Matrix coefficients of the large discrete series representations of $Sp(2;R)$ as hypergeometric series of two variables (II)

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Citation
数理解析研究所講究録 (1999), 1094: 60-82

Issue Date
1999-04

URL
http://hdl.handle.net/2433/62988

Type
Departmental Bulletin Paper

Textversion
publisher
Matrix coefficients of the large discrete series representations of $Sp(2; \mathbb{R})$ as hypergeometric series of two variables II
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Note presented at a mini-conference, Kyoto Univ., RIMS on September 21-25, 1998

Introduction

This is a continuation of a former article by the same author with the same title in the RIMS Kokyuroku

"Automorphic Forms on $Sp(2; \mathbb{R})$ and $SU(2,2)"$, May, 1995

Also it solves the main problem, not solved completed in the former note.

In a previous paper, we had a partial result on an explicit formula for the $A$-radial part of the matrix coefficients with minimal $K$-type of a large discrete series representation of $Sp(2; \mathbb{R})$.

Our former result was not satisfactory, because we had the result only for the extremal components of the minimal $K$-type. At that time I do not yet understand the structure of the holonomic system (i.e. a system of difference-differential equations ) for the matrix coefficients of a large discrete series, perplexed by the apparently combinatorial complexity of the equations.

Now we can see that that system has rather simple structure, discussed in Section 4 of this note. This enable us to have a complete set of rather simple integral (and power series) expression of the $A$-radial parts of the matrix coefficients. They are written as a kind of Eulerian integral, and a special case of Appell's classical $F_2$-type hypergeometric function of two variables, though the origin of the system is different from the classical one (the classical origin is thrown up to infinity by birational transformation).

Because of the laziness of the author and also to lessen the burden to him, general setting explained in the former article and other papers are not refrained. The symbols for the Lie group and Lie algebra of $Sp(2; \mathbb{R})$ should be the same as Oda's former paper in Tôhoku Journal, 1994, and also they are compatible with that of Iida, Publ. of RIMS, Kyoto University, 1996.
In a paper in preparation, Section 1 should contain such generalities on the structure of Lie groups and Lie algebras, and Harish-Chandra's parametrization etc. Section 2 should explain the gradient operators and Schmid operators etc. These are the same as other papers, mutatis mutandis.

So we shall start from §3!

Before starting that, I would like to comment that a similar result is obtained for the large discrete series of $SU(2, 2)$. There appear almost the same functions discussed here, and the proof is almost parallel. So together with the classical result by Hua on the Bergmann kernel of the holomorphic discrete series, we have some explicit formulae for the $A$-radial parts of the minimal $K$-type matrix coefficients of all the discrete series of $SU(2, 2)$, not only those of $Sp(2; \mathbb{R})$.

We do not attach a complete reference. Related papers up to the time of the former article may be found in its reference, or in the reference of Iida's paper. Some of development on the spherical functions on $Sp(2; \mathbb{R})$ or on $SU(2, 2)$ is reviewed in a joint paper with Hayata and Koseki, referred below. The paper of Takayama below, is not logically related to this paper itself. But it has some important connection internally, and strongly related to (near) future work(s).

Add references


Iida, M.: 'Spherical functions of the principal series representations as hypergeometric functions of $C_2$-type', Publ. RIMS, Kyoto University, 32 (1996), 689-727

3 The explicit formula of Schmid operator

3.1 Presentation of the holonomic system

We prepare some macro symbols to denote our difference-differential equations. With the help of these symbols, some symmetries of the system become apparent.

Here are some symbols of "hyperbolic triangular functions". The functions are in the variables \((a_1, a_2) \in A\), i.e. on the split Catan subgroup \(A\). The are really hyperbolic triangular functions in the coordinates of the Lie algebra of \(A\).

Notation For \((a_1, a_2) \in A\),

\[
\begin{align*}
sh(a_i) &= (a_i - a_i^{-1})/2, \quad ch(a_i) = (a_i + a_i^{-1})/2 \quad (i = 1, 2) \\
th(a_i) &= sh(a_i)/ch(a_i), \quad ct(a_i) = ch(a_i)/sh(a_i) \\
D(a_1, a_2) &= sh^2(a_1) - sh^2(a_2) = ch^2(a_1) - ch^2(a_2).
\end{align*}
\]

Now we introduce some symbols concerning the indices \(I = (i, j) = (i_1, i_2)\) which belong to the product set \(\{0, d\}^2\), \(\{0, d\}\) being the set of numbers \(\{0, 1, \cdots, d-1, d\}\).

Notation For \(I\), we set

\[
\begin{align*}
s(I) &= \frac{1}{2}(d - i - j), \quad c_1(I) = \frac{1}{2}(i - j - L), \quad c_2(I) = \frac{1}{2}(j - i - L), \quad w(I) = |s(I)|.
\end{align*}
\]

Here we recall that we set \(L = l_1 + l_2\). The last number \(w(I)\) is called the weight of the index \(I\). Moreover for \(p \in \{1, 2\}\), we set

\[
\partial_p^\pm(I) := \partial_p \mp s(I)ct(a_p) - c_p(I)th(a_p).
\]

The functions \(e_{pq}(a)\) \((p, q \in \{1, 2\})\) are given by

\[
e_{pq}(a) = (-)^{p-1}sh(a_p)ch(a_q)/D(a_1, a_2).
\]
Proposition  Here are the differential-difference equations given by Schmid operators:

(a):  \textit{chirality equations}

\[(1.+):  \{\partial_{1}^{+}(I(0,2)) + (d-j-1)e_{11}(a)\}c_{I(0,2)} + (d-i)e_{12}(a)c_{I(1,1)}\]
\[+ (j+1)e_{22}c_{I}(a) + ie_{21}c_{I(-1,1)} = 0.\]

\[(1.-):  \{\partial_{1}^{-}(I)) + (i+1)e_{11}(a)\}c_{I} + je_{12}(a)c_{I(1,-1)}\]
\[+ (d-j)e_{21}c_{I(1,1)}(a) + (d-i-1)e_{22}c_{I(2,0)} = 0.\]

\[(2.+):  \{\partial_{2}^{+}(I(2,0)) + (d-i-1)e_{22}(a)\}c_{I(2,0)} + (d-j)e_{21}(a)c_{I(1,1)}\]
\[+ je_{12}(a)c_{I(1,-1)} + (i+1)e_{11}(a)c_{I} = 0.\]

\[(2.-) : \{\partial_{2}^{-}(I)) + (j+1)e_{22}(a)\}c_{I} + ie_{21}c_{I(1,-1)}\]
\[+ (d-i)e_{12}c_{I(1,1)}(a) + (d-j-1)e_{11}c_{I(0,2)} = 0.\]

(b)  \textit{adjoint equations}

\[(3r):  \{\partial_{2}^{+} + 2c_{2}(I)th(a_{2}) + 2(i+1)e_{22}(a)\}c_{I} + 2je_{21}c_{I(1,-1)}\]
\[+ \{\partial_{1}^{+}(I(2,0))+2c_{1}(I(2,0))th(a_{1})+2(d-i-1)e_{11}(a)\}c_{I(2,0)} + 2(d-j)e_{12}c_{I(1,1)} = 0.\]

\[(3l):  \{\partial_{1}^{-} + 2c_{1}(I)th(a_{1}) + 2(j+1)e_{11}(a)\}c_{I} + 2ie_{12}c_{I(-1,1)}\]
\[+ \{\partial_{2}^{-}(I(0,2))+2c_{2}(I(0,2))th(a_{2})+2(d-j-1)e_{22}(a)\}c_{I(0,2)} + 2(d-i)e_{21}c_{I(1,1)} = 0.\]

