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The generalized Whittaker functions for the discrete series representations of $SU(3,1)$

Yoshi-hiro Ishikawa

**Problem.** Let $G$ be a semi-simple Lie group and $\pi_\Lambda$ its discrete series representation. What kind of models does $\pi_\Lambda$ have? Exactly when the models exist, with how many multiplicity? What explicit form do functions corresponding to the model have?

More precisely, let $R$ be a closed subgroup of $G$. For $\pi_\Lambda \in \hat{G}_d$ and a representation $\eta$ of $R$, evaluate the upper bound of

$$\dim c\mathrm{Hom}(\pi_\Lambda^*, \mathrm{Ind}_{R}^{G}\eta),$$

where $\pi_\Lambda^*$ is a contragredient of $\pi_\Lambda$. When the dimension does not equal to zero, write down explicitly the functions describing the intertwiners.

Let $G = NAK$ be the Iwasawa decomposition of $G$. When $R$ is the maximal unipotent subgroup $N$ of $G$ and $\eta$ a non-degenerate character of $N$, $\mathrm{Ind}_{R}^{G}\eta$ is the Gelfand-Graev representation, hence the problem above is a traditional problem on Whittaker model, considered from various point of view. Here we treat the special unitary group of isometry for the Hermitian form of signature $(3+, 1-)$. The multiplicity one property is not valid for this model. We replace the Gelfand-Graev representation for the reduced generalized Gelfand-Graev representation and consider the generalized Whittaker model. That is, we take an irreducible infinite dimensional unitary representation of $N$, note $N$ is a Heisenberg group, as $\eta$ and a bigger group containing $N$ as $R$. We investigate this model and give an explicit form of generalized Whittaker functions. By fixing coordinate on $G$ and explicit realization of representations, we reduce the problem to solving a certain system of difference-differential equations for the coefficient functions of generalized Whittaker functions.

We put some remarks. In the case of the group $SU(2,1)$, we obtained an explicit form and the multiplicity one result for generalized Whittaker functions for the standard representations previously [I] from a motivation of automorphic forms. And this is just an "étude" for the work on generalized Whittaker functions on $SU(n,1)$, which will come soon. Main difference from the case of $SU(2,1)$ is in troublesome combinatorial calculation of $K$-types $S$-types.
<Groups and algebras>

We fix a coordinate on subgroups of $G$ as follows,

$$K = \left\{ \begin{pmatrix} k & \det k \end{pmatrix} \middle| k \in U(3) \right\}, \quad A = \left\{ a_r := \begin{pmatrix} 1_z & c \\ c & s \\ s & c \end{pmatrix} \middle| s = (r - r^{-1})/2, \right\}$$

$$N = \exp n = \left\{ \begin{pmatrix} 1 & \bar{z}_1 - \bar{z}_1 \\ \bar{z}_2 & -\bar{z}_2 \\ -z_1 & -z_2 \end{pmatrix} \middle| \alpha = 1 - |z_1|^2 - \frac{|z_2|^2}{2} + it, \quad \beta = 1 + \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2} - it \right\}.$$

Here the Lie algebra $n$ of $N$ is given by

$$n := \bigoplus_{p=1}^{2} (\mathbb{R}X_p + \mathbb{R}Y_p) \oplus \mathbb{R}W,$$

$$X_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} -i & i \\ -i & -i \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -i & i \\ -i & -i \end{pmatrix}, \quad W = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix},$$

where $i$ denotes the complex unity $\sqrt{-1}$. By natural isomorphisms we identify these groups as

$$K \cong U(3), \quad N \cong H(\mathbb{C}^2).$$

Here $H(\mathbb{C}^2)$ denotes the real Heisenberg group of dimension 5. The center $Z(N)$ of $N$ is of the form

$$Z(N) = \{ z_t := \begin{pmatrix} 1_z & 1 + it & -it \\ -it & 1 - it \end{pmatrix} \middle| t \in \mathbb{R} \}.$$ 

The Cartan decomposition of $g = \text{Lie } G$ is given by $g = p \oplus \mathfrak{t}$, with $\mathfrak{t} = \text{Lie } K$, 

$$p = \left\{ \begin{pmatrix} O_2 & X \\ X & 0 \end{pmatrix} \middle| X \in \mathbb{C}^3 \right\}.$$ 

The action of the Levi subgroup $L$ of $P := \exp p$ on $N$ is naturally extended to that on $\overline{N}$. By Ston-von Neumann theorem, the unitary dual $\overline{N}$ of $N$ is exhausted by unitary characters and infinite dimensional irreducible unitary representations. And the infinite dimensional ones $\rho$ are determined by their central characters $\psi$. Hence the stabilizer $S$ of $\rho$ in $L$ is the centralizer of $Z(N)$ and of the following form

$$S = \{ \text{diag}(m, d, d) \in G \mid m \in U(2), \quad d = (\det m)^{-1/2} \},$$

which coincides with the Levi part of $P$. Using this $S$, we define the group $R$ as

$$R := S \ltimes N \cong U(2) \ltimes H(\mathbb{C}^2).$$
Let \( \mathfrak{t} := \{ \text{diag}(ih_1, ih_2, ih_3, ih_4) \mid h_j \in \mathbb{R}, h_1 + \cdots + h_4 = 0 \} \) be a Cartan subalgebra of \( \mathfrak{k} \) and define roots \( \beta_{ij} : \mathfrak{t} \ni \text{diag}(ih_1, ih_2, ih_3, ih_4) \mapsto t_i - t_j \in \mathbb{C} \).

