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The generalized Whittaker functions for the discrete series representations of $SU(3,1)$

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Problem. Let G be a semi-simple Lie group and π_Λ its discrete series representation. What kind of models does π_Λ has? Exactly when the models exist, with how many multiplicity? What explicit form do functions corresponding to the model have?
More precisely, let R be a closed subgroup of G. For $\pi_\Lambda \in \widehat{G}_d$ and a representation η of R, evaluate the upper bound of

$$\dim_{\mathbb{C}}\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, \mathrm{Ind}_R^G\eta),$$

where π_Λ^* is a contragredient of π_Λ. When the dimension does not equal to zero, write down explicitly the functions describing the intertwiners.

Let $G = NAK$ be the Iwasawa decomposition of G. When R is the maximal unipotent subgroup N of G and η a non-degenerate character of N, $\mathrm{Ind}_R^G\eta$ is *the Gelfand-Graev representation*, hence the problem above is a traditional problem on Whittaker model, considered from various point of view. Here we treat the special unitary group of isometry for the Hermitian form of signature $(3+, 1-)$ realized by

$$G = SU(3,1) := \{g \in SL(4,\mathbb{C}) | {}^t\bar{g}I_{3,1}g = I_{3,1}\}.$$

Whittaker model for discrete series representations of this group G was investigated by Taniguchi [Ta], where he obtained a formula for dimension of the space of the intertwiners and an explicit form of corresponding functions (Whittaker function for π_Λ). His formula tells that the dimension of the Whittaker model not necessarily smaller than one even if the growth condition on the corresponding functions is imposed: The multiplicity one property is not valid for this model. We replace the Gelfand-Graev representation for *the reduced generalized Gelfand-Graev representation* and consider the generalized Whittaker model. That is, we take an irreducible infinite dimensional unitary representation of N, note N is a Heisenberg group, as η and a bigger group containing N as R. We investigate this model and give an explicit form of generalized Whittaker functions. By fixing coordinate on G and explicit realization of representations, we reduce the problem to solving a certain system of difference-differential equations for the coefficient functions of generalized Whittaker functions.

We put some remarks. In the case of the group $SU(2,1)$, we obtained an explicit form and the multiplicity one result for generalized Whittaker functions for the standard representations previously [I] from a motivation of automorphic forms. And this is just an "étude" for the work on generalized Whittaker functions on $SU(n,1)$, which will come soon. Main difference from the case of $SU(2,1)$ is in troublesome combinatorial calculation of K-types S-types.

<**Groups and algebras**>

We fix a coordinate on subgroups of G as follows,

$$K=\{\begin{pmatrix} k & \\ & \det k^{-1}\end{pmatrix} \mid k\in U(3)\},\qquad A=\{a_r:=\begin{pmatrix} 1_2 & & \\ & c & s \\ & s & c\end{pmatrix} \mid \begin{array}{l} c=(r+r^{-1})/2,\\ s=(r-r^{-1})/2,\\ r\in\mathbb{R}_{>0}\end{array}\},$$

$$N=\exp\mathfrak{n}=\{\begin{pmatrix} 1 & & \bar z_1 & -\bar z_1 \\ & 1 & \bar z_2 & -\bar z_2 \\ -z_1 & -z_2 & \alpha & \beta \\ -z_1 & -z_2 & \alpha & \beta \end{pmatrix} \mid \begin{array}{l} \alpha=1-\frac{|z_1|^2}{2}-\frac{|z_2|^2}{2}+it,\\ \beta=1+\frac{|z_1|^2}{2}+\frac{|z_2|^2}{2}-it,\\ z_1,z_2\in\mathbb{C},t\in\mathbb{R}\end{array}\}.$$

Here the Lie algebra $\mathfrak{n}$ of N is given by

$$\mathfrak{n}:=\bigoplus_{p=1}^{2}(\mathbb{R}X_p+\mathbb{R}Y_p)\oplus\mathbb{R}W,$$

$$X_1=\begin{pmatrix} & & 1 & -1 \\ & & & \\ -1 & & & \\ -1 & & & \end{pmatrix},\quad Y_1=\begin{pmatrix} & & -i & i \\ & & & \\ -i & & & \\ -i & & & \end{pmatrix},$$

$$X_2=\begin{pmatrix} & & & \\ & & 1 & -1 \\ & -1 & & \\ & -1 & & \end{pmatrix},\quad Y_2=\begin{pmatrix} & & & \\ & & -i & i \\ & -i & & \\ & -i & & \end{pmatrix},\quad W=\begin{pmatrix} & & & \\ & & & \\ & & i & -i \\ & & i & -i \end{pmatrix},$$

where i denotes the complex unity $\sqrt{-1}$. By natural isomorphisms we identify these groups as

$$K\cong U(3),\qquad N\cong H(\mathbb{C}^2).$$

Here $H(\mathbb{C}^2)$ denotes the real Heisenberg group of dimension 5. The center $Z(N)$ of N is of the form

$$Z(N)=\{z_t:=\begin{pmatrix} 1_2 & & \\ & 1+it & -it \\ & it & 1-it\end{pmatrix} \mid t\in\mathbb{R}\}$$

The Cartan decomposition of $\mathfrak{g}=\mathrm{Lie}\,G$ is given by $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$, with $\mathfrak{k}=\mathrm{Lie}\,K$,

$$\mathfrak{p}=\{\begin{pmatrix} O_2 & X \\ {}^t\bar X & 0\end{pmatrix} \mid X\in\mathbb{C}^3\}.$$

