak{p}_\pm$:

\[
V_\alpha \otimes \mathfrak{p}_+ \cong V_{\lambda + \beta_4} \oplus V_{\lambda + \beta_2} \oplus V_{\lambda + \beta_3}, \quad V_\alpha \otimes \mathfrak{p}_- \cong V_{\lambda + \beta_4} \oplus V_{\lambda + \beta_2} \oplus V_{\lambda + \beta_3}.
\]

The decompositions of $V_\alpha \otimes \mathfrak{p}_\pm$ induce the following projectors:

\[
\begin{align*}
p^{+\beta_4} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda + \beta_4}, \\
p^{-\beta_4} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda - \beta_4}, \\
p^{+\beta_2} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda + \beta_2}, \\
p^{-\beta_2} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda - \beta_2}, \\
p^{+\beta_3} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda + \beta_3}, \\
p^{-\beta_3} : V_\alpha \otimes \mathfrak{p}_C &\rightarrow V_{\lambda - \beta_3}.
\end{align*}
\]

In terms of $\{v(Q)\}$, they are expressed as follows:
Proposition 1 The projectors are described as follows.

(1) \[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{41}}) = A_{1}^{-} \sqrt{k - \mu_{2}} v(Q) v(Q_{-}^{-}e_{1}) - B_{1}^{-} \sqrt{\mu_{1} + 1 - k} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{42}}) = A_{1}^{-} \sqrt{\mu_{1} - k} v(Q) v(Q_{-}^{-}e_{1}) + B_{1}^{-} \sqrt{k + 1 - \mu_{2}} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{\mu_{1} + 1 - \mu_{2}}(\lambda_{2} - \mu_{1}) v(Q)
\]
with coefficients
\[
A_{1}^{-} = \left| \frac{(\lambda_{2} - \mu_{1})(\lambda_{2} - \mu_{1} - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_{2} + 1 - \lambda_{3}}, \quad B_{1}^{-} = \left| \frac{(\lambda_{2} + 1 - \mu_{2})(\lambda_{2} - \mu_{1})}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_{1} - \mu_{1}}.
\]

(2) \[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{41}}) = -A_{2}^{-} \sqrt{k - \mu_{2}} v(Q) v(Q_{-}^{-}e_{1}) - B_{2}^{-} \sqrt{\mu_{1} + 1 - k} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{42}}) = -A_{2}^{-} \sqrt{\mu_{1} - k} v(Q) v(Q_{-}^{-}e_{1}) + B_{2}^{-} \sqrt{k + 1 - \mu_{2}} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{\mu_{2} + 1 - \lambda_{3}}(\mu_{2} + 1 - \lambda_{3}) v(Q)
\]
with coefficients
\[
A_{2}^{-} = -\left| \frac{(\lambda_{2} - \mu_{1})(\lambda_{2} - \mu_{1} - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_{2} - \mu_{2}}, \quad B_{2}^{-} = -\left| \frac{(\lambda_{2} + 1 - \mu_{2})(\lambda_{2} - \mu_{1})}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_{1} + 1 - \lambda_{2}}.
\]

(3) \[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{41}}) = -A_{3}^{-} \sqrt{k - \mu_{2}} v(Q) v(Q_{-}^{-}e_{1}) - B_{3}^{-} \sqrt{\mu_{1} + 1 - k} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{42}}) = -A_{3}^{-} \sqrt{\mu_{1} - k} v(Q) v(Q_{-}^{-}e_{1}) + B_{3}^{-} \sqrt{k + 1 - \mu_{2}} v(Q_{-}^{-}e_{2})
\]
\[
p^{-\beta_{4}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{\mu_{2} + 1 - \lambda_{3}}(\mu_{2} + 1 - \lambda_{3}) v(Q)
\]
with coefficients
\[
A_{3}^{-} = -\left| \frac{(\lambda_{2} - \mu_{1})(\lambda_{2} - \mu_{1} - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_{2} - \mu_{2}}, \quad B_{3}^{-} = -\left| \frac{(\lambda_{2} + 1 - \mu_{2})(\lambda_{2} + 1 - \mu_{2})}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_{1} + 2 - \lambda_{3}}.
\]