We denote \( \Sigma_c \) and \( \Sigma_n \) the sets of compact and noncompact roots, respectively. In our choice of coordinate,

\[
\Sigma_c = \{ \beta_{12}, \beta_{13}, \beta_{23}, \beta_{21}, \beta_{31}, \beta_{32} \}, \quad \Sigma_n = \{ \beta_{14}, \beta_{24}, \beta_{34}, \beta_{41}, \beta_{42}, \beta_{43} \},
\]

and matrix element \( E_{ij} (1 \leq i, j \leq 3) \) generates the root space \( \mathfrak{g}_{\beta_{ij}} \). We put

\[
X_{\beta_{ij}} = \begin{cases} -E_{ij} & \text{when } (i, j) = (2, 1), (3, 1), (3, 2); \\
E_{ij} & \text{otherwise}, \end{cases}
\]

and take it as a root vector in \( \mathfrak{g}_{\beta_{ij}} \). These root vectors decompose with respect to the Iwasawa decomposition as

\[
X_{\beta_{34}} = \frac{1}{2} H_{34} + \frac{1}{2} H + \frac{i}{2} W; \quad X_{\beta_{43}} = -\frac{1}{2} H_{34} + \frac{1}{2} H - \frac{i}{2} W;
\]

\[
X_{\beta_{14}} = X_{\beta_{13}} - \frac{1}{2} X_1 - \frac{i}{2} Y_1; \quad X_{\beta_{41}} = X_{\beta_{31}} - \frac{1}{2} X_1 + \frac{i}{2} Y_1;
\]

\[
X_{\beta_{24}} = X_{\beta_{23}} - \frac{1}{2} X_2 - \frac{i}{2} Y_2; \quad X_{\beta_{42}} = X_{\beta_{32}} - \frac{1}{2} X_2 + \frac{i}{2} Y_2,
\]

where \( H_{34} \) is a generator \((0_{2} \quad 1 \quad 0_{2})\) of \( \alpha := \log A \). This is used to calculate the action of Schmid operators.

<Representations>

We fix realization of representations of groups.

**Parameterization of irreducible \( K \)-modules**

In this subsection, we recall the Gel’fand-Zetlin basis, which gives a nice realization of irreducible representation of \( K \). The set \( L^+_T \) of \( \Sigma^+_c \)-dominant \( T \)-integral weights is given by \( L^+_T = \{ (l, m, n) \in \mathbb{Z}^3 \mid l \geq m \geq n \} \). For a given \( \Sigma^+_c \)-dominant \( T \)-integral weight \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \in L^+_T \), let \( V_\lambda \) be a complex vector space spanned by \( v(Q) \)'s.

\[
V_\lambda := \bigoplus_{Q \in GZ(\lambda)} \mathbb{C} v(Q).
\]

Here the index set \( GZ(\lambda) \) is the set of the Gel’fand-Zetlin schemes with top raw \( \lambda \):

\[
GZ(\lambda) := \left\{ Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & k \end{pmatrix} \mid \begin{array}{c} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3, \\ \mu_1 \geq k \geq \mu_2, \lambda_1, \mu_1, k \in \mathbb{Z} \end{array} \right\}.
\]

The \( \mathfrak{t}_c \)-module structure defined by

\[
\tau_\lambda(H_{14})v(Q) = kv(Q), \quad \tau_\lambda(H_{24}')v(Q) = (|\mu| - k)v(Q), \quad \tau_\lambda(H_{34}')v(Q) = (|\lambda| - |\mu|)v(Q),
\]

\[
\tau_\lambda(X_{\beta_{34}})v(Q) = a_{1}^{1}(Q)v(Q^{+\epsilon_1}) + a_{2}^{1}(Q)v(Q^{+\epsilon_2}), \quad \tau_\lambda(X_{\beta_{13}})v(Q) = a_{2}^{1}(Q)v(Q_{+1}),
\]

\[
\tau_\lambda(X_{\beta_{23}})v(Q) = b_{1}^{1}(Q)v(Q_{-1}) + b_{2}^{1}(Q)v(Q_{-2}), \quad \tau_\lambda(X_{\beta_{21}})v(Q) = b_{1}^{1}(Q)v(Q_{-1})
\]
gives an irreducible $K$-module $(\tau_\lambda, V_\lambda)$ via the highest weight theory. The coefficients appearing above are given as follows.

\[
\begin{align*}
  a_{2}^{1+}(Q) &= \left(\frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - 1)(\lambda_3 - \mu_1 - 2)}{(d_\mu' + 1)(d_\mu' + 2)}\right)^{1/2} \sqrt{\mu_1 + 1 - k}, \\
  a_{2}^{2+}(Q) &= \left(\frac{- (\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)(\lambda_3 - \mu_2 - 1)}{d_\mu'(d_\mu' + 1)}\right)^{1/2} \sqrt{k - \mu_2}, \\
  a_{1}^{+}(Q) &= \left(\frac{(\mu_1 - k)(k + 1 - \mu_2)}{d_\mu'(d_\mu' + 1)}\right)^{1/2}, \\
  b_{2}^{1-}(Q) &= -\left(\frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)(\lambda_3 - \mu_2)}{(d_\mu' + 1)(d_\mu' + 2)}\right)^{1/2} \sqrt{k + 1 - \mu_2}, \\
  b_{2}^{2-}(Q) &= -\left(\frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)(\lambda_3 - \mu_2)}{(d_\mu' + 1)(d_\mu' + 2)}\right)^{1/2} \sqrt{k + 1 - \mu_2}, \\
  b_{1}^{-}(Q) &= -\sqrt{(\mu_1 + 1 - k)(k - \mu_2)}.
\end{align*}
\]

And the indices $Q_{e_1}^{\pm}, Q_{e_2}^{\pm}, Q_{\pm 1}$ mean

\[
Q_{e_1}^{\pm} = \left(\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 \pm 1 & \mu_2 & k
\end{array}\right), \quad Q_{e_2}^{\pm} = \left(\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 & \mu_2 \pm 1 & k
\end{array}\right),
Q_{\pm 1} = \left(\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 & \mu_2 & k \pm 1
\end{array}\right),
\]

respectively. The basis $\{v(Q) \mid Q \in GZ(\lambda)\}$ prescribed above is called the Gef'fand-Zetlin basis of $(\tau_\lambda, V_\lambda)$.