The action of the Levi subgroup L of $P:=\exp\mathfrak{p}$ on N is naturally extended to that on $\widehat{N}$. By Ston-von Neumann theorem, the unitary dual $\widehat{N}$ of N is exhausted by unitary characters and infinite dimensional irreducible unitary representations. And the infinite dimensional ones ρ are determined by their central characters ψ. Hence the stabilizer S of ρ in L is the centralizer of $Z(N)$ and of the following form

$$S=\{\mathrm{diag}(m,d,d)\in G \mid m\in U(2),\ d=(\det m)^{-1/2}\},$$

which coincides with the Levi part of P. Using this S, we define the group R as

$$R:=S\ltimes N\ \cong\ U(2)\ltimes H(\mathbb{C}^2).$$

Let $\mathfrak{t} := \{\mathrm{diag}(ih_1, ih_2, ih_3, ih_4) |\ h_j \in \mathbb{R}, h_1 + \cdots + h_4 = 0\}$ be a Cartan subalgebra of $\mathfrak{k}$ and define roots $\beta_{ij} : \mathfrak{t}_{\mathbb{C}} \ni \mathrm{diag}(ih_1, ih_2, ih_3, ih_4) \mapsto t_i - t_j \in \mathbb{C}$. We denote Σ_c and Σ_n the sets of compact and noncompact roots, respectively. In our choice of coordinate,

$$\Sigma_c = \{\beta_{12}, \beta_{13}, \beta_{23}, \beta_{21}, \beta_{31}, \beta_{32}\}, \quad \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{34}, \beta_{41}, \beta_{42}, \beta_{43}\},$$

and matrix element E_{ij} $(1 \leq i, j \leq 3)$ generates the root space $\mathfrak{g}_{\beta_{ij}}$. we put

$$X_{\beta_{ij}} = \begin{cases} -E_{ij} & \mathrm{when}(i,j) = (2,1), (3,1), (3,2); \\ E_{ij} & \text{otherwise}, \end{cases}$$

and take it as a root vector in $\mathfrak{g}_{\beta_{ij}}$. These root vectors decompose with respect to the Iwasawa decomposition as

$$X_{\beta_{34}} = \frac{1}{2}H'_{34} + \frac{1}{2}H + \frac{i}{2}W; \quad X_{\beta_{43}} = \frac{-1}{2}H'_{34} + \frac{1}{2}H - \frac{i}{2}W;$$

$$X_{\beta_{14}} = X_{\beta_{13}} - \frac{1}{2}X_1 - \frac{i}{2}Y_1; \quad X_{\beta_{41}} = X_{\beta_{31}} - \frac{1}{2}X_1 + \frac{i}{2}Y_1;$$

$$X_{\beta_{24}} = X_{\beta_{23}} - \frac{1}{2}X_2 - \frac{i}{2}Y_2; \quad X_{\beta_{42}} = X_{\beta_{32}} - \frac{1}{2}X_2 + \frac{i}{2}Y_2,$$

where H'_{34} is a generator $\begin{pmatrix} 0_2 & & \\ & & 1 \\ & 1 & \end{pmatrix}$ of $\mathfrak{a} := \log A$. This is used to calculate the action of Schmid operators.

<**Representations**>

We fix realization of representations of groups.

Parameterization of irreducible K-modules

In this subsection, we recall the Gel'fand-Zetlin basis, which gives a nice realization of irreducible representation of K. The set L_T^+ of $\Sigma_{c,+}$-dominant T-integral weights is given by $L_T^+ = \{(l, m, n) \in \mathbb{Z}^{\oplus 3} | l \geq m \geq n\}$. For a given Σ_c^+-dominant T-integral weight $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in L_T^+$, let V_λ be a complex vector space spanned by $v(Q)$'s.

$$V_\lambda := \bigoplus_{Q \in GZ(\lambda)} \mathbb{C} v(Q).$$

Here the index set $GZ(\lambda)$ is the set of *the Gel'fand-Zetlin schemes* with top raw λ:

$$GZ(\lambda) := \left\{ Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & k & & \end{pmatrix} \middle| \begin{array}{l} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3, \\ \mu_1 \geq k \geq \mu_2,\ \lambda_i, \mu_j, k \in \mathbb{Z} \end{array} \right\}.$$

The $\mathfrak{k}_{\mathbb{C}}$-module structure defined by

$$\tau_\lambda(H'_{14})v(Q) = kv(Q), \quad \tau_\lambda(H'_{24})v(Q) = (|\mu| - k)v(Q), \quad \tau_\lambda(H'_{34})v(Q) = (|\lambda| - |\mu|)v(Q),$$

$$\tau_\lambda(X_{\beta_{23}})v(Q) = a^1{}_2^+(Q)v(Q^{+e_1}) + a^2{}_2^+(Q)v(Q^{+e_2}), \quad \tau_\lambda(X_{\beta_{12}})v(Q) = a_1^+(Q)v(Q_{+1}),$$

$$\tau_\lambda(X_{\beta_{32}})v(Q) = b^1{}_2^-(Q)v(Q^{-e_1}) + b^2{}_2^-(Q)v(Q^{-e_2}), \quad \tau_\lambda(X_{\beta_{21}})v(Q) = b_1^-(Q)v(Q_{-1})$$

gives an irreducible K-module $(\tau_\lambda, V_\lambda)$ via the highest weight theory. The coefficients appearing above are given as follows.

$$\begin{aligned}
a^{1+}_{\ 2}(Q) &= \Big(\frac{(\lambda_1-\mu_1)(\lambda_2-\mu_1-1)(\lambda_3-\mu_1-2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big)^{1/2}\sqrt{\mu_1+1-k},\\
a^{2+}_{\ 2}(Q) &= \Big(-\frac{(\lambda_1+1-\mu_2)(\lambda_2-\mu_2)(\lambda_3-\mu_2-1)}{d_{\mu'}(d_{\mu'}+1)}\Big)^{1/2}\sqrt{k-\mu_2},\\
a^{+}_{1}(Q) &= \sqrt{(\mu_1-k)(k+1-\mu_2)},\\
b^{1-}_{\ 2}(Q) &= -\Big(\frac{(\lambda_1+1-\mu_1)(\lambda_2-\mu_1)(\lambda_3-\mu_1-1)}{d_{\mu'}(d_{\mu'}+1)}\Big)^{1/2}\sqrt{\mu_1-k},\\
b^{2-}_{\ 2}(Q) &= -\Big(-\frac{(\lambda_1+2-\mu_2)(\lambda_2+1-\mu_2)(\lambda_3-\mu_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big)^{1/2}\sqrt{k+1-\mu_2},\\
b^{-}_{1}(Q) &= -\sqrt{(\mu_1+1-k)(k-\mu_2)}.
\end{aligned}$$

And the indices $Q^{\pm e_1}, Q^{\pm e_2}, Q_{\pm 1}$ mean

$$Q^{\pm e_1} = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 \pm 1 & & \mu_2 & \\ & & k & & \end{pmatrix}, \quad Q^{\pm e_2} = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 \pm 1 & \\ & & k & & \end{pmatrix},$$

$$Q_{\pm 1} = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & k \pm 1 & & \end{pmatrix},$$

respectively. The basis $\{v(Q) \mid Q \in GZ(\lambda)\}$ prescribed above is called *the Gef'fand-Zetlin basis* of $(\tau_\lambda, V_\lambda)$.