(4) \[
p^{+\beta_{4}}(v(Q) \otimes X_{\beta_{41}}) = A_{3}^{+} \sqrt{\mu_{1} - k} v(Q) v(Q_{+}^{e_{1}}) - B_{3}^{+} \sqrt{k - \mu_{2}} v(Q_{+}^{e_{1}})
\]
\[
p^{+\beta_{4}}(v(Q) \otimes X_{\beta_{42}}) = -A_{3}^{+} \sqrt{k - \mu_{2}} v(Q) v(Q_{+}^{e_{1}}) - B_{3}^{+} \sqrt{\mu_{1} + 1 - k} v(Q_{+}^{e_{1}})
\]
\[
p^{+\beta_{4}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{\mu_{2} + 1 - \lambda_{3}}(\mu_{2} + 1 - \lambda_{3}) v(Q)
\]
with coefficients
\[
A_{3}^{+} = \left| \frac{(\lambda_{2} - \mu_{1})(\lambda_{2} - \mu_{2})}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_{1} + 1 - \lambda_{3}}, \quad B_{3}^{+} = \left| \frac{(\lambda_{1} - \mu_{1})(\lambda_{2} - \mu_{1} - 1)}{d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_{2} - \lambda_{3}}.
\]
\[
\square
\]
Here we denote Gel'fand-Zetlin schemata with top raw $\lambda \pm \beta$

$$
\begin{pmatrix}
\mu_1 \pm 1 \\
\mu_2 \pm 1
\end{pmatrix}
$$

by $\tilde{Q} \in GZ(\lambda \pm \beta)$. Note $\beta_{14}, \beta_{24}, \beta_{34}$ is $(2, 1, 1), (1, 2, 1), (1, 1, 2)$ respectively. And other schemata mean as follows.

$$
\overline{Q}_{\pm 1}^{\pm e_1} =, \overline{Q}_{\pm 1}^{\pm e_2} =
$$

Representations of $S$

By identifying the group $S$ with $U(2)$, for each dominant weight $\mu' = (\mu_1', \mu_2')$, relations

$$
\sigma_\mu'(H_{14}' - H_{24}' - H_{34})w_{k'} = |\mu'|w_{k'}, \quad \sigma_\mu'(H_{14}' - H_{24}')w_{k'} = (2k' - |\mu'|)w_{k'},
$$

$$
\sigma_\mu'(X_{\beta_{14}})w_{k'} = \sqrt{(\mu_1' - k')(k' + 1)}w_{k'}, \quad \sigma_\mu'(X_{\beta_{24}})w_{k'} = \sqrt{(\mu_1' + 1 - k')(k' - \mu_2')}w_{k'}
$$

define a representation $\sigma_\mu'$ of $S$ on $W_{\mu'} := \bigoplus_{k=\mu}^{\mu_1'} \mathbb{C}w_{k'}$.

The Fock representation of $n$

Here we realize the infinite dimensional unitary representation $\rho$ with central character $\psi : Z(N) \ni z_t \mapsto e^{\sqrt{-1}st} \in \mathbb{C}$, $s \in \mathbb{R}\setminus\{0\}$, on $\mathbb{C}[z_1, z_2]$ by

$$
\rho_{\psi} : H(\mathbb{C}_J^2) \rightarrow \text{Aut}(\mathcal{F}_J),
$$

$$
\rho_{\psi}(X_i) := \sqrt{s}(\partial/\partial z_i + z_i), \quad \rho_{\psi}(Y_i) := -\sqrt{s}(\partial/\partial z_i - z_i),
$$

$$
\rho_{\psi}(W) := \sqrt{-1}s,
$$

when $s$ is positive. We choose the monomials $f_{j_1, j_2} := z_1^{j_1}z_2^{j_2}$, $j_i = 0, 1, 2, \ldots$ of two variables, abbreviated by $f_j$, as a base of $\mathbb{C}[z_1, z_2]$.

Representations of $R$ with nontrivial central characters

By natural identification $R = S \ltimes N$ is isomorphic to $U(2) \ltimes \mathcal{H}(\mathbb{C}^2)$ and can be regarded as a subgroup of $\overline{Sp}_2(\mathbb{R}) \ltimes \mathcal{H}(\mathbb{R}^4)$. From the theory of Weil representations, we have the canonical extension

$$
\omega_{\psi} \times \rho_{\psi} : \overline{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2]).
$$

Let $\tilde{R}$ be the pullback $\tilde{R} := \overline{S} \ltimes N \cong \overline{U}(2) \ltimes H(\mathbb{R}^4)$ of $R$ by the covering

$$
pr \times id : \overline{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4).
$$

Then tensoring an odd character $\tilde{\chi}_{1/2}$ of $\overline{U}(2)$ to $(\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}}$, we have a representation of $R$

$$
\tilde{\chi}_{1/2} \otimes (\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}} : \quad R = S \ltimes N \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2]).
$$
A result of Wolf (|Wolf| Prop 5.7.) says that all representations of \( R \) which come from infinite dimensional representation of \( H(\mathbb{C}^2) \) are exhausted by the representations of the form of this representation tensored by representations of \( U(2) \). That is

\[
\tilde{R}_{	ext{citichr} \neq 1} = \{ \sigma_{\mu'} \otimes \overline{\chi}_{1/2} \otimes (\omega_{\psi} \times \rho_{\psi}) |_{\tilde{R}} | \sigma_{\mu'} \in \tilde{U}(2) \}.
\]

We denote this representation by \((\eta, \mathbb{C}[z_1, z_2])\).