**Tensor products with $p_C$**

We regard the 6-dimensional vector space $p_C$ as a $\mathfrak{t}_C$-module via the adjoint representation. Then $p_+$ and $p_-$ are invariant subspaces, and

\[
p_+ := CX_{\beta_{14}} \oplus CX_{\beta_{24}} \oplus CX_{\beta_{34}} \cong V_{\beta_{14}}, \quad p_- := CX_{\beta_{14}} \oplus CX_{\beta_{24}} \oplus CX_{\beta_{34}} \cong V_{\beta_{14}}.
\]

Given an irreducible $K$-module $V_\lambda$, Clebsch-Gordan's theorem tells us the following decomposition of $V_\lambda \otimes_{C} p_+:

\[
V_\lambda \otimes_{C} p_+ \cong V_{\lambda + \beta_{14}} \oplus V_{\lambda + \beta_{24}} \oplus V_{\lambda + \beta_{34}}, \quad V_\lambda \otimes_{C} p_- \cong V_{\lambda + \beta_{14}} \oplus V_{\lambda + \beta_{24}} \oplus V_{\lambda + \beta_{34}}.
\]

The decompositions of $V_\lambda \otimes p_C$ induce the following projectors:

\[
p^{+\beta_{14}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda + \beta_{14}}, \quad p^{-\beta_{14}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda - \beta_{14}}, \\
p^{+\beta_{24}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda + \beta_{24}}, \quad p^{-\beta_{24}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda - \beta_{24}}, \\
p^{+\beta_{34}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda + \beta_{34}}, \quad p^{-\beta_{34}} : V_\lambda \otimes_{C} p_C \rightarrow V_{\lambda - \beta_{34}}.
\]

In terms of $\{v(Q)\}$, they are expressed as follows:
Proposition 1 The projectors are described as follows.

(1) \[ p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{41}}) = A_{1}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} - B_{1}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{42}}) = A_{1}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} + B_{1}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{(\lambda_{1} - 1 - \mu_{2})(\lambda_{1} - \mu_{1})}v(\bar{Q}) \]

with coefficients
\[ A_{1}^{-} = \left| \frac{(\lambda_{2} - \mu_{1})(\lambda_{3} - \mu_{1} - 1)}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\lambda_{1} - \mu_{1}}, \quad B_{1}^{-} = \left| \frac{(\lambda_{2} + 1 - \mu_{2})(\lambda_{3} - \mu_{2})}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{1} - 1 - \lambda_{2}}. \]

(2) \[ p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{41}}) = -A_{2}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} - B_{2}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{42}}) = -A_{2}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} + B_{2}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{(\lambda_{2} - \mu_{2})(\mu_{1} + 1 - \lambda_{2})}v(\bar{Q}) \]

with coefficients
\[ A_{2}^{-} = \left| \frac{(\lambda_{1} + 1 - \mu_{1})(\lambda_{3} - \mu_{1} - 1)}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{2} - \lambda_{3}}, \quad B_{2}^{-} = \left| \frac{(\lambda_{1} + 2 - \mu_{2})(\lambda_{3} + 1 - \mu_{3})}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{1} + 2 - \lambda_{3}}. \]

(3) \[ p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{41}}) = -A_{3}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} + B_{3}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{42}}) = -A_{3}^{-}\sqrt{k - \mu_{2}v(\bar{Q}_{-1}^{-1})} - B_{3}^{-}\sqrt{1 - k\mu_{1}}v(\overline{Q}_{-1}^{-1}) \]
\[ p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{(\mu_{2} + 1 - \lambda_{3})(\mu_{1} + 2 - \lambda_{3})}v(\bar{Q}) \]

with coefficients
\[ A_{3}^{-} = \left| \frac{(\lambda_{1} + 1 - \mu_{1})(\lambda_{3} - \mu_{1})}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{3} + 2 - \lambda_{3}}, \quad B_{3}^{-} = \left| \frac{(\lambda_{1} + 2 - \mu_{2})(\lambda_{3} + 1 - \mu_{3})}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{1} + 2 - \lambda_{3}}. \]

(4) \[ p^{+\beta_{14}}(v(Q) \otimes X_{\beta_{41}}) = A_{3}^{+}\sqrt{\mu_{1} - k\mu_{2}v(\bar{Q}_{+1}^{+1})} + B_{3}^{+}\sqrt{k - \mu_{2} + 1}v(\overline{Q}_{+1}^{+1}) \]
\[ p^{+\beta_{14}}(v(Q) \otimes X_{\beta_{42}}) = -A_{3}^{+}\sqrt{k - \mu_{2}v(\bar{Q}_{+1}^{+1})} + B_{3}^{+}\sqrt{\mu_{1} + 1 - k\mu_{2}}v(\overline{Q}_{+1}^{+1}) \]
\[ p^{+\beta_{14}}(v(Q) \otimes X_{\beta_{43}}) = \sqrt{(\mu_{1} + 1 - \lambda_{3})(\mu_{2} - \lambda_{3})}v(\bar{Q}) \]

with coefficients
\[ A_{3}^{+} = \left| \frac{(\lambda_{1} + 1 - \mu_{2})(\lambda_{2} - \mu_{2})}{(d_{\mu'}+1)} \right|^{1/2} \sqrt{\mu_{3} + 1 - \lambda_{3}}, \quad B_{3}^{+} = \left| \frac{(\lambda_{1} - \mu_{1})(\lambda_{2} - \mu_{1} + 1)}{(d_{\mu'}+1)(d_{\mu}+2)} \right|^{1/2} \sqrt{\mu_{2} - \lambda_{3}}. }
Here we denote Gel'fand-Zetlin schemata with top raw $\lambda \pm \beta$
\[
\begin{pmatrix}
\mu_1 & \pm \beta \\
\beta_1 & \mu_2 & \pm 1 \\
\end{pmatrix}
\]
by $\tilde{Q} \in GZ(\lambda \pm \beta)$. Note $\beta_{14}, \beta_{24}, \beta_{34}$ is $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$ respectively. And other schemata mean as follows.

