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THE EXTENT OF STRENGTH IN THE CLUB FILTERS

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1. Introduction

This paper gives a number of partial results towards the following conjectures. Unless otherwise noted, κ is a regular, uncountable cardinal and λ is an infinite cardinal ($\lambda \geq \kappa$).

Conjecture 1. *The club filter on $\mathcal{P}_\kappa\lambda$ is not precipitous — unless λ is regular.*

Conjecture 2. *The club filter on $\mathcal{P}_\kappa\lambda$ is not pre-saturated — unless $\kappa = \aleph_1$ and λ is regular or $\kappa = \lambda$ is weakly inaccessible.*

The corresponding conjecture for saturation has been established by Foreman and Magidor:

Theorem (Foreman-Magidor). *The club filter on $\mathcal{P}_\kappa\lambda$ is not saturated — unless $\kappa = \lambda = \aleph_1$.*

The results of section 2 of this paper are the authors partial results towards the above theorem. Shortly after the results of this paper were announced, Foreman and Magidor proved the above theorem. Their proof does not use any of the results of this paper, and in fact in the case covered by Theorem 2.10, they establish the stronger result that the club filter is not even λ^{++} saturated.

Remarks.
1. [She87] It is consistent that the club filter on $\aleph_1$ is saturated (assuming the consistency of a Woodin cardinal).
2. [Git95] It is consistent that the club filter on κ, κ weakly inaccessible, is pre-saturated (assuming the consistency of an up-repeat point).
3. [Gol92] If δ is Woodin then for every regular λ ($\aleph_1 \leq \lambda < \delta$),
$$V^{Col(\lambda,<\delta)} \models \text{“the club filter on } \mathcal{P}_{\aleph_1}\lambda \text{ is pre-saturated”}.$$
4. [Gol] If δ is Woodin then for every regular $\kappa < \lambda$ ($\aleph_1 \leq \kappa \leq \lambda < \delta$),
$$V^{Col(\lambda,<\delta)} \models \text{“the club filter on } \mathcal{P}_{\kappa}\lambda \text{ is precipitous”}.$$

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We now give our basic definitions and conventions.
$\mathcal{F}$ is a *normal filter* on $\mathcal{P}(\lambda)$ if

1. $\mathcal{F} \subseteq \mathcal{PP}(\lambda)$ is a filter.
2. (fine) $\forall \alpha \in \lambda \{ a \subseteq \lambda \mid \alpha \in a \} \in \mathcal{F}$.
3. (normal) If $C_\alpha \in \mathcal{F}$ $(\alpha \in \lambda)$, then $\{ a \subseteq \lambda \mid \forall \alpha \in a (a \in C_\alpha) \} \in \mathcal{F}$.

Throughout this paper, *filter* will mean normal filter.
$\mathcal{F}^+ =_{\text{def}} \{ A \subseteq \mathcal{P}(\lambda) \mid \forall C \in \mathcal{F} \ (C \cap A \neq \varnothing) \}$. $\mathcal{F}^+$ has an associated partial ordering: $A \leq B$ iff $A \subseteq B$.

A filter $\mathcal{F}$ on $\mathcal{P}(\lambda)$ is *saturated* if every antichain in $\mathcal{F}^+$ has size $\leq \lambda$. $\mathcal{F}$ is *pre-saturated* if given antichains $\mathcal{A}_\alpha$ $(\alpha < \lambda)$ and $S \in \mathcal{F}^+$, there is a $T \leq S$ such that for all $\alpha < \lambda$, $|\{ A \in \mathcal{A}_\alpha \mid A \cap T \in \mathcal{F}^+ \}| \leq \lambda$.

Forcing with $\mathcal{F}^+$ extends $\mathcal{F}$ to a V-normal, V-ultrafilter $\mathcal{G}$—so we get a generic embedding $j : V \to \text{Ult}(V, \mathcal{G}) \subseteq V[\mathcal{G}]$.

$\mathcal{F}$ is *precipitous* if this ultrapower is always well-founded. If $\mathcal{F}$ is pre-saturated, then $\mathcal{F}$ is precipitous and the ultrapower is closed under λ sequences in $V[\mathcal{G}]$. For more on the basic facts about generic embeddings see [For86].

The club filter on $\mathcal{P}(\lambda)$ ($\text{CF}_{\mathcal{P}(\lambda)}$ or just CF) consists of all $A \subseteq \mathcal{P}(\lambda)$ such that $\exists f : \lambda^{<\omega} \to \lambda$ with $\text{cl}_f \subseteq A$ ($\text{cl}_f = \{ a \subseteq \lambda \mid f''a^{<\omega} \subseteq a \}$). Sets in CF^+ are called stationary. CF is the smallest normal filter on $\mathcal{P}(\lambda)$.

If $S \in \mathcal{F}^+$, then $\mathcal{F} \restriction S =_{\text{def}} \{ A \subseteq \mathcal{P}(\lambda) \mid (\exists C \in \mathcal{F}) \ C \cap S \subseteq A \}$ is a normal filter. If $S \in \text{CF}^+$, then the club filter on S, $\text{CF} \restriction S$, is the smallest normal filter on $\mathcal{P}(\lambda)$ containing S.