The action of \( \tilde{S} \) on \( \mathbb{C}[z_1, z_2] \) through \( \omega_{\psi} \) is given infinitesimally as follows

\[
\omega_{\psi}(H'_{14} - H'_{24} - H'_{34})f_j = -(j_1 + j_2 + 2)f_j, \quad \omega_{\psi}(H'_{14} - H'_{24})f_j = -(j_1 - j_2)f_j,
\]

\[
\omega_{\psi}(X_{\beta_{12}})f_j = -j_1 f_{j-e_1+e_2}, \quad \omega_{\psi}(X_{\beta_{21}})f_j = -j_2 f_{j+e_1-e_2}.
\]

Here is a diagram explaining the above construction

\[
\begin{array}{ccc}
\tilde{R} = \tilde{S} \ltimes N & \xrightarrow{pr \times id} & \mathbb{R}_{\text{Cl}} \ltimes H(\mathbb{R}^4) \\
R = S \ltimes N & \longrightarrow & Sp_{2}(\mathbb{R}) \ltimes H(\mathbb{R}^4).
\end{array}
\]

The discrete series representations of \( G \)

By a theorem of Harish-Chandra, there is a one-to-one correspondence between \( \Sigma \)-regular \( \Sigma_{\text{c}, +} \)-dominant \( T \)-integral weight \( \Lambda \in \Xi \) and equivalence class of discrete series representations \( \pi_\Lambda \in \hat{G}_d \) of \( G \). The parameter set \( \Xi = \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} | \Lambda_1 > \Lambda_2 > \Lambda_3, \Lambda_1 \Lambda_2 \Lambda_3 \neq 0 \} \) decomposes into four disjoint subsets \( \Xi(J = I, II, III, IV) \) correspond to positive root systems \( \Sigma_\lambda^I := \Sigma_{\text{c}, +} \cup \{\beta_{14}, \beta_{24}, \beta_{34}\}, \Sigma_\lambda^II := \Sigma_{\text{c}, +} \cup \{\beta_{14}, \beta_{24}, \beta_{43}\}, \Sigma_\lambda^IV := \Sigma_{\text{c}, +} \cup \{\beta_{41}, \beta_{42}, \beta_{43}\} \). By the inner product induced from the Killing form we can see

\[
\begin{align*}
\Xi^I & = \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} | \Lambda_1 \Lambda_2 > \Lambda_3 > 0 \}, \\
\Xi^II & = \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} | \Lambda_2 \Lambda_3 > \Lambda_1 > 0 \}, \\
\Xi^III & = \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} | \Lambda_1 \Lambda_3 > \Lambda_2 > 0 \}, \\
\Xi^IV & = \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} | 0 > \Lambda_1 \Lambda_2 > \Lambda_3 \}.
\end{align*}
\]

Representations parameterized by \( \Xi^I \) (resp. \( \Xi^IV \)) are called the holomorphic discrete series representations (resp. the antiholomorphic discrete series representations). In the remaining case, discrete series representations whose Harish-Chandra parameters \( \Lambda \)'s belong to \( \Xi^{II}, \Xi^{III} \) are the large discrete series representations in the sense of Vogan [Vo].

**The space of generalized Whittaker functions of the discrete series**

Under the setting above, our main concern \( I_{\pi, \eta} := \text{Hom}_{(\text{c}, K)}(\pi_\Lambda^*, \text{Ind}_R^G \eta) \) is called the space of the algebraic generalized Whittaker functionals. Specifying a \( K \)-type of \( \pi \)

\[
\text{Hom}_{(\text{c}, K)}(\pi_\Lambda^*, \text{Ind}_R^G \eta) \ni l \mapsto \iota^*_\tau(l) \in \text{Hom}_K(\tau_\Lambda^*, \text{Ind}_R^G \eta|_K),
\]

where \( \iota_\tau : \tau_\Lambda \hookrightarrow \pi \), we define a function \( F \) through next identification \( \text{Hom}_K(\tau_\Lambda^*, \text{Ind}_R^G \eta|_K) \cong (\text{Ind}_R^G \eta|_K \otimes \tau_\Lambda)^K \). The latter space \( (\text{C}^\infty(\pi, G) \otimes_c V_\Lambda)^K \) is defined by