3.2  \underline{Symmetry with respect to the indices \(I\)}

\textit{Definition}  We define an involutive automorphism on the set \(\{1, d\}^{2}\) of indices by

\(I = (i, j) \rightarrow I' = (d-j, d-i).\)

Obviously we have

\(s(I') = -s(I), c_{k}(I') = c_{k}(I)quad(k = 1, 2).\)

Hence

\(\partial_{k}^{\pm}(I') = \partial_{k}^{\mp}(I) \quad (k = 1, 2).\)

Also for \(\epsilon_{1}, \epsilon_{2} \in \{0, \pm 1, \pm 2\},\)

\(\{I(\epsilon_{1}, \epsilon_{2})\}' = I'(-\epsilon_{2}, -\epsilon_{1}).\)
Apply the involution "" to the equations of our holonomic system, say, to (1.±). Then it is transformed to similar equations (1.±). Therefore we have the following

**Observation**  If we replace the functions \( \{c_I\} \) by another system \( \{\tilde{c}_I\} \) defined

\[
\tilde{c}_I = c_{I'} = c_{d-j,d-i} \quad \text{for each } I = (i, j).
\]

Then they satisfy the same holonomic system as for \( \{c_I\} \).

### 3.3 The order of zeros of \( c_I \) at the origin

**Lemma**  Let \( \{c_I\} \) be a system of solutions of our holonomic system in (3.1), such that all the \( c_I \) are regular at the origin \( (a_1, a_2) = (1, 1) \). Then each \( c_I \) is divisible by \( \{\text{sh}(a_1)\text{sh}(a_2)\}^{w(I)} \) in the ring of germs of analytic functions at the origin. Here \( w(I) = |s(I)| \) is the weight of \( I \).
4 Reduction of the holonomic system

4.1 Change of functions and change of variables

Up to the previous section, we had an explicit system of differential equations in terms of the coefficients \( \{c_I\} \). In order to make this holonomic system simpler to handle, we introduce new functions \( \{h_I\} \), which differ from \( \{c_I\} \) by simple multiplicators. Also we change the variables \( a_i \) by \(-sh^2(a_i)\).

Definition (multiplicator) We set

\[
\mu_I^\pm(a) = \{sh(a_1)sh(a_2)\}^{\pm s(I)} \prod_{i=1}^2 ch(a_i)^{c_i(I)}
\]

\[
= \{sh(a_1)sh(a_2)\}^{\pm \frac{1}{2}(d-i-j)} \{ch(a_1)ch(a_2)\}^{\frac{1}{2}L(ch(a_1)/ch(a_2))^{\frac{1}{2}(i-j)}}.
\]

Definition (new functions) We set

\[
c_I(a) = \mu_I^+(a)h_I^+(a) = \mu_I^-(a)h_I^-(a).
\]

We consider mainly only \( h_I^+ \) in the later sections for the reason of symmetry between the systems \( \{h_I^\pm\} \). Therefore we drop the superscript "+" in the symbol from now on.

Remark. (Observation on symmetry) For \( I' = (d-j, d-i) \),

\[
\mu_I^-(a) = \mu_I^+(a).
\]

Definition (change of variables) We set

\[
x_i = -sh^2(a_i) \quad (i = 1, 2).
\]

Note that

\[
ch^2(a_i) = 1 - x_i = -(x_i - 1), \quad \frac{\partial}{\partial x_i} = -\frac{1}{sh(a_i^2)} \partial_i
\]

\[
= -\frac{1}{2sh(a_i)ch(a_i)} \partial_i \quad (i = 1, 2).
\]

Now we rewrite the equations for \( \{c_I\} \) in terms of new functions \( \{h_I\} \) and new variables \( x_i \).

Remark (Symmetry) By the involution \( I \to I' = (d-j, d-i) \), the system of equations for \( \{h_I\} \) \((s(I) \geq 0)\) can be regarded as a system of equations for \( \{h_I^-\} \) \((s(I') \leq 0)\).
4.2 The holonomic system for \( \{h_I\} \)

Proposition

(a) < chirality equations >

\[
(1+) \quad \frac{\partial}{\partial x_1} + \frac{d-j-1}{2(x_1-x_2)} h_{I(0,2)} + \frac{d-i}{2(x_1-x_2)} h_{I(1,1)}
\]

\[
+ \frac{(j+1)x_2}{2(x_1-x_2)} h_I + \frac{ix_2}{2(x_1-x_2)} h_{I(-1,1)} = 0.
\]

\[
(1-) \quad x_1 \frac{\partial}{\partial x_1} + s(I) + \frac{(i+1)x_1}{2(x_1-x_2)} h_I + \frac{jx_1}{2(x_1-x_2)} h_{I(1,-1)}
\]

\[
+ \frac{d-j}{2(x_1-x_2)} h_{I(1,1)} + \frac{d-i-1}{2(x_1-x_2)} h_{I(2,0)} = 0.
\]

\[
(2+) \quad \frac{\partial}{\partial x_2} - \frac{d-i-1}{2(x_1-x_2)} h_{I(2,0)} - \frac{d-j}{2(x_1-x_2)} h_{I(1,1)}
\]

\[
- \frac{jx_1}{2(x_1-x_2)} h_{I(1,-1)} - \frac{(i+1)x_1}{2(x_1-x_2)} h_{I} = 0.
\]

\[
(2-) \quad x_2 \frac{\partial}{\partial x_2} + s(I) - \frac{(j+1)x_2}{2(x_1-x_2)} h_I - \frac{ix_2}{2(x_1-x_2)} h_{I(-1,1)}
\]

\[
- \frac{d-i}{2(x_1-x_2)} h_{I(1,1)} - \frac{d-j-1}{2(x_1-x_2)} h_{I(0,2)} = 0.
\]

(b) < adjoint equations >

\[
(3r) \quad (x_2-1) \left( x_2 \frac{\partial}{\partial x_2} + s(I) + c_2(I) \frac{x_2}{x_2-1} - \frac{(i+1)x_2}{x_1-x_2} \right) h_I
\]

\[
- j \frac{(x_1-1)x_2}{x_1-x_2} h_{I(1,-1)} - (d-j) \frac{x_2-1}{x_1-x_2} h_{I(1,1)}
\]

\[
- (x_1-1) \left[ \frac{\partial}{\partial x_1} - \frac{c_1(I)(2,0)}{x_1-1} + \frac{d-i-1}{x_1-x_2} \right] h_{I(2,0)} = 0.
\]

\[
(3l) \quad (x_1-1) \left[ x_1 \frac{\partial}{\partial x_1} + s(I) + c_1(I) \frac{x_1}{x_1-1} + \frac{(j+1)x_1}{x_1-x_2} \right] h_I
\]

\[
+ i \frac{x_1(x_2-1)}{x_1-x_2} h_{I(-1,1)} + (d-i) \frac{x_1-1}{x_1-x_2} h_{I(1,1)}
\]

\[
- (x_2-1) \left[ \frac{\partial}{\partial x_2} + \frac{c_2(I)(0,2)}{x_2-1} - \frac{d-j-1}{x_1-x_2} \right] h_{I(0,2)} = 0.
\]

Proof) It is immediately derived from the system for \( \{c_I\} \). We should note that identity

\[
\mu_I^+(a)^{-1} \cdot \partial_i \{ \mu_I^+(a)f(a) \} = \{ \partial_i + s(I)ct(a_i) + c_1(I)th(a_i) \} f.
\]

The other tasks are careful computation. \( \text{(q.e.d)} \)
4.3 Inductive equations

We derive from the basic system of the previous section for \( \{h_I\} \), some simpler equations in this subsection. We call them inductive equations because they will be used to find solutions for general \( I \) from the peripheral entries like \( h_{0,0}, h_{0,1} \).