$\mathcal{P}_\kappa\lambda =_{\text{def}} \{ a \subseteq \lambda \mid |a| < \kappa \ \& \ a \cap \kappa \in \kappa \}$. This definition is slightly non-standard: usually the condition "$a \cap \kappa \in \kappa$" is dropped. The set $\mathcal{P}_\kappa\lambda$ is stationary in $\mathcal{P}(\lambda)$. If $\mathcal{F}$ is a filter on $\mathcal{P}(\lambda)$ and $\mathcal{P}_\kappa\lambda \in \mathcal{F}$, then $\mathcal{F}$ is κ-complete, and so $\forall s \in \mathcal{P}_\kappa\lambda$, $\{ a \in \mathcal{P}_\kappa\lambda \mid s \subseteq a \} \in \mathcal{F}$.

If $a \subseteq \text{Ord}$, then $\text{cof}(a)$ is the cofinality of the order type of a. A $\diamond_{\kappa,\lambda}$ sequence is a set $\langle s_a \subseteq a : a \in \mathcal{P}_\kappa\lambda \rangle$ such that for all $A \subseteq \lambda$, $\{ a \in \mathcal{P}_\kappa\lambda \mid a \cap A = s_a \}$ is stationary.

The following fact was proved in [BTW77] for filters on cardinals. A similar proof works here.

Fact 1.1. *Assume $\mathcal{F}$ is a filter on $\mathcal{P}(\lambda)$. $\mathcal{F}$ is saturated iff for all filters $\mathcal{G} \supseteq \mathcal{F}$, $\exists S \in \mathcal{F}^+$ such that $\mathcal{G} = \mathcal{F} \restriction S$.*

Corollary 1.2. *Suppose the club filter on S is saturated. Then every filter on S is saturated.*

2. Saturation

One of the first results about the failure of saturation is a theorem of Shelah ([She82], p. 440) that says, for example, if $\mathcal{F}$ is a saturated filter on ω_2, then $\{ \alpha < \omega_2 \mid \operatorname{cof}(\alpha) = \omega_1 \} \in \mathcal{F}$. The proof of this uses the following result (with $\lambda = \omega_2$). We also use this result to get similar facts about saturated filters on $\mathcal{P}_\kappa\lambda$.

Theorem 2.1. ([She82],[Cum97]) *Assume $V \subseteq W$ are inner models of ZFC, λ is a cardinal of V, ρ is a cardinal of W, and $\lambda^+_V = \rho^+_W$. Assuming (*), $W \models \operatorname{cof}(\lambda) = \operatorname{cof}(\rho)$.*

(*) *λ is regular, or (λ is singular and) there is a good scale on λ, or (λ is singular and) W is a λ^+-cc forcing extension of V.*

See the next section for the definition of good scale.

Definition 2.2. $S_\lambda =_{def} \{ a \subseteq \lambda \mid \operatorname{cof}(a) = \operatorname{cof}(|a|) \}$.

Theorem 2.3. *Assume $\mathcal{F}$ is a saturated filter on $\mathcal{P}(\lambda)$. Then $S_\lambda \in \mathcal{F}$.*

Proof. Suppose not. So we get $j : V \to M \subseteq V[G]$ with $\mathcal{P}(\lambda) \setminus S_\lambda \in G$. Since $\mathcal{P}(\lambda) \setminus S_\lambda \in G$, $M \models \operatorname{cof}(\lambda) \neq \operatorname{cof}(|\lambda|)$. Since $M^\lambda \subseteq M$ in $V[G]$, $V[G] \models \operatorname{cof}(\lambda) \neq \operatorname{cof}(|\lambda|)$. This contradicts Theorem 2.1 since $V[G]$ is a λ^+-cc generic extension of V. □

Lemma 2.4. *Assume $\kappa = \rho^+$, $\operatorname{cof}(\lambda) < \kappa$, and $\operatorname{cof}(\lambda) \neq \operatorname{cof}(\rho)$. Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is non-stationary.*

Proof. Let $a \in \mathcal{P}_\kappa\lambda$. On a club, $|a| = \rho$ and so $\operatorname{cof}(|a|) = \operatorname{cof}(\rho)$. Since $\operatorname{cof}(\lambda) < \kappa$, on a club $\sup(a) = \lambda$ and so $\operatorname{cof}(a) = \operatorname{cof}(\lambda)$. Therefore $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is non-stationary. □

Corollary 2.5. *For κ, λ as above, there is no saturated filter on $\mathcal{P}_\kappa\lambda$.*

Remark. If $\kappa = \rho^+$ and $\operatorname{cof}(\lambda) = \operatorname{cof}(\rho)$, then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$.