\[
\text{C}^\infty_{\eta, \tau_\Lambda}(R \backslash G/K) := \{ \varphi : G \to \mathbb{C}[z] \otimes_c V_\Lambda | \varphi \text{ is a } C^\infty\text{-function satisfying } \varphi(rgk) = (\eta(r)\tau_\Lambda(k)^{-1}) \varphi(g), \forall r \in R, \forall g \in G, \forall k \in K \}.\]
We call the function $F^*_\eta \in C^\infty_{\eta,\tau}(R\backslash G/K)$ representing $\iota_\ast^*(l)$ the algebraic generalized Whittaker function associated to the discrete series representation $\pi_\Lambda$ with $K$-type $\tau$. By definition, \( I(v^*) (g) = \langle v^*, F(g) \rangle_K \), $v^* \in V^*_\tau$. Here $\langle \cdot, \cdot \rangle_K$ means the canonical pairing of $K$-modules $V^*_\tau$ and $V^*_\tau$.

Yamashita’s fundamental result tells that the algebraic generalized Whittaker functions $F$ are characterized by a system of differential equations.

**Proposition 2 ([Ya] Theorem 2.4.)** Let $\pi_\Lambda$ be a discrete series representation of $G$ with Harish-Chandra parameter $\Lambda \in \Xi_J$ and $\lambda$ be the Blattner parameter $\Lambda + \rho_J - 2\rho_c$ of $\pi_\Lambda$. Assume $\Lambda$ is far from walls, then the image of $\text{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^\ast, \text{Ind}_R^G \eta)$ in $C^\infty_{\eta,\tau}(R\backslash G/K)$ by the correspondence above is characterized by

\[
(D) : \quad \mathcal{D}_{\eta,\tau}^{-\beta} F = 0 \quad (\forall \beta \in \Sigma^+_J \cap \Sigma_n).
\]

Here the differential operators

\[
\mathcal{D}_{\eta,\tau}^{-\beta} : C^\infty_{\eta,\tau}(R\backslash G/K) \to C^\infty_{\eta,\tau} \otimes^\beta (R\backslash G/K).
\]

are defined by $\mathcal{D}_{\tau,\tau}^{-\beta} \phi (g) := p^{-\beta} (\nabla_{\tau,\tau} \phi (g))$, $\nabla_{\tau,\tau} \phi := \sum_{i=1}^6 R_{X_i} \phi \otimes X_i$. Here $\{X_i \mid i = 1, \ldots, 6\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and $R_{X_i}$ means the right differential of function $\phi$ by $X \in \mathfrak{g}$ : $R_{X} \phi (g) = \frac{d}{dt} \phi (g \exp t X)|_{t=0}$. We call the space

\[
Wh^\eta_{\tau}(\pi_\Lambda) := \{ F \in C^\infty_{\eta,\tau}(R\backslash G/K) \lvert I(v^*) = \langle v^*, F(\cdot) \rangle_K, \ l \in I_{\pi,\eta}, \ v^* \in V^*_\tau \}.
\]

the generalized Whittaker model for the representation $\pi_\Lambda$ of $G$ with $K$-type $\tau$ and the elements in this space the generalized Whittaker functions associated to the representation $\pi_\Lambda$ with $K$-type $\tau$.

**<Difference-differential equations for coefficients>**

**Radial part of Schmid operators**

For the representation $(\eta, C[z])$ of $R$ and for any finite dimensional $K$-module $V$, we denote the space of the smooth $C[z] \otimes_C V$-valued functions on $A$ by

\[
C^\infty(A; W_{\mu'} \otimes_C C[z] \otimes_C V) := \{ \phi : A \to W_{\mu'} \otimes_C C[z] \otimes_C V \mid C^\infty\text{-function} \}.
\]

Let

\[
\text{res}_A : C^\infty_{\eta,\tau}(R\backslash G/K) \to C^\infty(A; W_{\mu'} \otimes_C C[z] \otimes_C V),
\]

\[
\text{res}_{A, \pm} : C^\infty_{\eta,\tau} \otimes_{\text{Ad}_p} (R\backslash G/K) \to C^\infty(A; W_{\mu'} \otimes_C C[z] \otimes_C V \otimes \mathfrak{p}_p)
\]

be the restriction maps to $A$. Then we define the radial part $R(\nabla^\pm_{\eta,\tau})$ of $\nabla^\pm_{\eta,\tau}$ on the image of $\text{res}_A$ by

\[
R(\nabla^\pm_{\eta,\tau})(\text{res}_A^\ast \varphi) = \text{res}_{A, \pm}^\ast (\nabla^\pm_{\eta,\tau} \varphi).
\]