**Lemma** We have

\[
(i) : \quad \frac{\partial}{\partial x_1} h_{I(0,2)} + \frac{\partial}{\partial x_2} h_{I(1,1)} - \frac{i}{2} h_{I(-1,1)} - \frac{j+1}{2} h_I = 0 \\
\quad (i \in \{0, d-1\}, j \in \{-1, d-2\})
\]

\[
(ii) : \quad \frac{\partial}{\partial x_1} h_{I(1,1)} + \frac{\partial}{\partial x_2} h_{I(2,0)} - \frac{i+1}{2} h_I - \frac{j}{2} h_I(1) = 0 \\
\quad (i \in \{-1, d-2\}, j \in \{0, d-1\}).
\]

**Proof** Shift the index \( I \) in the formula (2.+) by \( I \rightarrow I(-1,1) \), and subtract the obtained formula from (1.+). The other (ii) is just the index shift \( I \rightarrow I(1, -1) \) in (i). It is written explicitly for the reference. \( \text{(q.e.d)} \)

**Proposition** (inductive equations of type I) We have

\[
(i) : \quad \frac{\partial}{\partial x_1} h_{I(0,2)} = -(x_2 \frac{\partial}{\partial x_2} + s(I)) h_I \quad (j \in \{0, d-2\})
\]

\[
(ii) : \quad \frac{\partial}{\partial x_2} h_{I(2,0)} = -(x_1 \frac{\partial}{\partial x_1} + s(I)) h_I \quad (i \in \{0, d-2\})
\]

\[
(iii) : \quad \frac{\partial}{\partial x_1} h_{I(1,1)} - \frac{j}{2} h_{I(1,-1)} - (x_1 \frac{\partial}{\partial x_1} + s(I) + \frac{i+1}{2}) h_I = 0 \\
\quad (i \in \{0, d-2\}, j \in \{0, d-1\})
\]

\[
(iv) : \quad \frac{\partial}{\partial x_2} h_{I(1,1)} - \frac{i}{2} h_{I(-1,1)} - (x_2 \frac{\partial}{\partial x_2} + s(I) + \frac{j+1}{2}) h_I = 0 \\
\quad (i \in \{0, d-1\}, j \in \{0, d-2\})
\]

**Proof** Add (1.+ and (2.-) to get (i); add (1.-) and (2.+ to get (ii). To show (iii) (resp. (iv)), subtract (ii) (resp. (i)) of the present Proposition from (ii) (resp. (i)) of the previous Lemma. \( \text{(q.e.d)} \)

**Remark** Note that up to this point, we use only the chirality equations.
We have more complicated inductive equations derived from the right and left adjoint equations.

**Proposition** (inductive equations of type II)

\[(v): (x_2 - 1)\left\{ \frac{\partial}{\partial x_2} + \frac{d - j - 1 + c_2(I(0,2))}{x_2 - 1} \right\}h_{I(0,2)} + ih_{I(-1,1)}

\[-(x_1 - 1)\left\{ \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + 3s(I) + j + 1 + \frac{c_1(I)x_1}{x_1 - 1} \right\}h_I = 0.\]

\[(vi): (x_1 - 1)\left\{ \frac{\partial}{\partial x_1} + \frac{d - i - 1 + c_1(I(0,2))}{x_1 - 1} \right\}h_{I(2,0)} + jh_{I(1,-1)}

\[-(x_2 - 1)\left\{ 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 3s(I) + i + 1 + \frac{c_2(I)x_2}{x_2 - 1} \right\}h_I = 0.\]

**Proof** Substract (3l) (resp. (3r)) from (2.-) × 2(x_1 - 1) (resp. (1.-) × 2(x_2 - 1)) to get (v) (resp. (iv)).

**(q.e.d)**

### 4.4 Linear relations for \( \{h_I\} \)

By eliminating the derivative terms in the equations by addition and subtraction among them, we have various linear relations of \( h_I \)'s.

**Lemma** (6-term relations) We have

\[(i): s(I)x_2h_I + s(I)x_1h_{I(1,-1)} + \frac{j - 1}{2}x_1x_2h_{I(0,-2)} + \frac{i}{2}x_1x_2h_{I(-1,-1)}

\[ \frac{d - j}{2}h_{I(1)} - \frac{d - i - 1}{2}h_{I(2,0)} = 0 \quad (i \in \{0, d - 1\}, j \in \{1, d\})\]

**Proof** Apply the shift \( I \rightarrow I(0,-2) \) to (1.+). Then we have

\[\left\{ \frac{\partial}{\partial x_1} + \frac{d + 1 - j}{2(x_1 - x_2)} \right\}h_I + \frac{d - i}{2(x_1 - x_2)}h_{I(1,-1)} + \frac{(j - 1)x_2}{2(x_1 - x_2)}h_{I(0,2)}

\[+ \frac{ix_2}{2(x_1 - x_2)}h_{I(-1,-1)} = 0 \quad (j \in \{1, d\}).\]

Multiply \( x_1 \) to this, and substract (1.-). Note that \( d - i - j = 2s(I) \). Then we have our Lemma.

**(q.e.d)**

**Remark** By eliminating \( \partial_2 \) in place of \( \partial_1 \) of the proof of above lemma, we have a similar 6-terms linear relation. It is identical with the above lemma up to the shift. For the convenience of reference, we write it explicitly here:

\[(ii): s(I)x_2h_{I(1,-1)} + s(I)x_1h_I + \frac{j - 1}{2}x_1x_2h_{I(0,-2)} + \frac{i}{2}x_1x_2h_{I(-1,-1)}

\[\frac{d - j - 1}{2}h_{I(0,2)} - \frac{d - i}{2}h_{I(1,1)} = 0 \quad (i \in \{1, d\}, j \in \{0, d - 1\})\]
Lemma (9-terms linear relations)

\((i)\) \[ x_1 x_2 \{2(i - 1)(x_2 - 1)h_{I(-2,0)} - (j - 1)(x_1 - 1)h_{I(0,-2)} \]
\[ + \{(2j - i)(x_1 - 1) + (i - j - L)(x_1 - x_2)\}h_{I(-1,-1)} \}
\[ + 2s(I)(x_1 + x_2)(x_2 - 1)h_{I(-1,1)} + 2s(I)(x_1 - 1)x_2 h_I - s(I)(x_1 - 1)x_2 h_{I(1,-1)} \]
\[ + (d - i - 1)(x_1 - 1)h_{I(2,0)} - 2(d - j - 1)(x_2 - 1)h_{I(0,2)} \]
\[- \{(d - 2i + j)(x_1 - 1) + (i - j - L)(x_1 - x_2)\}h_{I(1,1)} = 0. \quad (i \in \{1, d-1\}, j \in \{1, d-1\}) \]