$\beta_{14}, \beta_{24}, \beta_{34}$ is $(2, 1, 1), (1, 2, 1), (1, 1, 2)$ respectively.

Representations of $S$

By identifying the group $S$ with $U(2)$, for each dominant weight $\mu' = (\mu_1', \mu_2')$, relations

$\sigma_{\mu'}(H_{14}' - H_{24}' - H_{34}')w_{k'} = |\mu'|w_{k'}$,  
$\sigma_{\mu'}(H_{14}' - H_{24}')w_{k'} = (2k' - |\mu'|)w_{k'}$,

$\sigma_{\mu'}(X_{\beta_{14}})w_{k'} = \sqrt{(\mu_1' - k')(k_1' + 1 - \mu_2')}w_{k'}$,  
$\sigma_{\mu'}(X_{\beta_{24}})w_{k'} = \sqrt{(\mu_1' + 1 - k')(k_1' - \mu_2')}w_{k'}$

define a representation $\sigma_{\mu'}$ of $S$ on $W_{\mu'} := \bigoplus_{k=\mu_1'}^{\mu_1'} \mathbb{C}w_{k'}$.

The Fock representation of $n$

Here we realize the infinite dimensional unitary representation $\rho$ with central character $\psi : Z(N) \ni z_t \mapsto e^{\sqrt{-1}st} \in \mathbb{C}$, $s \in \mathbb{R}\setminus\{0\}$, on $\mathbb{C}[z_1, z_2]$ by

$\rho_{\psi} : H(\mathbb{C}^2) \rightarrow \text{Aut}(\mathcal{F}_J),$

$\rho_{\psi}(X_i) := \sqrt{s}(\frac{\partial}{\partial z_i} + z_i)$,  
$\rho_{\psi}(Y_i) := -\sqrt{s}(\frac{\partial}{\partial z_i} - z_i)$,

$\rho_{\psi}(W) := \sqrt{-1}s$,

when $s$ is positive. We choose the monomials $f_{j_1, j_2} := z_1^{j_1}z_2^{j_2}$, $j_i = 0, 1, 2, \ldots$ of two variables, abbreviated by $f_{j}$, as a base of $\mathbb{C}[z_1, z_2]$.

Representations of $R$ with nontrivial central characters

By natural identification $R = S \ltimes N$ is isomorphic to $U(2) \ltimes H(\mathbb{C}_2)$ and can be regarded as a subgroup of $\overline{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4)$. From the theory of Weil representations, we have the canonical extension

$\omega_{\psi} \times \rho_{\psi} : \overline{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2])$.

Let $\tilde{R}$ be the pullback $\tilde{R} := \tilde{S} \ltimes N \cong \tilde{U}(2) \ltimes H(\mathbb{R}^4)$ of $R$ by the covering

$pr \times id : \overline{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4)$.

Then tensoring an odd character $\tilde{\chi}_{1/2}$ of $\tilde{U}(2)$ to $(\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}}$, we have a representation of $\tilde{R}$

$\tilde{\chi}_{1/2} \otimes (\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}} : R = S \ltimes N \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2])$. 
A result of Wolf ([Wolf] Prop. 5.7.) says that all representations of $R$ which come from infinite dimensional representation of $H(C^2)$ are exhausted by the representations of the form of this representation tensored by representations of $U(2)$. That is

$$\tilde{R}_{\text{ctlc}r\neq 1} = \{\sigma_{\mu'} \otimes \tilde{\chi}_{1/2} \otimes (\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}} \mid \sigma_{\mu'} \in \hat{U}(2)\}.$$  

We denote this representation by $(\eta, C[z_1, z_2])$.

The action of $\tilde{S}$ on $C[z_1, z_2]$ through $\omega_{\psi}$ is given infinitesimally as follows

$$\omega_{\psi}(H'_{14} - H'_{24} - H'_{34})f_j = -(j_1 + j_2 + 2)f_j, \quad \omega_{\psi}(H'_{14} - H'_{24})f_j = -(j_1 - j_2)f_j,$$

$$\omega_{\psi}(X_{\beta_{12}})f_j = -j_1f_{j-e_1+e_2}, \quad \omega_{\psi}(X_{\beta_{21}})f_j = -j_2f_{j+e_1-e_2}.$$

Here is a diagram explaining the above construction

$$\begin{array}{ccc}
\tilde{R} = \tilde{S} \times N & \xrightarrow{pr \times id} & H(\mathbb{R}^4) \\
\downarrow & & \downarrow \\
R = S \times N & \longrightarrow & Sp_2(\mathbb{R}) \times H(\mathbb{R}^4). \\
\end{array}$$