Tensor products with $\mathfrak{p}_{\mathbb{C}}$

We regard the 6-dimensional vector space $\mathfrak{p}_{\mathbb{C}}$ as a $\mathfrak{k}_{\mathbb{C}}$-module via the adjoint representation. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are invariant subspaces, and

$$\mathfrak{p}_+ := \mathbb{C}X_{\beta_{14}} \oplus \mathbb{C}X_{\beta_{24}} \oplus \mathbb{C}X_{\beta_{34}} \cong V_{\beta_{14}}, \qquad \mathfrak{p}_- := \mathbb{C}X_{\beta_{41}} \oplus \mathbb{C}X_{\beta_{42}} \oplus \mathbb{C}X_{\beta_{43}} \cong V_{\beta_{43}}.$$

Given an irreducible K-module V_λ Clebsch-Gordan's theorem tells us the following decomposition of $V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm$:

$$V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_+ \cong V_{\lambda+\beta_{14}} \oplus V_{\lambda+\beta_{24}} \oplus V_{\lambda+\beta_{34}}, \quad V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_- \cong V_{\lambda+\beta_{41}} \oplus V_{\lambda+\beta_{42}} \oplus V_{\lambda+\beta_{43}}.$$

The decompositions of $V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}$ induce the following projectors:

$$p^{+\beta_{14}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda+\beta_{14}}, \quad p^{-\beta_{14}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda-\beta_{14}},$$

$$p^{+\beta_{24}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda+\beta_{24}}, \quad p^{-\beta_{24}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda-\beta_{24}},$$

$$p^{+\beta_{34}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda+\beta_{34}}, \quad p^{-\beta_{34}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \to V_{\lambda-\beta_{34}},$$

In terms of $\{v(Q)\}$, they are expressed as follows:

Proposition 1 *The projectors are described as follows.*
(1)

$$p^{-\beta_{14}}(v(Q)\otimes X_{\beta_{41}}) = A_1^-\sqrt{k-\mu_2}v(\tilde{Q}_{-1}^{-e_1}) - B_1^-\sqrt{\mu_1+1-k}v(\tilde{Q}_{-1}^{-e_2})$$
$$p^{-\beta_{14}}(v(Q)\otimes X_{\beta_{42}}) = A_1^-\sqrt{\mu_1-k}v(\tilde{Q}^{-e_1}) + B_1^-\sqrt{k+1-\mu_2}v(\tilde{Q}^{-e_2})$$
$$p^{-\beta_{14}}(v(Q)\otimes X_{\beta_{43}}) = \sqrt{(\lambda_1+1-\mu_2)(\lambda_1-\mu_1)}v(\tilde{Q})$$

with coefficients

$$A_1^- = \left|\frac{(\lambda_2-\mu_1)(\lambda_3-\mu_1-1)}{d_{\mu'}(d_{\mu'}+1)}\right|^{1/2}\sqrt{\lambda_1+1-\mu_2},\quad B_1^- = \left|-\frac{(\lambda_2+1-\mu_2)(\lambda_3-\mu_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\right|^{1/2}\sqrt{\lambda_1-\mu_1}.$$

(2)

$$p^{-\beta_{24}}(v(Q)\otimes X_{\beta_{41}}) = -A_2^-\sqrt{k-\mu_2}v(\tilde{Q}_{-1}^{-e_1}) - B_2^-\sqrt{\mu_1+1-k}v(\tilde{Q}_{-1}^{-e_2})$$
$$p^{-\beta_{24}}(v(Q)\otimes X_{\beta_{42}}) = -A_2^-\sqrt{\mu_1-k}v(\tilde{Q}^{-e_1}) + B_2^-\sqrt{k+1-\mu_2}v(\tilde{Q}^{-e_2})$$
$$p^{-\beta_{24}}(v(Q)\otimes X_{\beta_{43}}) = \sqrt{(\lambda_2-\mu_2)(\mu_1+1-\lambda_2)}v(\tilde{Q})$$

with coefficients

$$A_2^- = \left|-\frac{(\lambda_1+1-\mu_1)(\lambda_3-\mu_1-1)}{d_{\mu'}(d_{\mu'}+1)}\right|^{1/2}\sqrt{\lambda_2-\mu_2},\quad B_2^- = \left|-\frac{(\lambda_1+2-\mu_2)(\lambda_3-\mu_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\right|^{1/2}\sqrt{\mu_1+1-\lambda_2}.$$

(3)

$$p^{-\beta_{34}}(v(Q)\otimes X_{\beta_{41}}) = -A_3^-\sqrt{k-\mu_2}v(\tilde{Q}_{-1}^{-e_1}) + B_3^-\sqrt{\mu_1+1-k}v(\tilde{Q}_{-1}^{-e_2})$$
$$p^{-\beta_{34}}(v(Q)\otimes X_{\beta_{42}}) = -A_3^-\sqrt{\mu_1-k}v(\tilde{Q}^{-e_1}) - B_3^-\sqrt{k+1-\mu_2}v(\tilde{Q}^{-e_2})$$
$$p^{-\beta_{34}}(v(Q)\otimes X_{\beta_{43}}) = \sqrt{(\mu_2+1-\lambda_3)(\mu_1+2-\lambda_3)}v(\tilde{Q})$$

with coefficients

$$A_3^- = \left|-\frac{(\lambda_1+1-\mu_1)(\lambda_2-\mu_1)}{d_{\mu'}(d_{\mu'}+1)}\right|^{1/2}\sqrt{\mu_2+1-\lambda_3},\quad B_3^- = \left|\frac{(\lambda_1+2-\mu_2)(\lambda_2+1-\mu_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\right|^{1/2}\sqrt{\mu_1+2-\lambda_3}$$