\((ii)\) \[ x_1 x_2 \{2(j - 1)(x_1 - 1)h_{I(0,-2)} - (i - 1)(x_2 - 1)h_{I(-2,0)} \]
\[ + \{(j - 2i)(x_2 - 1) + (j - i - L)(x_1 - x_2)\}h_{I(-1,-1)} \}
\[ + 2s(I)x_1 (x_2 - 1)h_{I(-1,1)} - 2s(I)x_1 (x_2^{-1})h_{I(1,1)} \]
\[ + \{(d - 2j + i)(x_2 - 1) - (j - i - L)(x_1 - x_2)\}h_{I(1,1)} = 0. \quad (i, j \in \{1, d-1\}) \]

\textbf{Proof}

\textbf{Theorem} (5-terms relation) For \(i, j \in \{1, d-1\}\) we have
\[(d - i - j)\{2h_I + (x_1 - 1)h_{I(1,-1)} + (x_2 - 1)h_{I(-1,1)} \}
\[ + (i + j - L)x_1 x_2 h_{I(1,-1)} + (i + j + L - 2d)h_{I(1,1)} = 0. \]

\textbf{Remark} We may rewrite the above relation as
\[ s(I)\{2h_I(x_1 - 1)h_{I(1,-1)} + (x_2 - 1)h_{I(-1,1)} \}
\[ + \{(d - L)/2 - s(I)\}x_1 x_2 h_{I(1,-1)} - \{(d - L)/2 + s(I)\}h_{I(1,1)} = 0. \]

\textbf{Proof} Compute first a combination of 6-terms relations:
\[ 2(x_1 - 1) \times (6 - i)4(x_2 - 1) \times (6 - ii), \]
which equals to

**

After that, add the 9-terms relation \((i)\) of the previous lemma. \(\text{(q.e.d)}\)

\textbf{Corollary} \textbf{(initial values)} For \(i, j \in \{1, d-1\}\), we have
\[ s(I)\{2h_I(0) - h_{I(1,-1)}(0) - h_{I(-1,1)}(0) \} = \{s(I) + \frac{d - L}{2}\}h_{I(1,1)}(0). \]
4.5 The initial values $h_I(0,0)$

We determine the values of $h_I(x_1, x_2)$ at the origin $(0,0)$. We start with the case of diagonal elements.

**Lemma** If $s(I) = 0$ (i.e. $j = d - i$), then

$$h_I(0,0) = h_{i,d-i}(0) = c_0(-1)^{c_1(I)}  \binom{d}{i}^{-1} = c_0(-1)^{c_1(I)} \frac{i!j!}{d!},$$

where $c_0$ is a constant independent of $i$.

**Proof** The normalization condition for the matrix coefficient $\Phi(a_1, a_2)$ should be

$$\Phi(1,1) = 1_{d+1} \quad \text{(the unit matrix of size } d),$$

if $\Phi$ is written in term of some basis in the representation space of the minimal $K$-type and its dual basis. In our formulation, we use the standard basis in both sides of $K$-types. In of the relation between the standard basis and the dual basis for them, this condition is equivalent to

$$c_{i,d-i} = c_0(-1)^{\frac{i!j!}{d!}}, \quad \text{if } i + j = d,$$

$$c_{i,j} = 0, \quad \text{if } i + j \neq d.$$

Note finally that

$$h_{i,d-i}(0) = c_{i,d-i}(0) \text{ if } s(I) = 0.$$

**Remark** The constant $c_0$ depends on the choice of the identification of the standard basis and dual standard basis. The is canonical way to specify it completely. Even if one fix the length of the standard basis and the dual standard basis, $c = 0$ still has ambiguity up to a complex number of modulus 1.

Next we consider the relation between $h_I(0)$ with the same $s(I)$.

**Lemma**

$$(d - j)h_{i,j+1}(0) + (d - i)h_{i+1,j}(0) = 0.$$

**Proof** Input $x_1 = 0, x_2 = 0$ in the 6-terms linear relation (Lemma (***)). Then

$$(d - j)h_{i+1,j+1}(0) + (d - i - 1)h_{i+2,j}(0) = 0.$$

Replace $i$ by $i - 1$ to get our lemma. (q.e.d)

Now we determine the value $h_I(0)$ for general $I$ if $s(I) \geq 0$. By the Corollary of Theorem (****) (5-terms relation), we have

$$s(I)\{2h_I(0) - h_{I(-1,1)}(0) - h_{I(1,-1)}(0)\} = \left(\frac{1}{2}(d - L) + s(I)\right)h_{I(1,1)}(0).$$
On the other hand, the previous lemma shows that

\[
2h_I(0) - h_{I(-1,1)} - h_{I(1,-1)}
\]

\[
= \frac{1}{(d-i)(d-j)} \{2(d-i)(d-j) + (d-i)(d-i+1) + (d-j+1)(d-j)\} h_I(0)
\]

\[
= \frac{(2d+1-i-j)(2d-i-j)}{(d-i)(d-j)} h_I(0).
\]

Hence for \( I = (i,j) \),

\[
h_I(0) = \frac{(s(I) + \frac{1}{2}(d-L))(d-i)(d-j)}{s(I)(2d+1-i-j)(2d-i-j)} h_{I(1,1)}(0).
\]

Thus we have the following

**Proposition** If \( s(I) \geq 0 \),

\[
\frac{1}{d+1} h_I(0) = c_0(0)(-1)^{(i-j+d)/2} \left( \frac{1}{2}(d-L) + s(I) \right) \frac{(d-i)!(d-j)!}{(d+1)!}.
\]

*Proof* When \( s(I) = 0 \), the right hand side \( RHS(I) \), which is a function in \( I \), equals to

\[
c_0(-1)^i \frac{(d-i)!(d-j)!}{(d+1)!} = \frac{1}{d+1} h_{i,d-i}(0).
\]

So this case is settled. For the case \( s(I) > 0 \), consider the ratio

\[
RHS(I(-1,-1))/RHS(I) = \frac{s(I) + \frac{1}{2}(d-L) + 1}{s(I) + 1} \frac{(d+1-i)(d+1-j)}{(2d+3-i-j)(2d+2-i-j)}
\]

\[
= h_{I(-1,-1)}(0)/h_I(0),
\]

as shown in the Corollary (***) . Hence our Proposition is shown by induction with respect to \( s(I) \). (q.e.d)
5 Modified system of $F_2$ for extremal entries

We want to show that the extremal entries $h_I$ with $s(I) = \pm \left\lfloor \frac{d}{2} \right\rfloor$ are solution of certain holonomic system of rank 4 with singularities along the divisor $x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 - x_2) = 0$ and at infinity, which is called the modified system of (Appell's) $F_2$ in [Takayama II, §2, pp. ***], that consists of an Euler-Darboux equation and Poisson equation. These last two equations are deduced from a part of the inductive equations in Section (**.*).

We treat the Euler-Darboux equation first.

5.1 Euler-Darboux equation for extremal entries

If the index $I$ attains the possible highest weight, i.e. if $w(I) = |s(I)| = \left\lfloor \frac{d}{2} \right\rfloor$,

$I = (0, 0)$ or $I = (d, d)$ if $d$ is even, and

$I = (0, 1), (1, 0), (d, d-1), \text{ or } (d-1, d)$ if $d$ is odd.

By symmetry, it suffices to consider only positive $s(I)$.