Let us denote by $\phi$ and $\partial$ the restriction to $A$ of $\varphi \in C^\infty_{\eta,\tau}(R\backslash G/K)$ and the generator $H$ of $\mathfrak{a}$, respectively, $\partial \phi = (H.\varphi)|_A$. We remark $\partial = \frac{d}{dt}$ the Euler operator in variable $r$. By using the Iwasawa decomposition of root vectors, we have next proposition.
Proposition 3 Let \( \phi \) be the above element in \( C^\infty (A; W_\mu \otimes_C \mathbb{C}[z] \otimes_C V_\lambda) \). Then the radial part \( R(\nabla^+_\eta, r, \lambda) \) of \( \nabla^+_\eta, r, \lambda \) is given by

\[
(i) \quad R(\nabla^+_\eta, r, \lambda) . \phi = \frac{1}{2} \{(\partial - \sqrt{-1}r^2 \eta(W) - 6) (\phi \otimes X_{\beta_{43}}) + \frac{1}{2} (\tau_{\lambda} \otimes Ad_{p_+})(H^4_{s_{1}}) (\phi \otimes X_{\beta_{43}}) \}
\]

\[
- \frac{1}{2} r \eta(X_1 - \sqrt{-1}Y_1) (\phi \otimes X_{\beta_{41}}) - (\tau_{\lambda} \otimes Ad_{p_+})(X_{\beta_{13}}) (\phi \otimes X_{\beta_{14}})
\]

\[
- \frac{1}{2} r \eta(X_2 - \sqrt{-1}Y_2) (\phi \otimes X_{\beta_{23}}) - (\tau_{\lambda} \otimes Ad_{p_+})(X_{\beta_{32}}) (\phi \otimes X_{\beta_{24}}).
\]

Similarly for the radial part \( R(\nabla^-_{\eta, r, \lambda}) \) of \( \nabla^-_{\eta, r, \lambda} \), we have

\[
(ii) \quad R(\nabla^-_{\eta, r, \lambda}) . \phi = \frac{1}{2} \{(\partial + \sqrt{-1}r^2 \eta(W) - 6) (\phi \otimes X_{\beta_{43}}) - \frac{1}{2} (\tau_{\lambda} \otimes Ad_{p_-})(H^4_{s_{1}}) (\phi \otimes X_{\beta_{43}}) \}
\]

\[
- \frac{1}{2} r \eta(X_1 + \sqrt{-1}Y_1) (\phi \otimes X_{\beta_{41}}) - (\tau_{\lambda} \otimes Ad_{p_-})(X_{\beta_{13}}) (\phi \otimes X_{\beta_{14}})
\]

\[
- \frac{1}{2} r \eta(X_2 + \sqrt{-1}Y_2) (\phi \otimes X_{\beta_{23}}) - (\tau_{\lambda} \otimes Ad_{p_-})(X_{\beta_{32}}) (\phi \otimes X_{\beta_{24}}).
\]

Compatibility of \( S \)-type and \( K \)-type

Here we note the compatibility of the action of \( S \) from left hand side and the action of \( K \) or \( M \) from right hand side on the function \( \phi = \text{res}_A \varphi, \varphi \in C^\infty_{\eta, r, \lambda} (R \backslash G/K) \). If we write \( \phi = \varphi |_{A} \in C^\infty (A; W_\mu \otimes_C \mathbb{C}[z] \otimes_C V_\lambda) \) as

\[
\phi(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{j=0}^{\infty} \sum_{Q \in GZ(\lambda)} c_{j,k}^{k',\mu}(a_r) ((w_{k'}^m \otimes f_j) \otimes v(Q))
\]

in terms of basis \( \{w_{k'}^m | k = 0, \cdots, d_{\mu'} \}, \{f_j | j \in \mathbb{N}^2 \} \) and \( \{v(Q) | Q \in GZ(\lambda) \} \) of \( W_\mu, \mathbb{C}[z_1, z_2] \) and \( V_\lambda \) respectively, the compatibility of \( S \)-action and \( K \)-action implies of the vanishing of many coefficients \( c_{j,k}^{k',\mu} \). Actually by calculating \( \phi(mam^{-1}) \), \( m \in S = M, a \in A \) in two ways, we have next lemma.