(4)

$$p^{+\beta_{34}}(v(Q)\otimes X_{\beta_{14}}) = A_3^+\sqrt{\mu_1-k}v(\tilde{Q}_{+1}^{+e_2}) - B_3^+\sqrt{k-\mu_2+1}v(\tilde{Q}_{+1}^{+e_1})$$
$$p^{+\beta_{34}}(v(Q)\otimes X_{\beta_{24}}) = -A_3^+\sqrt{k-\mu_2}v(\tilde{Q}^{+e_2}) - B_3^+\sqrt{\mu_1+1-k}v(\tilde{Q}^{+e_1})$$
$$p^{+\beta_{34}}(v(Q)\otimes X_{\beta_{34}}) = \sqrt{(\mu_1+1-\lambda_3)(\mu_2-\lambda_3)}v(\tilde{Q})$$

with coefficients

$$A_3^+ = \left|\frac{(\lambda_1+1-\mu_2)(\lambda_2-\mu_2)}{d_{\mu'}(d_{\mu'}+1)}\right|^{1/2}\sqrt{\mu_1+1-\lambda_3},\quad B_3^+ = \left|-\frac{(\lambda_1-\mu_1)(\lambda_2-\mu_1-1)}{(d_{\mu'}+1)(d_{\mu'}+2)}\right|^{1/2}\sqrt{\mu_2-\lambda_3}.$$

□

Here we denote Gel'fand-Zetlin schemata with top raw $\lambda \pm \beta$

$$\begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 1 \\ & k \pm 1 & \end{pmatrix}$$

by $\tilde{Q} \in GZ(\lambda \pm \beta)$. Note $\beta_{14}, \beta_{24}, \beta_{34}$ is $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ respectively. And other schemata mean as follows.

$$\tilde{Q}^{\pm e_1}_{\pm 1} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 2 & & \mu_2 \pm 1 \\ & k \pm 2 & \end{pmatrix}, \quad \tilde{Q}^{\pm e_2}_{\pm 1} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 2 \\ & k \pm 2 & \end{pmatrix},$$

$$\tilde{Q}^{\pm e_1} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 2 & & \mu_2 \pm 1 \\ & k \pm 1 & \end{pmatrix}, \quad \tilde{Q}^{\pm e_2} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 2 \\ & k \pm 1 & \end{pmatrix}.$$

Representations of S

By identifying the group S with $U(2)$, for each dominant weight $\mu' = (\mu'_1, \mu'_2)$, relations

$$\sigma_{\mu'}(H'_{14} - H'_{24} - H_{34})w_{k'} = |\mu'|w_{k'}, \quad \sigma_{\mu'}(H'_{14} - H'_{24})w_{k'} = (2k' - |\mu'|)w_{k'},$$

$$\sigma_{\mu'}(X_{\beta_{12}})w_{k'} = \sqrt{(\mu'_1 - k')(k' + 1 - \mu'_2)}w_{k'}, \quad \sigma_{\mu'}(X_{\beta_{21}})w_{k'} = \sqrt{(\mu'_1 + 1 - k')(k' - \mu'_2)}w_{k'}$$

define a representation $\sigma_{\mu'}$ of S on $W_{\mu'} := \oplus_{k'=\mu'_2}^{\mu'_1} \mathbb{C}w_{k'}$.

The Fock representation of $\mathfrak{n}$

Here we realize the infinite dimensional unitary representation ρ with central character $\psi_s : Z(N) \ni z_t \mapsto e^{\sqrt{-1}st} \in \mathbb{C}^{(1)}$, $s \in \mathbb{R}\backslash\{0\}$, on $\mathbb{C}[z_1, z_2]$ by

$$\rho_{\psi_s} : H(\mathbb{C}^2_J) \to \mathrm{Aut}(\mathcal{F}_J),$$

$$\rho_{\psi_s}(X_i) := \sqrt{s}(\frac{\partial}{\partial z_i} + z_i), \qquad \rho_{\psi_s}(Y_i) := -\sqrt{-s}(\frac{\partial}{\partial z_i} - z_i),$$

$$\rho_{\psi_s}(W) := \sqrt{-1}s,$$

when s is positive. We choose the monomials $f_{j_1,j_2} := z_1^{j_1} z_2^{j_2}$, $j_i = 0, 1, 2, \ldots$ of two variables, abbreviated by f_j, as a base of $\mathbb{C}[z_1, z_2]$.

Representations of R with nontrivial central characters

By natural identification $R = S \ltimes N$ is isomorphic to $U(2) \ltimes H(\mathbb{C}^2)$ and can be regarded as a subgroup of $\widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4)$. From the theory of Weil representations, we have the canonical extension

$$\omega_\psi \times \rho_\psi : \ \widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \to \mathrm{Aut}(\mathbb{C}[z_1, z_2]).$$

Let $\tilde{R}$ be the pullback $\tilde{R} := \tilde{S} \ltimes N \cong \tilde{U}(2) \ltimes H(\mathbb{R}^4)$ of R by the covering

$$pr \times id \ : \widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \twoheadrightarrow Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4).$$

Then tensoring an odd character $\tilde{\chi}_{1/2}$ of $\tilde{U}(2)$ to $(\omega_\psi \times \rho_\psi)|_{\tilde{R}}$, we have a representation of R

$$\tilde{\chi}_{1/2} \otimes (\omega_\psi \times \rho_\psi)|_{\tilde{R}} \ : \ R = S \ltimes N \ \to \ \mathrm{Aut}(\mathbb{C}[z_1, z_2]).$$

A result of Wolf ([Wolf] Prop 5.7.) says that all representations of R which come from infinite dimensional representation of $H(\mathbb{C}^2)$ are exhausted by the representations of the form of this representation tensored by representations of $U(2)$. That is

$$\widehat{R}_{\mathrm{ctlchr}\neq 1} = \{\sigma_{\mu'} \otimes \widetilde{\chi}_{1/2} \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}} \mid \sigma_{\mu'} \in \widehat{U}(2)\}.$$

We denote this representation by $(\eta, \mathbb{C}[z_1, z_2])$.
The action of $\widetilde{S}$ on $\mathbb{C}[z_1, z_2]$ through ω_ψ is given infinitesimally as follows

$$\omega_\psi(H'_{14} - H'_{24} - H'_{34})f_j = -(j_1 + j_2 + 2)f_j, \quad \omega_\psi(H'_{14} - H'_{24})f_j = -(j_1 - j_2)f_j,$$

$$\omega_\psi(X_{\beta_{12}})f_j = -j_1 f_{j-e_1+e_2}, \qquad \omega_\psi(X_{\beta_{21}})f_j = -j_2 f_{j+e_1-e_2}.$$

Here is a diagram explaining the above construction

$$\begin{array}{ccccc} \widetilde{R} = \widetilde{S} \ltimes N & & \widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) & \xrightarrow{\omega_\psi \times \rho_\psi} & \mathrm{Aut}(\mathbb{C}[z_1, z_2]) \\ \downarrow & & \downarrow{\scriptstyle pr \times id} & & \\ R = S \ltimes N & \longrightarrow & Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4). & & \end{array}$$