**Proposition** (Euler-Darboux equations)

(a) If $d$ is even, both $h_{0,0}$ and $h_{1,1}$ satisfy the equation:

$$\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+1}{2} \frac{1}{x_1 - x_2} \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) h_{i,i} = 0 \quad (i = 0, 1).$$

(b) If $d$ is odd, $h_{0,1}$ and $h_{1,0}$ satisfy the equations:

$$\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+2}{2} \frac{1}{x_1 - x_2} \frac{\partial}{\partial x_1} + \frac{d}{2} \frac{1}{x_1 - x_2} \frac{\partial}{\partial x_2} \right) h_{0,1} = 0$$

and

$$\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+2}{2} \frac{1}{x_1 - x_2} \frac{\partial}{\partial x_1} + \frac{d}{2} \frac{1}{x_1 - x_2} \frac{\partial}{\partial x_2} \right) h_{1,0} = 0,$$

respectively.

**Proof** If $d$ is even, we have

$$\frac{\partial}{\partial x_i} h_{1,1} = (x_i \frac{\partial}{\partial x_i} + \frac{d+1}{2})h_{0,0} \quad (i = 1, 2),$$

by setting $I = (0, 0)$ in the formula (iii) among the inductive equations of type I (Proposition (**.*)).

**Personal Memo to Recall:**

$$\frac{\partial}{\partial x_1} h_{I(1,1)} - \frac{j}{2} h_{I(1,-1)} - (x_1 \frac{\partial}{\partial x_1} + s(I) + \frac{i+1}{2})h_I = 0.$$
For $h_{1,1}$, we have
\[
\frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} = (x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2} \frac{\partial}{\partial x_2}) h_{0,0} = x_2 \left( x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2} \right) \left( \frac{\partial}{\partial x_2} h_{1,1} - \frac{d+1}{2} h_{0,0} \right).
\]
Multiply $x_2$ to the above formula to have
\[
x_2 \frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} = (x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2} \frac{\partial}{\partial x_2}) \left( \frac{\partial}{\partial x_2} h_{1,1} - \frac{d+1}{2} \frac{\partial}{\partial x_1} h_{1,1} \right) = x_1 \frac{\partial}{\partial x_1 \partial x_2} h_{1,1} - \frac{d+1}{2} (\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}) h_{1,1},
\]
which is the equation in question.

For $h_{0,0}$ we have
\[
\frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} = \left( \frac{\partial}{\partial x_1} (x_2 \frac{\partial}{\partial x_2} + \frac{d+1}{2} \frac{\partial}{\partial x_1}) + \frac{d+1}{2} \frac{\partial}{\partial x_2} \right) h_{0,0} = x_2 \left( \frac{\partial}{\partial x_1 \partial x_2} + \frac{d+1}{2} \frac{\partial}{\partial x_1} \right) h_{0,0}.
\]
Change the role of the variables $x_1, x_2$ to have a symmetric formula, and substract it from the original one. Then we have asymmetric equality:
\[
\left\{ (x_1 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2} \frac{\partial}{\partial x_2}) - (x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2} \frac{\partial}{\partial x_1}) \right\} h_{0,0} = 0,
\]
which is the desired equation.

Now we settle the case $d$ odd. Set $I = (0, -1)$ in (1.+), and $I = (-1, 0)$ in (2.+). Then
\[
\left( \frac{\partial}{\partial x_1} + \frac{d}{2} \frac{1}{x_1 - x_2} \right) h_{0,1} + \frac{d}{2} \frac{1}{x_1 - x_2} h_{1,0} = 0
\]
and
\[
\left( \frac{\partial}{\partial x_2} - \frac{d}{2} \frac{1}{x_1 - x_2} \right) h_{1,0} - \frac{d}{2} \frac{1}{x_1 - x_2} h_{0,1} = 0.
\]
Eliminate $h_{1,0}$ in the second formula by using the first one. Then we have the equation for $h_{0,1}$. Eliminate $h_{0,1}$ in the first formula by the second formula. The we have the other equation for $h_{1,0}$. (q.e.d)

5.2 Poisson equations for the peripheral and extremal entries

We deduce the other partial differential equations for the extremal $h_I$. We start with an equation valid for the peripheral entries, i.e. for $h_I$ with $i = 0$ (or $j = 0$ by symmetry).

Lemma If $I = (0, j)$ i.e. $i = 0$, we have an equation
\[
(\#)_j : \sum_{k=1}^{2} x_i (x_i - 1) \frac{\partial^2}{\partial x_i^2} + 2(x_1 - 1)x_2 \frac{\partial^2}{\partial x_1 \partial x_2} h_I
\]
\[ \sum_{k=1}^{2} \left\{ \frac{d-L}{2} + d-j+3 \right\} x_i - (s(I) + 1) \frac{\partial}{\partial x_i} h_I \]

\[ -(d+1) \frac{\partial}{\partial x_1} h_I + (\frac{d-j}{2}+1) (\frac{d-L}{2} + \frac{d-j}{2}+1) h_I = 0. \]

**Proof**  Set \( i = 0 \) in the formula (v) of the inductive equation (Proposition).

Then we have

\[ (x_2-1)(\frac{\partial}{\partial x_2} + \frac{d-j-1+c_2(I(0,2))}{x_2-1})h_{I(0,2)} \]

\[ -(x_1-1)(x_1 \frac{\partial}{\partial x_1} 2x_2 \frac{\partial}{\partial x_2} + 3s(I) + j + 1 + \frac{c_1(I)x_1}{x_1-x_2})h_I = 0. \]

Apply the operator \( \frac{\partial}{\partial x_1} \) to the above formula. Then, since the operator on the function \( h_{I(0,2)} \) depends only on the variable \( x_2 \), it commutes with \( \frac{\partial}{\partial x_1} \).

Recalling the inductive equation

\[ \frac{\partial}{\partial x_1} h_{I(0,2)} = -(x_2 \frac{\partial}{\partial x_2} + s(I))h_I, \]

we have an equation:

\[ \frac{\partial}{\partial x_1} \left\{ (x_1-1)(x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + 3s(I) + j + 1 + \frac{c_1(I)x_1}{x_1-1})h_I \right\} \]

\[ + \left\{ (x_2-1) \frac{\partial}{\partial x_2} + d-j-1+c_2(I(0,2)) \right\}(x_2 \frac{\partial}{\partial x_2} + s(I))h_I = 0. \]

Note here

\[ 3s(I) + j + 1 + c_1(I) = 2s(I) + j + 1 + \{ s(I) + c_1(I) \}, \]

\[ = 2s(I) + \frac{d-L}{2} + 1 = \frac{d-L}{2} + d-j+1. \]

Then the last equation is found to be the desired equation by direct computation. (q.e.d)

The following Poisson equations are obtained from the above lemma.