The discrete series representations of $G$

By a theorem of Harish-Chandra, there is a one-to-one correspondence between $\Sigma$-regular $\Sigma_{c,+}$-dominant $T$-integral weight $\Lambda \in \Xi$ and equivalence class of discrete series representations $\pi_{\Lambda} \in \hat{G}_d$ of $G$. The parameter set $\Xi = \{\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\geq 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3, \Lambda_1\Lambda_2\Lambda_3 \neq 0\}$ decomposes into four disjoint subsets $\Xi_J (J = I, II, III, IV)$ correspond to positive root systems $\Sigma_J := \Sigma_{c,+} \cup \{\beta_{14}, \beta_{24}, \beta_{34}\}$. The $\Sigma_{c,+}$ and $\Sigma_{c,+}$ are given by $\Sigma_{c,+}$ and $\Sigma_{c,+} \cup \{\beta_{14}, \beta_{24}, \beta_{34}\}$. By the inner product induced from the Killing form we can see

$$\begin{align*}
\Xi_I^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\geq 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3 > 0\}, \\
\Xi_{II}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\geq 3} \mid \Lambda_1 > \Lambda_2 > 0 > \Lambda_3\}, \\
\Xi_{III}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\geq 3} \mid 0 > \Lambda_1 > \Lambda_2 > \Lambda_3\}, \\
\Xi_{IV}^+ &= \{(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\geq 3} \mid 0 > \Lambda_1 > \Lambda_2 > \Lambda_3\}.
\end{align*}$$

Representations parameterized by $\Xi_I^+$ (resp. $\Xi_{IV}^+$) are called the holomorphic discrete series representations (resp. the antiholomorphic discrete series representations). In the remaining case, discrete series representations whose Harish-Chandra parameters $\Lambda$'s belong to $\Xi_{II}^+$, $\Xi_{III}^+$ are the large discrete series representations in the sense of Vogan [Vo].

The space of generalized Whittaker functions of the discrete series

Under the setting above, our main concern $I_{\pi, \eta} := \text{Hom}_{(c, K)}(\pi_{\Lambda}^*, \text{Ind}_R^G \eta)$ is called the space of the algebraic generalized Whittaker functionals. Specifying a $K$-type of $\pi$

$$\text{Hom}_{(c, K)}(\pi_{\Lambda}^*, \text{Ind}_R^G \eta) \ni \iota \mapsto \iota^* (l) \in \text{Hom}_K(\tau_{\Lambda}^*, \text{Ind}_R^G \eta|_K),$$

where $\iota_{\tau} : \tau_{\Lambda} \hookrightarrow \pi$, we define a function $F$ through next identification $\text{Hom}_K(\tau_{\Lambda}^*, \text{Ind}_R^G \eta|_K) \cong (\text{Ind}_R^G \eta|_K \otimes \tau_{\Lambda})^K$. The latter space $(C^\infty_\eta(R \backslash G) \otimes_C V_{\Lambda})^K$ is defined by

$$C^\infty_{\eta, \tau_{\Lambda}}(R \backslash G/K) := \left\{ \varphi : G \to C[z] \otimes_C V_{\Lambda} \mid \varphi \text{ is a } C^\infty \text{-function satisfying} \\
\varphi(r g k) = \eta(r) r_{\Lambda}^{-1}(k) \varphi(g), \quad \forall r \in R, \forall g \in G, \forall k \in K \right\}.$$
We call the function $F^r_\eta \in C^\infty_{\eta,\tau}(R\backslash G/K)$ representing $\psi(r)$ the algebraic generalized Whittaker function associated to the discrete series representation $\pi_\Lambda$ with $K$-type $\tau$. By definition, $l(v^*)(g) = \langle v^*, F(g) \rangle_K$, $v^* \in V^*_r$. Here $\langle \cdot, \cdot \rangle_K$ means the canonical pairing of $K$-modules $V^*_r$ and $V_r$.

Yamashita's fundamental result tells that the algebraic generalized Whittaker functions $F$ are characterized by a system of differential equations.

**Proposition 2 ([Ya] Theorem 2.4.)** Let $\pi_\Lambda$ be a discrete series representation of $G$ with Harish-Chandra parameter $\Lambda \in \Xi_J$, and $\lambda$ be the Blattner parameter $\Lambda + \rho_J - 2\rho_c$ of $\pi_\Lambda$. Assume $\Lambda$ is far from walls, then the image of $\text{Hom}(\psi:\eta,K)(\pi^\Lambda,\text{Ind}_K^G[\eta])$ in $C^{\infty}_{\eta,\tau}(R\backslash G/K)$ by the correspondence above is characterized by

\[(D) : \quad D^{-\beta}_{\eta,\tau}F = 0 \quad (\forall \beta \in \Sigma^+_J \cap \Sigma_n).\]

Here the differential operators

\[D^{-\beta}_{\eta,\tau} : C^{\infty}_{\eta,\tau}(R\backslash G/K) \to C^{\infty}_{\eta,\tau-\beta}(R\backslash G/K).\]

are defined by $D^{-\beta}r_\tau \varphi(g) := p^{-\beta}(\nabla_{r_\tau \varphi}(g))$, $\nabla_{r_\tau} \varphi := \sum_{i=1}^6 R_X i, \varphi \otimes X_i$. Here $\{X_i (i = 1, \ldots, 6)\}$ is an orthonormal basis of $p$ with respect to the Killing form on $g$ and $R_X \varphi$ means the right differential of function $\varphi$ by $X \in g : R_X \varphi(g) = \frac{d}{dt} \varphi(g \exp X)|_{t=0}$. We call the space

\[Wh^\tau_\eta(\pi_\Lambda) := \{F \in C^{\infty}_{\eta,\tau}(R\backslash G/K) | l(v^*) = \langle v^*, F(\cdot) \rangle_K, l \in I_{\pi,\eta}, v^* \in V^*_r\}.\]

the generalized Whittaker model for the representation $\pi_\Lambda$ of $G$ with $K$-type $\tau$ and the elements in this space the generalized Whittaker functions associated to the representation $\pi_\Lambda$ with $K$-type $\tau$.

**<Difference-differential equations for coefficients>**

**Radial part of Schmid operators**

For the representation $(\eta, C[z])$ of $R$ and for any finite dimensional $K$-module $V$, we denote the space of the smooth $C[z] \otimes_C V$-valued functions on $A$ by

\[C^\infty(A; W\mu' \otimes_C C[z] \otimes_C V) := \{\phi : A \to W\mu' \otimes_C C[z] \otimes_C V \mid C^\infty\text{-function}\}.