The discrete series representations of G

By a theorem of Harish-Chandra, there is a one-to-one correspondence between Σ-regular $\Sigma_{c,+}$-dominant T-integral weight $\Lambda \in \Xi$ and equivalence class of discrete series representations $\pi_\Lambda \in \widehat{G}_d$ of G. The parameter set $\Xi = \{\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3,\ \Lambda_1\Lambda_2\Lambda_3 \neq 0\}$ decomposes into four disjoint subsets $\Xi_J (J = I, II, III, IV)$ correspond to positive root systems $\Sigma_I^+ := \Sigma_{c,+} \cup \{\beta_{14}, \beta_{24}, \beta_{34}\}$, $\Sigma_{II}^+ := \Sigma_{c,+} \cup \{\beta_{14}, \beta_{24}, \beta_{43}\}$, $\Sigma_{III}^+ := \Sigma_{c,+} \cup \{\beta_{14}, \beta_{42}, \beta_{43}\}$, $\Sigma_{IV}^+ := \Sigma_{c,+} \cup \{\beta_{41}, \beta_{42}, \beta_{43}\}$. By the inner product induced from the Killing form we can see

$$\begin{aligned} \Xi_I^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3 > 0\}, \\ \Xi_{II}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > 0 > \Lambda_3\}, \\ \Xi_{III}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > 0 > \Lambda_2 > \Lambda_3\}, \\ \Xi_{IV}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid 0 > \Lambda_1 > \Lambda_2 > \Lambda_3\}. \end{aligned}$$

Representations parameterized by Ξ_I^+ (resp.Ξ_{IV}^+) are called the holomorphic discrete series representations (resp. the antiholomorphic discrete series representations). In the remaining case, discrete series representations whose Harish-Chandra parameters Λ's belong to Ξ_{II}^+, Ξ_{III}^+ are the large discrete series representations in the sense of Vogan [Vo].

<The space of generalized Whittaker functions of the discrete series >

Under the setting above, our main concern $I_{\pi,\eta} := \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, \mathrm{Ind}_R^G \eta)$ is called *the space of the algebraic generalized Whittaker functionals.* Specifying a K-type of π

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, \mathrm{Ind}_R^G \eta) \ni l \mapsto \iota_\tau^*(l) \in \mathrm{Hom}_K(\tau_\lambda^*, \mathrm{Ind}_R^G \eta|_K),$$

where $\iota_\tau : \tau_\lambda \hookrightarrow \pi$, we define a function F through next identification $\mathrm{Hom}_K(\tau_\lambda^*, \mathrm{Ind}_R^G \eta|_K) \cong (\mathrm{Ind}_R^G \eta|_K \otimes \tau_\lambda)^K$. The latter space $\left(C_\eta^\infty(R\backslash G) \otimes_{\mathbb{C}} V_\lambda\right)^K$ is defined by

$$C_{\eta,\tau_\lambda}^\infty(R\backslash G/K) := \left\{ \varphi : G \to \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda \;\middle|\; \begin{array}{l} \varphi \text{ is a } C^\infty\text{-function satisfying} \\ \varphi(rgk) = \eta(r)\tau_\lambda(k)^{-1}.\varphi(g), \\ \forall r \in R, \forall g \in G, \forall k \in K \end{array} \right\}.$$

We call the function $F_\eta^\tau \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ representing $\iota_\tau^*(l)$ *the algebraic generalized Whittaker function associated to the discrete series representation* π_Λ *with* K*-type* τ_λ. By definition, $l(v^*)(g) = \langle v^*, F(g)\rangle_K$, $v^* \in V_\tau^*$. Here $\langle\ ,\ \rangle_K$ means the canonical pairing of K-modules V_τ^* and V_τ.

Yamashita's fundamental result tells that the algebraic generalized Whittaker functions F are characterized by a system of differential equations.

Proposition 2 ([Ya] Theorem 2.4.) *Let* π_Λ *be a discrete series representation of* G *with Harish-Chandra parameter* $\Lambda \in \Xi_J$*, and* λ *be the Blattner parameter* $\Lambda + \rho_J - 2\rho_c$ *of* π_Λ*. Assume* Λ *is far from walls, then the image of* $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, \mathrm{Ind}_R^G\eta)$ *in* $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ *by the correspondence above is characterized by*

$$(D)\ :\qquad \mathcal{D}^{-\beta}_{\eta,\tau_\lambda}.F \;=\; 0 \qquad (\forall\beta \in \Sigma_J^+ \cap \Sigma_n).$$

Here the differential operators

$$\mathcal{D}^{-\beta}_{\eta,\tau_\lambda} : C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) \to C^\infty_{\eta,\tau_{\lambda-\beta}}(R\backslash G/K).$$

are defined by $\mathcal{D}^{-\beta}_{\tau_\lambda}\varphi(g) := p^{-\beta}(\nabla_{\tau_\lambda}\varphi(g))$, $\nabla_{\tau_\lambda}\varphi := \sum_{i=1}^6 R_{X_i}\varphi \otimes X_i$. Here $\{X_i\ (i = 1,\ldots,6)\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and $R_X\varphi$ means the right differential of function φ by $X \in \mathfrak{g}$: $R_X\varphi(g) = \frac{d}{dt}\varphi(g\exp tX)|_{t=0}$. We call the space

$$Wh_\eta^\tau(\pi_\Lambda) \;:=\; \{F \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) | l(v^*) = \langle v^*, F(\cdot)\rangle_K,\ l \in I_{\pi,\eta},\ v^* \in V_\lambda^*\}.$$

the generalized Whittaker model for the representation π_Λ *of* G *with* K*-type* τ and the elements in this space *the generalized Whittaker functions associated to the representation* π_Λ *with* K*-type* τ.