**Theorem**

(a) If \( d \) is even,

\[ \{ \sum_{k=1}^{2} x_k(x_k-1) \frac{\partial^2}{\partial x_k^2} + (d+1) \frac{x_1(x_1-1)}{x_1-x_2} \frac{\partial}{\partial x_1} - (d+1) \frac{x_2(x_2-1)}{x_1-x_2} \frac{\partial}{\partial x_2} \]

\[ - \sum_{k=1}^{2} \left\{ \frac{d-L}{2} + 2x_k - (\frac{d}{2}+1) \right\} \frac{\partial}{\partial x_k} + (d+1)(\frac{d-L}{2} + \frac{d}{2}+1) \} h_{0,0} = 0. \]

(b) If \( d \) is odd,

\[ \{ \sum_{k=1}^{2} x_k(x_k-1) \frac{\partial^2}{\partial x_k^2} + (d+2) \frac{x_1(x_1-1)}{x_1-x_2} \frac{\partial}{\partial x_1} - d \frac{x_2(x_2-1)}{x_1-x_2} \frac{\partial}{\partial x_2} \]
\[
\sum_{k=1}^{2}\left\{ \left( \frac{d-L}{2}+2k-2 \right)x_k - \left( \frac{d+2k-3}{2} \right) \right\} \frac{\partial}{\partial x_k} + \left( \frac{d-1}{2}+1 \right) \left( \frac{d-L}{2}+\frac{d-1}{2}+1 \right) h_{0,1} = 0.
\]

and
\[
\sum_{k=1}^{2}\left\{ \left( \frac{d-L}{2}+4-2k \right)x_k - \left( \frac{d+3-2k}{2} \right) \right\} \frac{\partial}{\partial x_k} + \left( \frac{d-1}{2}+1 \right) \left( \frac{d-L}{2}+\frac{d-1}{2}+1 \right) h_{1,0} = 0.
\]

**Proof** When \(d\) is even (resp. odd), set \(j=0\) (resp. \(j=1\)) in the formula \((\#)_{j}\) of the above lemma, and applying Euler-Darboux equation, replace the term \(2(x_1 - 1)x_2 \frac{\partial^2}{\partial x_1 \partial x_2}\) by
\[
(d+1+j) \frac{x_1(x_1 - 1)}{x_1 - x_2} \frac{\partial}{\partial x_1} - (d+1-j) \frac{x_1(x_1 - 1)}{x_1 - x_2} \frac{\partial}{\partial x_1}
\]
\[
= (d+1+j) \frac{x_1(x_1 - 1)}{x_1 - x_2} \frac{\partial}{\partial x_1} - (d+1-j) \frac{x_2(x_2 - 1)}{x_1 - x_2} \frac{\partial}{\partial x_1}
\]
\[
-(d+1+j) \frac{x_1 - 1}{x_1 - x_2} \frac{\partial}{\partial x_1} - (d+1-j) \frac{x_2}{x_1 - x_2} \frac{\partial}{\partial x_2}
\]

with \(j=0\) (resp. \(j=1\)). (q.e.d)

前回はこの節の方程式と、この”extremal entries”の解の級数表示・積分表示が可能であることを注意して終った。今回のノートの、”inductive equations”の部分が全く分からなかった。気が付けばコロンブスの卵である。
6 Solutions \( \{h_I\} \)

6.1 The solutions for the modified system of \( F_2 \)

We recall some basic facts on the regular solutions of the modified system of \( F_2 \) (cf.[Iida.§**]).

**Definition** We define an operator \( Q_{B_1,B_2} \), called *Euler-Darboux operator*, by

\[
Q_{B_1,B_2} = \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{B_2}{x_1 - x_2} \frac{\partial}{\partial x_1} + \frac{B_1}{x_1 - x_2} \frac{\partial}{\partial x_2}.
\]

Here \( B_1, B_2 \) are constants.

**Lemma** If a function \( h(x_1, x_2) \), analytic around the origin, satisfies an equation

\[
Q_{B_1,B_2} h = 0 \quad (B_1 > 0, B_2 > 0),
\]

it has a power series expansion of the form:

\[
h(x_1, x_2) = \sum_{m_1, m_2 \geq 0} \frac{(B_1)_{m_1}(B_2)_{m_2} \xi(m_1 + m_2)}{m_1! m_2!} x_1^{m_1} x_2^{m_2}.
\]

with some series \( \{\xi(k)\}_{k \in \mathbb{N}} \). Here we set

\[
(B)_k = \frac{\Gamma(B + k)}{\Gamma(B)}.
\]

Moreover, let \( F_h(z) \) be a function in one variable \( z \), regular around zero, defined by the restriction \( h \) to the diagonal:

\[
h(z, z) = \frac{\Gamma(B_1) \Gamma(B_2)}{\Gamma(B_1 + B_2)} F_h(z) = \sum_{k \geq 0} \frac{(B_1 + B_2)_k}{k!} \xi(k) z^k.
\]

Then the Eulerian integral formula for the beta function implies an integral

\[
h(x_1, x_2) = \int_0^1 F_h(t x_1 + (1 - t) x_2) t^{B_1-1} (1 - t)^{B_2-1} dt
\]

if the integral of the right hand side converges.

The modified \( F_2 \) system consists of the above \( Q_{B_1,B_2} \) and a Poisson operator \( P = P_{A,B_1,B_1,C;\lambda} \). The role of the Poisson operator is the following "intertwining property" for the series of the above type.

**Lemma** Assume that \( h \) satisfies the condition in the previous lemma \( :Q_{B_1,B_2} h = 0 \), and let \( F_h(z) \) be the associated power series of one variable defined there. Let

\[
P = \sum_{i=1}^i x_i (x_i - 1) \frac{\partial^2}{\partial x_i^2} + \{(A + B_1 - B_2 + 1)x_1 + B_2 - C + 2B_2 \frac{x_1(x_1 - 1)}{x_1 - x_2}\} \frac{\partial}{\partial x_1}
\]
\[ + \{(A - B_1 + B_2 + 1)x_2 + B_1 - C + 2B_1 \frac{x_2(x_2 - 1)}{x_1 - x_2} \} \frac{\partial}{\partial x_2} - \lambda. \]

Then \( h_1 = Ph \) also satisfies \( Q_{B_1, B_2} h_1 = 0 \) and for the operator

\[ L = z(z-1) \frac{d^2}{dz^2} - \{C - (A + B_1 + B_2 + 1)z\} \frac{d}{dz} - \lambda, \]

we have an interesting intertwining property:

\[ F_{Ph}(z) = L \cdot F_{h}(z). \]

Remark This is an analogy of Lemma (8.6) of [Iida, §8, p.***]. He formulated this for the integral expression of \( h \). But this expression requires extra-work to verify its convergence in the definite interval \([0, 1]\), strictly speaking. So we prefer to formulate in terms of (formal) power series.

Proof Since \( Q_{B_1, B_2} h = 0 \), we have an equality

\[ Ph = \{P + 2(x_1 - 1)x_2 Q_{B_1, B_2} - (x_1 - x_2) Q_{B_1, B_2}\} h. \]

Set

\[ P_0 h = \sum_{i=1}^{2} \sum_{j=1}^{2} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + (A + B_1 + B_2 + 1) \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i} \lambda, \]

and

\[ P_1 = -\sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} - (x_1 + x_2) \frac{\partial^2}{\partial x_1 \partial x_2} - C \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \]

\[ = -\left( \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i} + C \right) \cdot \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right). \]

Then by direct computation, we have an equality between operators:

\[ P + 2(x_1 - 1)x_2 Q_{B_1, B_2} - (x_1 - x_2) Q_{B_1, B_2} = P_0 + P_1. \]

The power series \( P_0 h \) is given by

\[ \sum \frac{(B_1)_{m_1} (B_2)_{m_2}}{m_1! m_2!} \{ (m_1 + m_2)^2 + (A + B_1 + B_2)(m_1 + m_2) - \lambda \} \xi(m_1 + m_2) x_1^{m_1} x_2^{m_2} \]

and \( P_1 h \) by

\[ -\sum \frac{(B_1)_{m_1} (B_2)_{m_2}}{m_1! m_2!} (m_1 + m_2 + B_1 + B_2)(m_1 + m_2 + C) \xi(m_1 + m_2 + 1) x_1^{m_1} x_2^{m_2}. \]

Hence \( h_1 = Ph \) is equal to

\[ \sum \frac{(B_1)_{m_1} (B_2)_{m_2}}{m_1! m_2!} \eta(m_1 + m_2) x_1^{m_1} x_2^{m_2}. \]
$\eta(k) = \{k^2 + (A + B_1 + B_2)k - \lambda\} \xi(k) - (k + B_1 + B_2)(k + C)\xi(k + 1)$ for each natural number $k$.