Lemma 4 (1) There is linear relations between indices of bases

\[
j_1 = -k - k' - |\mu|/2 + (|\lambda|/2 - 1), \quad j_2 = k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|).\]

And there are relations between coefficient functions

\[ -(j_1 + 1)c_{j_1+e_1}^{k',\mu} = \sqrt{(\mu_1' - k' + 1)(k' - \mu_2')} c_{j_1}^{k'-1,\mu} + \sqrt{(\mu_1 - k + 1)(k - \mu_2)} c_{j_1}^{k',\mu-1}, \]

\[ -(j_2 + 1)c_{j_2-e_1}^{k,\mu} = \sqrt{(\mu_1' - k')(k' + \mu_2 + 1)} c_{j_2}^{k+1,\mu} + \sqrt{(\mu_1 - k)(k - \mu_2 + 1)} c_{j_2}^{k,\mu+1}. \]

(2) If above relations are not satisfied, then the image of \( \text{res}_A \) in \( C^\infty (A; W_\mu \otimes_C \mathbb{C}[z] \otimes_C V_\lambda) \) is zero.

Difference-differential equations

Because an algebraic generalized Whittaker function \( F \) is determined by its \( A \)-radial part \( \phi = F|_{A} \), and \( \phi \) is determined by the coefficient functions \( c_{j,k}^{k',\mu}(a_r) \), we write down the \( A \)-radial part \( R(D_{\eta, r, \lambda}^\beta) \) of the \( \beta \)-shift operators \( D_{\eta, r, \lambda}^\beta \) in terms of coefficient functions of \( \phi \).
Proposition 5 Let \( \phi \) be any function in \( C^\infty(A; W_\mu \otimes_\CC C[z] \otimes_\CC V_\lambda) \) which is the A-radial part of \( \varphi \in C^\infty_{\eta,\tau} (R \backslash G/K) \). Then for an arbitrary noncompact root \( \beta \), the action of the A-radial part \( R(D_{\eta,x}^\beta) \) of the \( \beta \)-shift operator is given as follows:

\[
R(D_{\eta,x}^\beta) \phi(a_r) = \sum c_{j,k}^{k',\mu}[\beta](a_r) \left( \left( w_\mu^{k'} \otimes f_j \otimes v(\overline{Q}) \right) \right),
\]

with

\[
2c_{j,k}^{k',\mu}[\beta_{14}](a_r) = \sqrt{\left( \lambda_1 - \mu_1 \right) \left( \lambda_1 + 1 - \mu_2 \right)} \left( \vartheta - 6 - s r^2 + |\lambda| - 2 \lambda_1 + 2 - |\mu| \right) c_{j,k}^{k',\mu}(a_r)
- 2 \sqrt{s} \left| \frac{(\mu_1 + 1 - \lambda_2)(\mu_1 + 2 - \lambda_3)}{(d_\mu + 1)(d_\mu + 2)} \right|^{1/2} \sqrt{\lambda_1 + 1 - \mu_2}
\times \left( \sqrt{k + 1 - \mu_2(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
- 2 \sqrt{s} \left| \frac{(\lambda_2 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_\mu(d_\mu + 1)} \right|^{1/2} \sqrt{\lambda_1 - \mu_1}
\times \left( \sqrt{\mu_1 - k(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_2(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
\]

\[
2c_{j,k}^{k',\mu}[\beta_{24}](a_r) = \sqrt{\left( \lambda_2 - \mu_2 \right) \left( \mu_1 + 1 - \lambda_2 \right)} \left( \vartheta - 6 - s r^2 + |\lambda| - 2 \lambda_2 + 2 - |\mu| \right) c_{j,k}^{k',\mu}(a_r)
+ 2 \sqrt{s} \left| \frac{(\mu_1 - \lambda_1)(\mu_1 + 2 - \lambda_3)}{(d_\mu + 1)(d_\mu + 2)} \right|^{1/2} \sqrt{\lambda_2 - \mu_2}
\times \left( \sqrt{k + 1 - \mu_2(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
- 2 \sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_\mu(d_\mu + 1)} \right|^{1/2} \sqrt{\mu_1 + 1 - \lambda_2}
\times \left( \sqrt{\mu_1 - k(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_2(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
\]

\[
2c_{j,k}^{k',\mu}[\beta_{34}](a_r) = \sqrt{\left( \mu_1 - \lambda_3 \right) \left( \mu_1 + 1 - \lambda_3 \right)} \left( \vartheta - 6 - s r^2 + |\lambda| - 2 \lambda_3 + 4 + 2 - |\mu| \right) c_{j,k}^{k',\mu}(a_r)
+ 2 \sqrt{s} \left| \frac{(\lambda_1 - \mu_1)(\mu_1 + 1 - \lambda_2)}{(d_\mu + 1)(d_\mu + 2)} \right|^{1/2} \sqrt{\mu_2 + 1 - \lambda_3}
\times \left( \sqrt{k + 1 - \mu_2(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
+ 2 \sqrt{s} \left| \frac{(\lambda_1 + 2 - \mu_2)(\mu_2 - \lambda_3)}{d_\mu(d_\mu + 1)} \right|^{1/2} \sqrt{\mu_1 + 2 - \lambda_3}
\times \left( \sqrt{\mu_1 - k(j_1 + 1)r c_{j+1+1,k+1}^{k',\mu+1}(a_r) + \sqrt{\mu_2(j_2 + 1) r c_{j+2+1,k}^{k',\mu+1}(a_r)} \right)
\]