Let

\[
\begin{align*}
\text{res}_A & : C^{\infty}_{\eta,\tau}(R\backslash G/K) \to C^{\infty}(A; W\mu' \otimes_C C[z] \otimes_C V) \\
\text{res}_{A,\pm} & : C^{\infty}_{\eta,\tau,\pm}(R\backslash G/K) \to C^{\infty}(A; W\mu' \otimes_C C[z] \otimes_C V \pm) 
\end{align*}
\]

be the restriction maps to $A$. Then we define the radial part $R(\nabla^\pm_{\eta,\tau})$ of $\nabla^\pm_{\eta,\tau}$ on the image of $\text{res}_A$ by

\[R(\nabla^\pm_{\eta,\tau})(\text{res}_A \varphi) = \text{res}_{A,\pm}(\nabla^\pm_{\eta,\tau} \varphi).\]

Let us denote by $\phi$ and $\partial$ the restriction to $A$ of $\varphi \in C^{\infty}_{\eta,\tau}(R\backslash G/K)$ and the generator $H$ of $a$, respectively, $\partial \phi = (H.\varphi)|_A$. We remark $\partial = \tau \frac{d}{dr}$: the Euler operator in variable $r$.

By using the Iwasawa decomposition of root vectors, we have next proposition.
Proposition 3 Let \( \phi \) be the above element in \( C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} C[z] \otimes_{\mathbb{C}} V_{\lambda}) \). Then the radial part \( R(\nabla^+_n) \) of \( \nabla^+_n \) is given by

\[
(i) \quad R(\nabla^+_n).\phi = \frac{1}{2} \{ \partial - \sqrt{-1} r^2 \eta(W) - 6 \} (\phi \otimes X_{\beta_{34}}) + \frac{1}{2} (\tau_{\lambda} \otimes \text{Ad}_{p_{+}})(H_{34}')(\phi \otimes X_{\beta_{34}})
\]

\[
- \frac{1}{2} \eta(X_1 - \sqrt{-1} Y_1)(\phi \otimes X_{\beta_{43}}) - (\tau_{\lambda} \otimes \text{Ad}_{p_{+}})(X_{\beta_{43}})(\phi \otimes X_{\beta_{44}})
\]

\[
- \frac{1}{2} \eta(X_2 - \sqrt{-1} Y_2)(\phi \otimes X_{\beta_{43}}) - (\tau_{\lambda} \otimes \text{Ad}_{p_{+}})(X_{\beta_{43}})(\phi \otimes X_{\beta_{44}}).
\]

Similarly for the radial part \( R(\nabla^-_n) \) of \( \nabla^-_n \), we have

\[
(ii) \quad R(\nabla^-_n).\phi = \frac{1}{2} \{ \partial + \sqrt{-1} r^2 \eta(W) - 6 \} (\phi \otimes X_{\beta_{43}}) - \frac{1}{2} (\tau_{\lambda} \otimes \text{Ad}_{p_{-}})(H_{34}')(\phi \otimes X_{\beta_{34}})
\]

\[
- \frac{1}{2} \eta(X_1 + \sqrt{-1} Y_1)(\phi \otimes X_{\beta_{43}}) - (\tau_{\lambda} \otimes \text{Ad}_{p_{-}})(X_{\beta_{43}})(\phi \otimes X_{\beta_{44}})
\]

\[
+ \frac{1}{2} \eta(X_2 + \sqrt{-1} Y_2)(\phi \otimes X_{\beta_{43}}) - (\tau_{\lambda} \otimes \text{Ad}_{p_{-}})(X_{\beta_{43}})(\phi \otimes X_{\beta_{44}}).
\]

Compatibility of \( S \)-type and \( K \)-type

Here we note the compatibility of the action of \( S \) from left hand side and the action of \( K \) or \( M \) from right hand side on the function \( \phi = \text{res}_A \varphi \), \( \varphi \in C^\infty(G/K) \). If we write \( \phi = \varphi|_A \in C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} C[z] \otimes_{\mathbb{C}} V_{\lambda}) \) as

\[
\phi(a_r) = \sum_{\kappa'=0}^{d_{\mu'}} \sum_{j=0}^{\infty} \sum_{Q \in GZ(\lambda)} c_{j,k}^{k',\mu}(a_r)((w_{\kappa'} \otimes f_j) \otimes v(Q))
\]

in terms of basis \( \{ w_{\kappa'}|k = 0, \cdots, d_{\mu'} \}, \{ f_j|j \in \mathbb{N}^2 \} \) and \( \{ v(Q)|Q \in GZ(\lambda) \} \) of \( W_{\mu'}, C[z_1, z_2] \) and \( V_{\lambda} \) respectively, the compatibility of \( S \)-action and \( K \)-action implies of the vanishing of many coefficients \( c_{j,k}^{k',\mu} \). Actually by calculating \( \phi(mam^{-1}), \) \( m \in S = M, a \in A \) in two ways, we have next lemma.

Lemma 4 (1) There is linear relations between indices of bases

\[
j_1 = -k - k' - |\mu|/2 + (|\lambda|/2 - 1), \quad j_2 = k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|).
\]

And there are relations between coefficient functions

\[
-(j_1 + 1)c_{j_1+r_1-k'-e_3,k}^{k',\mu} = (\mu_1 - k' + 1)(k' - \mu_2)c_{j_1,k'-1}^{k'-1,\mu} + (\mu_1 - k + 1)(k - \mu_2)c_{j_1,k-1}^{k',\mu},
\]

\[
-(j_2 + 1)c_{j_2+r_1+k'-e_3,k}^{k,\mu} = (\mu_1 - k')(k' - \mu_2 + 1)c_{j_2,k+1}^{k+1,\mu} + (\mu_1 - k)(k - \mu_2 + 1)c_{j_2,k}^{k,\mu}.
\]

(2) If above relations are not satisfied, then the image of \( \text{res}_A \) in \( C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} C[z] \otimes_{\mathbb{C}} V_{\lambda}) \) is zero.