Now it is obvious that $Q_{B_1,B_2} h_1 = 0$ from the last power series expression of $h_1$, and for $F_{h_1}$ we have

$$\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1 + B_2)} F_{h_1}(z) = \sum_{k \geq 0} \frac{(B_1 + B_2)_k}{k!} \eta(k) z^k$$

by definition. In turn it is equal to the sum of

$$\sum \frac{B_1 + B_2}_k}{k!} \{k^2 + (A + B_1 + B_2)k - \lambda\} \xi(k) z^k = \{(z \frac{d}{dz} + (A + B_1 + B_2)(z \frac{d}{dz}) - \lambda\} \frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1 + B_2)} F_h(z)\}$$

and

$$- \sum \frac{B_1 + B_2}_k}{k!} (k + B_1 + B_2)(k + C)\xi(k + 1) z^k$$

$$= - \sum_{k = 0}^{\infty} \frac{(B_1 + B2)_{k+1}}{(k + 1)!} (k + 1 + C - 1)\xi(k + 1) \frac{d}{dz} (z^{k+1})$$

$$= - \frac{d}{dz} \{\sum_{n=0}^{\infty} (n + C - 1) \xi(n) z^n\}$$

$$= - \frac{d}{dz} (z^C + C - 1) \frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1 + B_2)} F_h(z).$$

Therefore, cancelling the same factor $\Gamma(B_1)\Gamma(B_2)/\Gamma(B_1 + B_2)$, we have

$$F_{h_1}(z) = \{(z \frac{d}{dz}^2 + (A + B_1 + B_2)(z \frac{d}{dz}) - \lambda - \frac{d}{dz} (z \frac{d}{dz} + C - 1)\} F_h(z),$$

as desired. (q.e.d)
6.2 The solutions for the extremal entries

Now we can describe the solutions for extremal entries \( H_{0,0} \) (when \( d \) is even) and \( h_{0,1}, h_{1,0} \) (when \( d \) is odd). We saw in the previous section (§§(∗∗)) that

\[
Q_{(d+1)/2,(d+1)/2} h_{0,0} = 0 \quad \text{and} \quad Q_{(d+1)/2,(d+1)/2} h_{1,1} = 0 \quad \text{for even} \quad d,
\]

and

\[
Q_{d/2,(d+2)/2} = 0 \quad \text{and} \quad Q_{(d+2)/2,d/2} h_{1,0} = 0 \quad \text{for odd} \quad d,
\]

respectively. Thus we have the following

**Lemma**

(i) If \( d \) is even, \( h_{0,0} \) and \( h_{1,1} \) are constant multiple of formal power series \( h_{0,0}^p \) and \( h_{1,1}^p \) of the form:

\[
h_{i,i}^p = \sum_{m_1 \geq 0, m_2 \geq 0} \frac{(B_1)_{m_1} (B_2)_{m_2}}{m_1! m_2!} \xi_{i,i} (m_1 + m_2) x_1^{m_1} x_2^{m_2} \quad (i = 0, 1)
\]

with \( B_1 = B_2 = \frac{d+1}{2} \), for some series \( \{\xi_{i,i}(k) \mid k \in \mathbb{N}\} \ (i = 0, 1) \).

(ii) If \( d \) is odd, \( h_{0,1} \) and \( h_{1,0} \) have power series expressions:

\[
h_{i,1-i}^p = \sum_{m_1, m_2 \geq 0} \frac{(B_1)_{m_1} (B_2)_{m_2}}{m_1! m_2!} \xi_{i,1-i} (m_1 + m_2) x_1^{m_1} x_2^{m_2} \quad (i = 0, 1)
\]

with \( B_1 = \frac{d}{2} + i, B_2 = \frac{d}{2} + 1 - i \), for some series \( \{\xi_{i,1-i}(k) \mid k \in \mathbb{N}\} \ (i = 0, 1) \).

Since \( h_{0,0} \) satisfies the Poisson equation

\[
Ph_{0,0} = 0
\]

with parameters

\[
A = \frac{d-L}{2} + 1, \quad B_1 = B_2 = (d+1)/2, \quad C = \frac{3}{2} d + 2, \quad \lambda = -(\frac{d}{2} + 1)(\frac{d-L}{2} + \frac{d}{2} + 1),
\]

the associated function \( F_{h_{0,0}}(z) \) should be a constant multiple of the hypergeometric series

\[
_{2}F_{1}(\frac{d}{2}, \frac{d-L}{2}; \frac{d}{2} + 1; \frac{3}{2} d + 2; z).
\]
6.3 The power series solutions for general $h_I$

We determine the (unique) power series expansion of the solutions of the holonomic system for $\{h_I|s(I) \geq 0\}$, which is regular at the origin.

**Theorem** The holonomic system for $\{h_I|d \equiv i + j(\text{mod } 2)\}$ given by Proposition (*,*,*) has a unique system of solutions regular at the origin $(x_1, x_2) = (0, 0)$, up to constant multiple. Moreover these power series expansions are given as follows:

(i) If $d$ is even, and $i + j \leq d$, for $I = (i, j)$

$$h_I(x_1, x_2) = c_0^i(-1)^{(i-j+d)/2} \sum_{m_1, m_2 \geq 0} (m_1!m_2!)^{-1} \left( \frac{d+1}{2} \right)_{m_1} \left( \frac{d+1}{2} \right)_{m_2} \prod_{\ell=1}^{[i/2]} (m_1 + s(I) + \ell) \prod_{\ell=1}^{[j/2]} (m_2 + s(I) + \ell) \xi_{00}(m_1 + m_2 - (i + j)/2) x_1^{m_1} x_2^{m_2}.$$  

Here

$$\xi_{00}(k) = \frac{\left( \frac{d+2}{2} \right)_k \left( \frac{d+1}{2} + \frac{d-L}{2} \right)_k}{\left( d + \frac{3}{2} \right)_k (d+1)_k} \quad \text{for } k \in \{-d/2, \infty\}.$$  

(ii) If $d$ is odd and $s(I) \geq 0$,

$$h_I(x_1, x_2) = c_1^i(-1)^{(i-j+d)/2} \sum_{m_1, m_2 \geq 0} (m_1!m_2!)^{-1} \left( \frac{d}{2} \right)_{m_1+1-[(j+1)/2]} \left( \frac{d}{2} \right)_{m_2+1-[(i+1)/2]} \prod_{\ell=1}^{[i/2]} (m_1 + s(I) + \ell) \prod_{\ell=1}^{[j/2]} (m_2 + s(I) + \ell) \xi_{01}(m_1 + m_2 - (i + j - 1)/2) x_1^{m_1} x_2^{m_2}.$$  

Here

$$\xi_{01}(k) = \frac{\left( \frac{d+1}{2} \right)_k \left( \frac{d+1}{2} + \frac{d-L}{2} \right)_k}{\left( d + \frac{1}{2} \right)_k (d+1)_k} \quad \text{for } k \in \{-d/2, \infty\}.$$  

Finally, $c_0^i, c_1^i$ are constants independent of $i, j$.