\[
2c_{j,k}^{k',\mu}[\beta_{43}](a_r) = \sqrt{\left( \mu_2 - \lambda_3 \right) \left( \mu_1 + 1 - \lambda_3 \right)} \left( \vartheta - 6 + s r^2 - |\lambda| + 2 \lambda_3 + 2 + |\mu| \right) c_{j,k}^{k',\mu}(a_r)
+ 2 \sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_1)(\mu_1 - \lambda_2)}{d_\mu(d_\mu + 1)} \right|^{1/2} \sqrt{\mu_2 - \lambda_3}
\]
\[ \times \left( \sqrt{k - \mu_2} r c_{j-1, k-1}^{k, \mu- \beta_1} (a_r) + \sqrt{\mu_1 - k} r c_{j-2, k}^{k, \mu- \beta_1} (a_r) \right) \]
\[ + \ 2 \sqrt{s} \left( \frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right)^{1/2} \sqrt{\mu_1 + 1 - \lambda_3} \]
\[ \times \left( -\sqrt{\mu_1 + 1 - k} r c_{j-1, k-1}^{k, \mu- \beta_1} (a_r) + \sqrt{k + 1 - \mu_2} r c_{j-2, k}^{k, \mu- \beta_1} (a_r) \right) \]

\[ \square \]

<An explicit formula>

By solving the system of difference-differential equations given above for coefficient functions, we can obtain an explicit form of the generalized Whittaker functions \( F \).

The case of holomorphic discrete series

Here we treat the holomorphic discrete series \( \pi_{\Lambda}, \Lambda \in \Xi^+_I \). In this case \( \Sigma^+_I \cup \Sigma_n = \{ \beta_{14}, \beta_{24}, \beta_{34} \} \). Hence the system \( (D) \) characterizing the generalized Whittaker function \( F \) associated to \( \pi_{\Lambda} \) with the minimal \( K \)-type turns into the system of difference-differential equations for coefficient functions

\[
\begin{align*}
\{ c_{j,k}^{k',\mu}[-\beta_{14}](a_r) &= 0 \\
\{ c_{j,k}^{k',\mu}[-\beta_{24}](a_r) &= 0 \\
\{ c_{j,k}^{k',\mu}[-\beta_{34}](a_r) &= 0.
\end{align*}
\]

This reduces to an ordinary differential equation of first order

\[ \{ \partial - sr^2 - |\lambda| + 2\mu_1 \} c_{j,k}^{k',\mu}(a_r) = 0, \]

and we obtain

\[ c_{j,k}^{k',\mu}(a_r) = (\text{const.}) \cdot r^{\lambda|2 - 2\mu_1} e^{sr^2/2}. \]

**Theorem 6** When \( \Lambda \in \Xi_I, \pi_{\Lambda} \) has multiplicity one property if and only if

\[ -k - k' - |\mu|/2 + (|\lambda|/2 - 1) \in \mathbb{Z}_{\geq 0}, \quad k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|) \in \mathbb{Z}_{\geq 0}. \]

Under this condition, the minimal \( K \)-type generalized Whittaker model \( Wh^\eta_{\tau_{\lambda}}(\pi_{\Lambda}) \) of \( \pi_{\Lambda} \) has a basis \( F_{\eta}^{\tau_{\lambda}} \) whose A-radial part is given by

\[ F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in sK(\mu',\lambda)} r^{\lambda|2 - 2\mu_1} e^{sr^2/2} \cdot \left( w_{k'}^j \otimes f_j \otimes v(Q) \right), \]

where the indices \( j \) run through nonnegative integers satisfying the constraint condition in lemma 4. \( \square \)

The case of large discrete series

In this case \( \Sigma^+_I \cup \Sigma_n = \{ \beta_{14}, \beta_{24}, \beta_{34} \} \) and we have

\[
\begin{align*}
\{ c_{j,k}^{k',\mu}[-\beta_{14}](a_r) &= 0 \\
\{ c_{j,k}^{k',\mu}[-\beta_{24}](a_r) &= 0 \\
\{ c_{j,k}^{k',\mu}[-\beta_{34}](a_r) &= 0.
\end{align*}
\]
for characterizing system of difference-differential equations of coefficient functions of generalized Whittaker functions. This system can be solved when the Gel'fand-Zetlin scheme is of the extremal form

\[ Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \mu_2 & \lambda_3 \\ \lambda_2 & \lambda_2 & \mu_2 \end{pmatrix}. \]