<Difference-differential equations for coefficients>

Radial part of Schmid operators

For the representation $(\eta, \mathbb{C}[z])$ of R and for any finite dimensional K-module V, we denote the space of the smooth $\mathbb{C}[z] \otimes_{\mathbb{C}} V$-valued functions on A by

$$C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V) \;:=\; \{\phi : A \to W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V \mid C^\infty\text{-function}\}.$$

Let

$$\begin{array}{lccc} \mathrm{res}_A & : & C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) & \to & C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda) \\ \mathrm{res}_{A,\pm} & : & C^\infty_{\eta,\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(R\backslash G/K) & \to & C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm) \end{array}$$

be the restriction maps to A. Then we define *the radial part* $R(\nabla^\pm_{\eta,\tau_\lambda})$ of $\nabla^\pm_{\eta,\tau_\lambda}$ on the image of res_A by

$$R(\nabla^\pm_{\eta,\tau_\lambda}).(\mathrm{res}_A\varphi) \;=\; \mathrm{res}_{A,\pm}(\nabla^\pm_{\eta,\tau_\lambda}.\varphi).$$

Let us denote by ϕ and ∂ the restriction to A of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ and the generator H of $\mathfrak{a}$, respectively, $\partial\phi = (H.\varphi)|_A$. We remark $\partial = r\frac{d}{dr}$: the Euler operator in variable r. By using the Iwasawa decomposition of root vectors, we have next proposition.

Proposition 3 *Let ϕ be the above element in $C^\infty(A; W_{\mu'} \otimes_{\mathbf{C}} \mathbf{C}[z] \otimes_{\mathbf{C}} V_\lambda)$. Then the radial part $R(\nabla^+_{\eta,\tau_\lambda})$ of $\nabla^+_{\eta,\tau_\lambda}$ is given by*

$$\begin{aligned}
\text{(i)}\quad R(\nabla^+_{\eta,\tau_\lambda}).\phi &= \frac{1}{2}\{\partial - \sqrt{-1}r^2\eta(W) - 6\}.(\phi \otimes X_{\beta_{34}}) + \frac{1}{2}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H'_{34}).(\phi \otimes X_{\beta_{34}}) \\
&\quad -\frac{1}{2}r\eta(X_1 - \sqrt{-1}Y_1).(\phi \otimes X_{\beta_{14}}) - (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X_{\beta_{31}}).(\phi \otimes X_{\beta_{14}}) \\
&\quad -\frac{1}{2}r\eta(X_2 - \sqrt{-1}Y_2).(\phi \otimes X_{\beta_{24}}) - (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X_{\beta_{32}}).(\phi \otimes X_{\beta_{24}}).
\end{aligned}$$

Similarly for the radial part $R(\nabla^-_{\eta,\tau_\lambda})$ of $\nabla^-_{\eta,\tau_\lambda}$, we have

$$\begin{aligned}
\text{(ii)}\quad R(\nabla^-_{\eta,\tau_\lambda}).\phi &= \frac{1}{2}\{\partial + \sqrt{-1}r^2\eta(W) - 6\}.(\phi \otimes X_{\beta_{43}}) - \frac{1}{2}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(H'_{34}).(\phi \otimes X_{\beta_{43}}) \\
&\quad -\frac{1}{2}r\eta(X_1 + \sqrt{-1}Y_1).(\phi \otimes X_{\beta_{41}}) - (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(X_{\beta_{13}}).(\phi \otimes X_{\beta_{41}}) \\
&\quad -\frac{1}{2}r\eta(X_2 + \sqrt{-1}Y_2).(\phi \otimes X_{\beta_{42}}) - (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(X_{\beta_{23}}).(\phi \otimes X_{\beta_{42}}).
\end{aligned}$$

□

Compatibility of S-type and K-type

Here we note the compatibility of the action of S from left hand side and the action of K or M from right hand side on the function $\phi = \mathrm{res}_A\varphi$, $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$. If we write $\phi = \varphi|_A \in C^\infty(A; W_{\mu'} \otimes_{\mathbf{C}} \mathbf{C}[z] \otimes_{\mathbf{C}} V_\lambda)$ as

$$\phi(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{j_i=0}^{\infty} \sum_{Q \in GZ(\lambda)} c^{k',\mu}_{j,k}(a_r)\Big((w^{\mu'}_{k'} \otimes f_j) \otimes v(Q)\Big)$$

in terms of basis $\{w^{\mu}_{k'} | k = 0, \cdots, d_{\mu'}\}$, $\{f_j | j \in \mathrm{N}^2\}$ and $\{v(Q) | Q \in GZ(\lambda)\}$ of $W_{\mu'}$, $\mathbf{C}[z_1, z_2]$ and V_λ respectively, the compatibility of S-action and K-action implies of the vanishing of many coefficients $c^{k',\mu}_{j,k}$. Actually by calculating $\phi(mam^{-1})$, $m \in S = M, a \in A$ in two ways, wa have next lemma.

Lemma 4 (1) *There is linear relations between indices of bases*

$$j_1 = -k - k' - |\mu|/2 + (|\lambda|/2 - 1), \quad j_2 = k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|).$$

And there are relations between coefficient functions

$$\begin{aligned}
-(j_1+1)c^{k',\mu}_{j+e_1-e_2,k} &= \sqrt{(\mu'_1 - k' + 1)(k' - \mu'_2)}c^{k'-1,\mu}_{j,k} + \sqrt{(\mu_1 - k + 1)(k - \mu_2)}c^{k',\mu}_{j,k-1}, \\
-(j_2+1)c^{k',\mu}_{j-e_1+e_2,k} &= \sqrt{(\mu'_1 - k')(k' - \mu'_2 + 1)}c^{k'+1,\mu}_{j,k} + \sqrt{(\mu_1 - k)(k - \mu_2 + 1)}c^{k',\mu}_{j,k+1}.
\end{aligned}$$

(2) *If above relations are not satisfied, then the image of* res_A *in* $C^\infty(A; W_{\mu'} \otimes_{\mathbf{C}} \mathbf{C}[z] \otimes_{\mathbf{C}} V_\lambda)$ *is zero.* □

Difference-differential equations

Because an algebraic generalized Whittaker function F is determined by its A-radial part $\phi = F|_A$, and ϕ is determined by the coefficient functions $c^{k',\mu}_{j,k}(a_r)$, we write down the A-radial part $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ of the β-shift operators $\mathcal{D}^{-\beta}_{\eta,\tau_\lambda}$ in terms of coefficient functions of ϕ.