**Proof** We have to prepare two lemmas to show our Theorem. Let $\{h_I^P\}$ denotes the system of the power series defined by the right hand sides of our Theorem. Then, the first one claims that the initial values $h_I(0)$ coincide with the constants terms $h_I^P(0)$ of the power series solutions $h_I^P$. The second one claims that the system $\{h_I^P\}$ is a special solution of our holonomic system (Proposition).

These are the substantial works in the proof. We omit exact formulation of these lemmata in this review.
6.4 Integral expression of solutions

In this subsection, we give an integral expression of Euler type for the system of solutions for \( \{h_I\} \). It is deduced from the power series expression given in the previous subsection. The first step in this procedure is to represent \( h_I \)'s by simpler power series.

**Lemma** Assume that \( s(I) \geq 0 \). Let \( c_0' \) (resp. \( c_1' \)) be the constant defined in the previous Theorem for even (resp. odd) \( d \). We set for \( I = (i_1, i_2) \)

\[
H_I(x_1, x_2) = \sum_{m_1, m_2 \geq 0} \frac{1}{m_1!m_2!} x^{m_1} y^{m_2} \]

where

\[
\left\{ \begin{array}{l}
\frac{d+1}{2} m_1 - \frac{i_1}{2} \left( \frac{d+1}{2} m_2 - \frac{i_2}{2} \right) \zeta_{00} (m_1 + m_2 - \frac{1}{2} (i+j)) x_1^{m_1} x_2^{m_2} & \text{if } d \text{ is even} \\
\frac{d}{2} (m_1+1) - \frac{(i_1+1)}{2} \xi_{01} (m_1 + m_2 - \frac{1}{2} (i+j-1)) x_1^{m_1} x_2^{m_2} & \text{if } d \text{ is odd}
\end{array} \right.
\]

Then

\[
h_I = c_0' \times (x_1 x_2)^{-s(I)} \left( \frac{\partial}{\partial x_1} \right)^{[i_1/2]} \left( \frac{\partial}{\partial x_2} \right)^{[i_2/2]} \{(x_1 x_2)^{s(I)} x_1^{[i_1/2]} x_2^{[i_2/2]} H_I \}.
\]

**Proof** It suffices to check the corresponding statement termwise:

\[
(x_1 x_2)^{-s(I)} \left( \frac{\partial}{\partial x_1} \right)^{[i_1/2]} \left( \frac{\partial}{\partial x_2} \right)^{[i_2/2]} \{(x_1 x_2)^{s(I)} x_1^{[i_1/2]} x_2^{[i_2/2]} H_I \}
\]

\[
= \prod_{t=1}^{[i_1/2]} \left( m_1 + s(I) + t \right) \prod_{t=1}^{[i_2/2]} \left( m_2 + s(I) + t \right) x_1^{m_1} x_2^{m_2}
\]

(q.e.d)

The next step is to find an integral expression for \( H_I \). Note that now it satisfies Euler-Darboux equations with

\[
B_1 = (d+1)/2 - [i_2/2], B_2 = (d+1)/2 - [i_1/2] \text{ if } d \text{ is even.}
\]

\[
B_1 = (d+2)/2 + [-i_2/2], B_2 = (d+2)/2 + [-i_1/2] \text{ if } d \text{ is odd.}
\]

**Theorem** We have the following integral expression for \( \{H_I\} \):

(a) The case of even \( d \): (i) If both \( i_1 \) and \( i_2 \) are even,

\[
H_I(x_1, x_2) = H_I(0) \frac{\Gamma(d+1-i_1+i_2)}{\Gamma(d+1-i_1)\Gamma(d+1-i_2)} x_1^{i_1} x_2^{i_2}
\]

\[
\int_0^1 2F_1(s(I)+1, \frac{d+3}{2}; d+1; s(I)+1; \frac{d+3}{2} + s(I); t x_1 + (1-t) x_2) t^{d+1-i_2-1} (1-t)^{d+1-i_1-1} dt.
\]

Here the constant \( H_I(0) \) is given by

\[
H_I(0) \frac{\Gamma(d+1-i_1+i_2)}{\Gamma(d+1-i_1)\Gamma(d+1-i_2)} = c_0' (-1)^{(i-j+d)/2} \frac{\Gamma(d+1-i_1+i_2)}{\Gamma(d+1)\Gamma(d+1)} \xi_{00}(\frac{i_1+i_2}{2})
\]
\[
\begin{align*}
&= c_0'(-1)^{(i-j+d)/2} \frac{\Gamma(d+1)}{\Gamma(d+1/2)^2} \frac{\Gamma(d+1-i_1-i_2)}{\Gamma(d+1/2)^2} \\
&\quad \times \int_0^1 G(tx_1 + (1-t)x_2) t^{\frac{d+3-i_2}{2}-1} (1-t)^{\frac{d+3-i_1}{2}-1} dt.
\end{align*}
\]

where
\[
G(z) = \left( \frac{d}{dz} + d + 1 - \frac{i_1 + i_2}{2} \right) _2F_1(s(I) + 1, \frac{1}{2}(d-L) + s(I) + 1; \frac{d+3}{2} + s(I); z).
\]

Moreover the constant \( H_I(0) \) is given by
\[
H_I(0) \frac{\Gamma(d+2 - \frac{i_1+i_2}{2})}{\Gamma(d+2-i_1)} \frac{\Gamma(d+1-i_1-i_2)}{\Gamma(d+1/2)^2} = c_0'(-1)^{(i-j+d)/2} \frac{\Gamma(d+1)}{\Gamma(d+1/2)^2} \xi_00(-\frac{i_1+i_2}{2})
\]

(b) The case of odd \( d \):
\[
\begin{align*}
&= c_0'(-1)^{(i-j+d)/2} \frac{\Gamma(d+1)}{\Gamma(d+1/2)^2} \frac{\Gamma(d+1-i_1-i_2)}{\Gamma(d+1/2)^2} \\
&\quad \times \int_0^1 _2F_1(s(I) + 1, \frac{1}{2}(d-L) + s(I) + 1; \frac{d+2}{2} + s(I); tx_1 + (1-t)x_2) \\
&\quad \quad \times t^{\frac{d+1-i_2}{2}-1} (1-t)^{\frac{d+1-i_1}{2}-1} dt. \quad \text{(if } i_1 \text{ even, } i_2 \text{ odd)}
\end{align*}
\]

The constant \( H_I(0) \) is given by
\[
H_I(0) \frac{\Gamma(d + \frac{3}{2} - \frac{i_1+i_2}{2})}{\Gamma(d + \frac{3}{2} - \frac{i_1+i_2}{2})} = c_0'(-1)^{(i-j+d)/2} \frac{\Gamma(d+3/2 - \frac{i_1+i_2}{2})}{\Gamma(d+3/2 - \frac{i_1+i_2}{2})} \xi_{01}(-\frac{i_1+i_2}{2} + \frac{1}{2})
\]