Difference-differential equations

Because an algebraic generalized Whittaker function \( F \) is determined by its \( A \)-radial part \( \phi = F|_A \), and \( \phi \) is determined by the coefficient functions \( c_{j,k}^{k,\mu}(a_r) \), we write down the \( A \)-radial part \( R(\mathcal{D}^-_{n}\beta) \) of the \( \beta \)-shift operators \( \mathcal{D}^-_{n}\beta \) in terms of coefficient functions of \( \phi \).
Proposition 5 Let $\phi$ be any function in $C^\infty(A; W_{\mu'} \otimes_C C[z] \otimes_C V_\lambda)$ which is the $A$-radial part of $\phi \in C^\infty_{\eta, \tau_\lambda}(R \backslash G/K)$. Then for an arbitrary noncompact root $\beta$, the action of the $A$-radial part $R(D_{\eta, \tau_\lambda}^{-\beta})$ of the $\beta$-shift operator is given as follows:

$$R(D_{\eta, \tau_\lambda}^{-\beta})\phi(a_r) = \sum c_{j,k}^{\beta'}(a_r) \left((w_{\mu'} \otimes f_j) \otimes v(\overline{Q})\right),$$

with

$$2c_{j,k}^{\beta'}[-\beta_{14}](a_r) = \sqrt{(\lambda_1 - \mu_1)(\lambda_1 + 1 - \mu_2)} \left(\vartheta - 6 - st^2 + |\lambda| - 2\lambda_1 + 2 - |\mu|\right)c_{j,k}^{\beta'}(a_r)$$

$$- 2\sqrt{s} \left| \frac{(\mu_1 + 1 - \mu_2)(\mu_1 + 2 - \lambda_3)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_1 + 1 - \mu_2}$$

$$\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)}r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{\mu_1 + 1 - k}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$- 2\sqrt{s} \left| \frac{(\lambda_2 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_1 - \lambda_2}$$

$$\times \left(- \sqrt{\mu_1 - k}(j_1 + 1) r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{k - \mu_2}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$2c_{j,k}^{\beta'}[-\beta_{24}](a_r) = \sqrt{(\lambda_2 - \mu_2)(\mu_1 + 1 - \lambda_2)} \left(\vartheta - 6 - st^2 + |\lambda| - 2\lambda_2 + 2 - |\mu|\right)c_{j,k}^{\beta'}(a_r)$$

$$+ 2\sqrt{s} \left| \frac{(\lambda_1 - \mu_1)(\mu_1 + 2 - \lambda_3)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_2 - \mu_2}$$

$$\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)}r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{\mu_1 + 1 - k}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$- 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_2 + 1 - \lambda_3}$$

$$\times \left(- \sqrt{\mu_1 - k}(j_1 + 1) r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{k - \mu_2}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$2c_{j,k}^{\beta'}[-\beta_{34}](a_r) = \sqrt{(\mu_1 - \lambda_3 + 2)(\mu_2 + 1 - \lambda_3)} \left(\vartheta - 6 - st^2 + |\lambda| - 2\lambda_3 + 4 + 2 - |\mu|\right)c_{j,k}^{\beta'}(a_r)$$

$$+ 2\sqrt{s} \left| \frac{(\lambda_1 - \mu_1)(\mu_1 + 1 - \lambda_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_2 + 1 - \lambda_3}$$

$$\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)}r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{\mu_1 + 1 - k}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$+ 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_1 + 2 - \lambda_3}$$

$$\times \left(- \sqrt{\mu_1 - k}(j_1 + 1) r c_{j+1,1,k+1}^{\beta'}(a_r) + \sqrt{k - \mu_2}(j_2 + 1) r c_{j+2,k}^{\beta'}(a_r)\right)$$

$$2c_{j,k}^{\beta'}[-\beta_{43}](a_r) = \sqrt{(\mu_2 - \lambda_3)(\mu_1 + 1 - \lambda_3)} \left(\vartheta - 6 + st^2 - |\lambda| + 2\lambda_2 + 2 + |\mu|\right)c_{j,k}^{\beta'}(a_r)$$

$$+ 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_1)(\mu_1 - \lambda_2)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_2 - \lambda_3}$$
<An explicit formula >

By solving the system of difference-differential equations given above for coefficient functions, we can obtain an explicit form of the generalized Whittaker functions $F$.

The case of holomorphic discrete series

Here we treat the holomorphic discrete series $\pi_{\Lambda}$, $\Lambda \in \Xi_{p}$. In this case $\Sigma_{I}^{\mp} \cup \Sigma_{n} = \{\beta_{14}, \beta_{24}, \beta_{34}\}$. Hence the system $(D)$ characterizing the generalized Whittaker function $F$ associated to $\pi_{\Lambda}$ with the minimal $K$-type turns into the system of difference-differential equations for coefficient functions

\[
\begin{align*}
\{ c_{j,k}^{k',\mu}[ -\beta_{14}](a_{r}) & = 0 \\
\{ c_{j,k}^{k',\mu}[ -\beta_{24}](a_{r}) & = 0 \\
\{ c_{j,k}^{k',\mu}[ -\beta_{34}](a_{r}) & = 0.
\end{align*}
\]

This reduces to an ordinary differential equation of first order

\[
\{ \partial - sr^{2} - |\lambda| + 2\mu_{1} \} c_{j,k}^{k',\mu}(a_{r}) = 0,
\]

and we obtain

\[
c_{j,k}^{k',\mu}(a_{r}) = (\text{const.}) \cdot r^{|\lambda| - 2\mu_{1}} e^{sr^{2}/2}.
\]