Proposition 5 *Let ϕ be any function in $C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda)$ which is the A-radial part of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$. Then for an arbitrary noncompact root β, the action of the A-radial part $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ of the β-shift operator is given as follows:*

$$R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})\phi(a_r) = \sum c^{k',\mu}_{j,k}[-\beta](a_r)\Big((w^{\mu'}_{k'} \otimes f_j) \otimes v(\tilde{Q})\Big),$$

with

$$\begin{aligned}
2c^{k',\mu}_{j,k}[-\beta_{14}](a_r) &= \sqrt{(\lambda_1-\mu_1)(\lambda_1+1-\mu_2)}\Big\{\partial - 6 - sr^2 + |\lambda| - 2\lambda_1 + 2 - |\mu|\Big\}c^{k',\mu}_{j,k}(a_r) \\
&- 2\sqrt{s}\Big|\frac{(\mu_1+1-\lambda_2)(\mu_1+2-\lambda_3)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big|^{1/2}\sqrt{\lambda_1+1-\mu_2} \\
&\quad \times\Big(\sqrt{k+1-\mu_2}(j_1+1)rc^{k',\mu+e_1}_{j+e_1,k+1}(a_r) + \sqrt{\mu_1+1-k}(j_2+1)rc^{k',\mu+e_1}_{j+e_2,k}(a_r)\Big) \\
&- 2\sqrt{s}\Big|\frac{(\lambda_2-\mu_2)(\mu_2+1-\lambda_3)}{d_{\mu'}(d_{\mu'}+1)}\Big|^{1/2}\sqrt{\lambda_1-\mu_1} \\
&\quad \times\Big(-\sqrt{\mu_1-k}(j_1+1)rc^{k',\mu+e_2}_{j+e_1,k+1}(a_r) + \sqrt{k-\mu_2}(j_2+1)rc^{k',\mu+e_2}_{j+e_2,k}(a_r)\Big)
\end{aligned}$$

$$\begin{aligned}
2c^{k',\mu}_{j,k}[-\beta_{24}](a_r) &= \sqrt{(\lambda_2-\mu_2)(\mu_1+1-\lambda_2)}\Big\{\partial - 6 - sr^2 + |\lambda| - 2\lambda_2 + 2 + 2 - |\mu|\Big\}c^{k',\mu}_{j,k}(a_r) \\
&+ 2\sqrt{s}\Big|\frac{(\lambda_1-\mu_1)(\mu_1+2-\lambda_3)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big|^{1/2}\sqrt{\lambda_2-\mu_2} \\
&\quad \times\Big(\sqrt{k+1-\mu_2}(j_1+1)rc^{k',\mu+e_1}_{j+e_1,k+1}(a_r) + \sqrt{\mu_1+1-k}(j_2+1)rc^{k',\mu+e_1}_{j+e_2,k}(a_r)\Big) \\
&- 2\sqrt{s}\Big|\frac{(\lambda_1+1-\mu_2)(\mu_2+1-\lambda_3)}{d_{\mu'}(d_{\mu'}+1)}\Big|^{1/2}\sqrt{\mu_1+1-\lambda_2} \\
&\quad \times\Big(-\sqrt{\mu_1-k}(j_1+1)rc^{k',\mu+e_2}_{j+e_1,k+1}(a_r) + \sqrt{k-\mu_2}(j_2+1)rc^{k',\mu+e_2}_{j+e_2,k}(a_r)\Big)
\end{aligned}$$

$$\begin{aligned}
2c^{k',\mu}_{j,k}[-\beta_{34}](a_r) &= \sqrt{(\mu_1-\lambda_3+2)(\mu_2+1-\lambda_3)}\Big\{\partial - 6 - sr^2 + |\lambda| - 2\lambda_3 + 4 + 2 - |\mu|\Big\}c^{k',\mu}_{j,k}(a_r \\
&+ 2\sqrt{s}\Big|\frac{(\lambda_1-\mu_1)(\mu_1+1-\lambda_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big|^{1/2}\sqrt{\mu_2+1-\lambda_3} \\
&\quad \times\Big(\sqrt{k+1-\mu_2}(j_1+1)rc^{k',\mu+e_1}_{j+e_1,k+1}(a_r) + \sqrt{\mu_1+1-k}(j_2+1)rc^{k',\mu+e_1}_{j+e_2,k}(a_r)\Big) \\
&+ 2\sqrt{s}\Big|\frac{(\lambda_1+1-\mu_2)(\lambda_2-\mu_2)}{d_{\mu'}(d_{\mu'}+1)}\Big|^{1/2}\sqrt{\mu_1+2-\lambda_3} \\
&\quad \times\Big(-\sqrt{\mu_1-k}(j_1+1)rc^{k',\mu+e_2}_{j+e_1,k+1}(a_r) + \sqrt{k-\mu_2}(j_2+1)rc^{k',\mu+e_2}_{j+e_2,k}(a_r)\Big)
\end{aligned}$$

$$\begin{aligned}
2c^{k',\mu}_{j,k}[-\beta_{43}](a_r) &= \sqrt{(\mu_2-\lambda_3)(\mu_1+1-\lambda_3)}\Big\{\partial - 6 + sr^2 - |\lambda| + 2\lambda_3 + 2 + |\mu|\Big\}c^{k',\mu}_{j,k}(a_r) \\
&+ 2\sqrt{s}\Big|\frac{(\lambda_1+1-\mu_1)(\mu_1-\lambda_2)}{d_{\mu'}(d_{\mu'}+1)}\Big|^{1/2}\sqrt{\mu_2-\lambda_3}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(\sqrt{k-\mu_2}\, r c^{k',\mu-e_1}_{j-e_1,k-1}(a_r) + \sqrt{\mu_1 - k}\, r c^{k',\mu-e_1}_{j-e_2,k}(a_r)\Big) \\
+ \; & 2\sqrt{s}\Big|\frac{(\lambda_1+2-\mu_2)(\lambda_2+1-\mu_2)}{(d_{\mu'}+1)(d_{\mu'}+2)}\Big|^{1/2}\sqrt{\mu_1+1-\lambda_3} \\
&\times \Big(-\sqrt{\mu_1+1-k}\, r c^{k',\mu-e_2}_{j-e_1,k-1}(a_r) + \sqrt{k+1-\mu_2}\, r c^{k',\mu-e_2}_{j-e_2,k}(a_r)\Big)
\end{aligned}
$$

□

<An explicit formula >

By solving the system of difference-differential equations given above for coefficient functions, we can obtain an explicit form of the generalized Whittaker functions F.