Actually when \( k = \mu_1 = \lambda_2 \), from the first line and the second one we have a two term relation

\[
\{\partial - sl^2 - 2 - \lambda_1 + \lambda_3 - \mu_2\} c_{j,\lambda_2}^{k',\mu}(a_r) = 2\sqrt{s} \frac{(\lambda_2 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)} (j_1 + 1) r c_{j+e_2,\lambda_2}^{k',\mu+e_2}(a_r).
\]

On the other hand the third line turns into

\[
\{\partial + sl^2 - 4 - \lambda_1 + \lambda_3 + \mu_2\} c_{j,\lambda_2}^{k',\mu}(a_r) = -2\sqrt{s} \frac{\lambda_1 + 1 - \mu_2}{(\lambda_2 + 2 - \mu_2)(\mu_2 - \lambda_3)} \left\{ - r c_{j,\lambda_2}^{k',\mu-e_2}(a_r) + \sqrt{\lambda_2 + 1 - \mu_2} r c_{j,\lambda_2}^{k',\mu-e_2}(a_r) \right\}.
\]

Here use the relation caused by the compatibility of \( S \)-action and \( K \)-action. For \( k' = \mu_2' \) the second relation in lemma4 is of the form

\[-(j_1 + 1) c_{j+e_1-e_2,k}^{\mu',\mu}(a_r) = \sqrt{(\mu_1 - k + 1)(k - \mu_2)} c_{j,k-1}^{\mu',\mu}.
\]

By this we can raise the \( k \) parameter and obtain

\[
\{\partial + sl^2 - 3 - \lambda_1 + \lambda_3 + \mu_2\} c_{j,\lambda_2}^{\mu',\mu+e_2}(a_r) = -2\sqrt{s} \frac{\lambda_1 + 1 - \mu_2}{(\lambda_2 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)} \sqrt{\lambda_2 - \mu_2} r c_{j,\lambda_2}^{\mu',\mu}(a_r).
\]

From these equations (1) and (2), we at last obtain the differential equation

\[
\left[\partial^2 - 2(\lambda_1 - \lambda_3 + 3)\partial - \{s^2 r^4 + 2\mu_2 s r^2 + (\mu_2 - 1)^2 - (\lambda_1 - \lambda_3 + 3)^2\}\right] c_{j,\lambda_2}^{\mu',\mu}(a_r)
\]

\[
= -4s \frac{(j_1 + \lambda_2 - \mu_2)(j_2 + 1)}{\lambda_2 - \mu_2} r c_{j,\lambda_2}^{\mu',\mu}(a_r).
\]

After some variable changes we have an explicit form of extremal coefficient functions.

**Theorem 7** When \( \Lambda \in \Xi_{II} \), the \( A \)-radial part of the minimal \( K \)-type generalized Whittaker function

\[ F(a_r) = \sum_{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu',\lambda)} c_{j,\lambda_2}^{\mu',\mu}(a_r) \cdot \left( (w_{k'} \otimes f_j) \otimes v(Q) \right) \]

for large discrete series representation \( \pi_{\Lambda} \) has extremal coefficient functions

\[ c_{j,\lambda_2}^{\mu',\mu}(a_r) = r^{\lambda_1 - \lambda_3 + 2} \{ c_1(\mu_2) \cdot W_{\kappa,\mu_2-1}(sr^2) + c_2(\mu_2) \cdot M_{\kappa,\mu_2-1}(sr^2) \}, \]

where \( \kappa = -\mu_2 - (j_1 + \lambda_2 - \mu_2)(j_2 + 1) \), \( W_{\kappa,m}, M_{\kappa,m} \) are the classical Whittaker functions and \( c_1(\mu_2), c_2(\mu_2) \) are constants depending only on \( \mu_2 \). Other coefficient functions are determined recursively by difference-differential relations between them.
References


[I] Ishikawa, Y., The generalized Whittaker functions for $SU(2,1)$ and the Fourier expansion of automorphic forms, preprint, (1997)


[Ta] Taniguchi, K., Embedding of discrete series into the space of Whittaker functions -The case of $Sp(1,1)$ and $SU(3,1)$-, preprint, (1995)


The Graduate School of Natural Science and Technology, Okayama University, Naka 3-1-1 Tushima Okayama, 700-8530, Japan

E-mail address: ishikawa@math.okayama-u.ac.jp