Theorem 6 When $\Lambda \in \Xi_{I}$, $\pi_{\Lambda}$ has multiplicity one property if and only if

\[-k - k' - |\mu|/2 + (|\lambda|/2 - 1) \in \mathbb{Z}_{\geq 0}, \quad k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|) \in \mathbb{Z}_{\geq 0}.
\]

Under this condition, the minimal $K$-type generalized Whittaker model $Wh_{\eta}^{\tau_{\lambda}}(\pi_{\Lambda})$ of $\pi_{\Lambda}$ has a basis $F_{\eta}^{\tau_{\lambda}}$ whose $A$-radial part is given by

\[
F(a_{r}) = \sum_{k'=0}^{d_{\mu}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu',\lambda)} r^{|\lambda| - 2\mu_{1}} e^{sr^{2}/2} \cdot \left( w_{k'}^{\mu'} \otimes f_{j} \otimes v(Q) \right),
\]

where the indices $j$ run through nonnegative integers satisfying the constraint condition in lemma 4.

The case of large discrete series

In this case $\Sigma_{II}^{\mp} \cup \Sigma_{n} = \{\beta_{14}, \beta_{24}, \beta_{34}\}$ and we have

\[
\begin{align*}
\{ c_{j,k}^{k',\mu}[ -\beta_{14}](a_{r}) & = 0 \\
\{ c_{j,k}^{k',\mu}[ -\beta_{24}](a_{r}) & = 0 \\
\{ c_{j,k}^{k',\mu}[ -\beta_{34}](a_{r}) & = 0.
\end{align*}
\]
for characterizing system of difference-differential equations of coefficient functions of generalized Whittaker functions. This system can be solved when the Gel'fand-Zetlin scheme is of the extremal form

$$Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \mu_2 & \\ \lambda_2 & & \mu_3 \end{pmatrix}. $$

Actually when \( k = \mu_1 = \lambda_2 \), from the first line and the second one we have a two term relation

$$\{ \partial - sr^2 - 2 - \lambda_1 + \lambda_3 - \mu_2 \} c_{j, \lambda_2}^{k', \mu}(a_r) = 2\sqrt{s} \left( \frac{(\lambda_2 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)} (j_1 + 1) r c_{j+e_2, \lambda_2}^{k', \mu+e_2}(a_r) \right). $$

On the other hand the third line turns into

$$\{ \partial + sr^2 - 4 - \lambda_1 + \lambda_3 + \mu_2 \} c_{j, \lambda_2}^{k', \mu}(a_r) = -2\sqrt{s} \left( \frac{\lambda_1 + 2 - \mu_2}{(\lambda_2 + 2 - \mu_2)(\mu_2 - \lambda_3)} \right) \left\{ - r c_{j, \lambda_2}^{k', \mu-e_2}(a_r) + \sqrt{\lambda_2 + 1 - \mu_2} r c_{j, \lambda_2}^{k', \mu-e_2}(a_r) \right\}. $$

Here use the relation caused by the compatibility of \( S \)-action and \( K \)-action. For \( k' = \mu'_2 \), the second relation in lemma4 is of the form

$$-(j_1 + 1) c_{j+e_1-e_2, k}^{\mu', \mu}(a_r) = \sqrt{(\mu_1 - k + 1)(k - \mu_2)} c_{j, k-1}^{\mu', \mu}. $$

By this we can raise the \( k \) parameter and obtain

$$\{ \partial + sr^2 - 3 - \lambda_1 + \lambda_3 + \mu_2 \} c_{j, \lambda_2}^{k', \mu+e_2}(a_r) = -2\sqrt{s} \left( \frac{\lambda_1 + 2 - \mu_2}{(\lambda_2 + 2 - \mu_2)(\mu_2 - \lambda_3)} \right) \left\{ - r c_{j+e_2, \lambda_2}^{k', \mu-e_2}(a_r) + \sqrt{\lambda_2 + 1 - \mu_2} r c_{j+e_2, \lambda_2}^{k', \mu-e_2}(a_r) \right\}. $$

From these equations (1) and (2), we at last obtain the differential equation

$$\left[ \partial^2 - 2(\lambda_1 - \lambda_3 + 3) \partial - \{ s^2 r^4 + 2\mu_2 s r^2 + (\mu_2 - 1)^2 - (\lambda_1 - \lambda_3 + 3)^2 \} \right] c_{j, \lambda_2}^{\mu', \mu}(a_r) = -4\sqrt{s} \frac{(j_1 + \lambda_2 - \mu_2)(j_2 + 1)}{\lambda_2 - \mu_2} r c_{j, \lambda_2}^{\mu', \mu}(a_r). $$

After some variable changes we have an explicit form of extremal coefficient functions.

**Theorem 7** When \( \Lambda \in \Xi_{II} \), the \( A \)-radial part of the minimal \( K \)-type generalized Whittaker function

$$F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu', \lambda)} c_{j, \lambda_2}^{k', \mu}(a_r) \cdot ((w'_{k'} \otimes f_j) \otimes v(Q)) $$

for large discrete series representation \( \pi_\Lambda \) has extremal coefficient functions

$$c_{j, \lambda_2}^{\mu', \mu}(a_r) = r^{\lambda_1 - \lambda_3 + 2} \{ c_1(\mu_2) \cdot W_{\kappa, \mu_2-1}(sr^2) + c_2(\mu_2) \cdot M_{\kappa, \mu_2-1}(sr^2) \}, $$

where \( \kappa = -\mu_2 - (j_1 + \lambda_3 - \mu_3)j_2 + 1 \), \( W_{\kappa,m}, M_{\kappa,m} \) are the classical Whittaker functions and \( c_1(\mu_2), c_2(\mu_2) \) are constants depending only on \( \mu_2 \). Other coefficient functions are determined recursively by difference-differential relations between them. \hfill \Box
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