The case of holomorphic discrete series

Here we treat the holomorphic discrete series π_Λ, $\Lambda \in \Xi_I^+$. In this case $\Sigma_I^+ \cup \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{34}\}$. Hence the system (D) characterizing the generalized Whittaker function F associated to π_Λ with the minimal K-type turns into the system of difference-differential equations for coefficient functions

$$
\begin{cases}
c^{k',\mu}_{j,k}[-\beta_{14}](a_r) = 0 \\
c^{k',\mu}_{j,k}[-\beta_{24}](a_r) = 0 \\
c^{k',\mu}_{j,k}[-\beta_{34}](a_r) = 0.
\end{cases}
$$

This reduces to an ordinary differential equation of first order

$$
\{\partial - sr^2 - |\lambda| + 2\mu_1\} c^{k',\mu}_{j,k}(a_r) = 0,
$$

and we obtain

$$
c^{k',\mu}_{j,k}(a_r) = (\text{const.}) \cdot r^{|\lambda|-2\mu_1} e^{sr^2/2}.
$$

Theorem 6 *When $\Lambda \in \Xi_I$, π_Λ has multiplicity one property if and only if*

$$
-k - k' - |\mu|/2 + (|\lambda|/2 - 1) \in \mathbb{Z}_{\geq 0}, \quad k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|) \in \mathbb{Z}_{\geq 0}.
$$

Under this condition, the minimal K-type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ of π_Λ has a basis $F_\eta^{\tau_\lambda}$ whose A-radial part is given by

$$
F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu',\lambda)} r^{|\lambda|-2\mu_1} e^{sr^2/2} \cdot \Big((w^{\mu'}_{k'} \otimes f_j) \otimes v(Q)\Big),
$$

where the indices j run through nonnegative integers satisfying the constraint condition in lemma 4. □

The case of large discrete series

In this case $\Sigma_{II}^+ \cup \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{43}\}$ and we have

$$
\begin{cases}
c^{k',\mu}_{j,k}[-\beta_{14}](a_r) = 0 \\
c^{k',\mu}_{j,k}[-\beta_{24}](a_r) = 0 \\
c^{k',\mu}_{j,k}[-\beta_{43}](a_r) = 0
\end{cases}
$$

for characterizing system of difference-differential equations of coefficient functions of generalized Whittaker functions. This system can be solved when the Gel'fand-Zetlin scheme is of the extremal form

$$Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_2 & & \mu_2 & \\ & & \lambda_2 & & \end{pmatrix}.$$

Actually when $k = \mu_1 = \lambda_2$, from the first line and the second one we have a two term relation

$$\begin{aligned} (1) \qquad & \{\partial - sr^2 - 2 - \lambda_1 + \lambda_3 - \mu_2\} c_{j,\lambda_2}^{k',\mu}(a_r) \\ & = 2\sqrt{s}\sqrt{\frac{(\lambda_2+1-\mu_2)(\mu_2+1-\lambda_3)}{(\lambda_1+1-\mu_2)(\lambda_2-\mu_2)}}(j_1+1) r c_{j+e_2,\lambda_2}^{k',\mu+e_2}(a_r). \end{aligned}$$

On the other hand the third line turns into

$$\begin{aligned} & \{\partial + sr^2 - 4 - \lambda_1 + \lambda_3 + \mu_2\} c_{j,\lambda_2}^{k',\mu}(a_r) \\ & = -2\sqrt{s}\sqrt{\frac{\lambda_1+2-\mu_2}{(\lambda_2+2-\mu_2)(\mu_2-\lambda_3)}}\Big\{ - r c_{j-e_1,\lambda_2-1}^{k',\mu-e_2}(a_r) + \sqrt{\lambda_2+1-\mu_2}\, r c_{j-e_2,\lambda_2}^{k',\mu-e_2}(a_r)\Big\}. \end{aligned}$$

Here use the relation caused by the compatibility of S-action and K-action. For $k' = \mu_2'$ the second relation in lemma4 is of the form

$$-(j_1+1) c_{j+e_1-e_2,k}^{\mu_2',\mu} = \sqrt{(\mu_1-k+1)(k-\mu_2)}\, c_{j,k-1}^{\mu_2',\mu}.$$

By this we can raise the k parameter and obtain

$$\begin{aligned} (2) \qquad & \{\partial + sr^2 - 3 - \lambda_1 + \lambda_3 + \mu_2\} c_{j+e_2,\lambda_2}^{\mu_2',\mu+e_2}(a_r) \\ & = -2\sqrt{s}\sqrt{\frac{\lambda_1+1-\mu_2}{(\lambda_2+1-\mu_2)(\mu_2+1-\lambda_3)}} \frac{j_1+\lambda_2-\mu_2}{\sqrt{\lambda_2-\mu_2}} r c_{j,\lambda_2}^{\mu_2',\mu}(a_r). \end{aligned}$$

From these equations (1) and (2), we at last obtain the differential equation

$$\begin{aligned} & \Big[\partial^2 - 2(\lambda_1-\lambda_3+3)\partial - \{s^2r^4 + 2\mu_2 sr^2 + (\mu_2-1)^2 - (\lambda_1-\lambda_3+3)^2\}\Big] c_{j,\lambda_2}^{\mu_2',\mu}(a_r) \\ & \qquad = -4s\frac{(j_1+\lambda_2-\mu_2)(j_2+1)}{\lambda_2-\mu_2} r^2 c_{j,\lambda_2}^{\mu_2',\mu}(a_r). \end{aligned}$$

After some variable changes we have an explicit form of extremal coefficient functions.

Theorem 7 *When $\Lambda \in \Xi_{II}$, the A-radial part of the minimal K-type generalized Whittaker function*

$$F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q\in GZ(\lambda)} \sum_{j\in SK(\mu',\lambda)} c_{j,k}^{k',\mu}(a_r) \cdot \Big((w_{k'}^{\mu'} \otimes f_j) \otimes v(Q)\Big)$$

for large discrete series representation π_Λ has extremal coefficient functions

$$c_{j,\lambda_2}^{\mu_2',\mu}(a_r) = r^{\lambda_1-\lambda_3+2}\{c_1(\mu_2)\cdot W_{\kappa,\frac{\mu_2-1}{2}}(sr^2) + c_2(\mu_2)\cdot M_{\kappa,\frac{\mu_2-1}{2}}(sr^2)\},$$

where $\kappa = -\frac{\mu_2}{2} - \frac{(j_1+\lambda_2-\mu_2)(j_2+1)}{\lambda_2-\mu_2}$, $W_{\kappa,m}, M_{\kappa,m}$ are the classical Whittaker functions and $c_1(\mu_2), c_2(\mu_2)$ are constants depending only on μ_2. Other coefficient functions are determined recursively by difference-differential relations between them. □

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