Lemma 2.6. *Assume $\kappa = \rho^+ \geq \aleph_2$ and $\operatorname{cof}(\lambda) \geq \kappa$. Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary, co-stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Let $f : \lambda^{<\omega} \to \lambda$. We may assume $a \in \mathcal{P}_\kappa\lambda \cap \mathrm{cl}_f$ implies $\operatorname{cof}(|a|) = \operatorname{cof}(\rho)$. For any regular $\delta < \kappa$ we can build a continuous increasing chain of length δ to find $a \in \mathcal{P}_\kappa\lambda$ closed under f with $\operatorname{cof}(a) = \delta$. Taking $\delta = \operatorname{cof}(\rho)$ shows that $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary. Taking $\delta \neq \operatorname{cof}(\rho)$ shows that $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$. □

Corollary 2.7. *For κ, λ as above, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Remark. If $\kappa = \aleph_1$, then for all $\lambda \geq \kappa$, $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$.

Lemma 2.8. *Assume κ is a regular limit cardinal and* $\mathrm{cof}(\lambda) \neq \kappa$. *Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary, co-stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Let $f : \lambda^{<\omega} \to \lambda$ and $\rho < \kappa$ a regular cardinal. It is easy to find $a \in \mathcal{P}_\kappa\lambda \cap \mathrm{cl}_f$ such that $|a| = |a \cap \kappa|$ and $\mathrm{cof}(|a \cap \kappa|) = \rho$ and, if $\mathrm{cof}(\lambda) > \kappa$, $\mathrm{cof}(a) = \rho$ (if $\mathrm{cof}(\lambda) < \kappa$, then for club many $a \in \mathcal{P}_\kappa\lambda$, $\mathrm{cof}(a) = \mathrm{cof}(\lambda)$). Hence $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary (take $\rho = \mathrm{cof}(\lambda)$ if $\mathrm{cof}(\lambda) < \kappa$). If $\mathrm{cof}(\lambda) < \kappa$ then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$—take $\rho \neq \mathrm{cof}(\lambda)$. Finally, assume $\mathrm{cof}(\lambda) > \kappa$. The idea for the following argument is from [Bau91]. Let $\delta = \mathrm{cof}(\lambda)$. Note that $\{ a \in \mathcal{P}_\kappa\lambda \mid \mathrm{cof}(a) = \mathrm{cof}(a \cap \delta) \}$ is club, so we may assume f witnesses this. Let $\bar{f} : \delta^{<\omega} \to \delta$ be such that $a \in \mathrm{cl}_{\bar{f}}$ implies $\mathrm{cl}_f(a) \cap \delta = a$. Define $g : \delta \to \delta$ by $g(\alpha) = \sup(\mathrm{cl}_{\bar{f}}(\alpha + 1))$. Now choose $a \in \mathcal{P}_\kappa\delta$ such that $a \in \mathrm{cl}_{\bar{f}}$, $a \in \mathrm{cl}_g$, $|a| = |a \cap \kappa|$, $\mathrm{cof}(|a \cap \kappa|) = \aleph_1$, $\mathrm{cof}(a) = \aleph_1$, and $\kappa \in a$. Let $a_0 = a \cap \kappa$. Given a_n, let $\beta \in a \setminus \sup(a_n)$, and $a_{n+1} = \mathrm{cl}_{\bar{f}}(a_n \cup \{\beta\})$. Let $a_\omega = \bigcup_{n\in\omega} a_n$. Then $a_\omega \cap \kappa = a \cap \kappa$, $a_\omega \in \mathrm{cl}_{\bar{f}}$ and $\mathrm{cof}(a_\omega) = \omega$. Let $b = \mathrm{cl}_f(a_\omega)$. Then $b \in \mathrm{cl}_f$ and $\mathrm{cof}(|b|) = \aleph_1$ and $\mathrm{cof}(b) = \omega$. Hence $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$. □

Corollary 2.9. *For κ, λ as above, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Remark. Assume $\mathrm{cof}(\lambda) = \kappa$ (κ regular limit). Then for club many $a \in \mathcal{P}_\kappa\lambda$, $\mathrm{cof}(a) = \mathrm{cof}(a \cap \kappa)$. So S_λ is club in the (stationary) set $\{ a \in \mathcal{P}_\kappa\lambda \mid |a| = |a \cap \kappa| \}$ and is non-stationary in the (possibly non-stationary) set $\{ a \in \mathcal{P}_\kappa\lambda \mid |a| = |a \cap \kappa|^+ \}$.

The above method does not handle the cases: (i) $\kappa = \aleph_1$, (ii) $\kappa = \rho^+$ and $\mathrm{cof}(\lambda) = \mathrm{cof}(\rho)$, and (iii) κ regular limit and $\mathrm{cof}(\lambda) = \kappa$. Case (ii) is handled in the following:

Theorem 2.10. *Assume* $\mathrm{cof}(\lambda) < \kappa$ *and* $\kappa \geq \aleph_2$. *Then the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Proof. Let $\langle f_\alpha : \alpha \in \lambda^+ \rangle$ be a scale on λ (see definition 3.3, so each $f_\alpha \in \Pi_{\xi<\mathrm{cof}(\lambda)}\rho_\xi$, where the ρ_ξ's are an increasing sequence of regular cardinals cofinal in λ with $\kappa < \rho_0$). Given $a \in \mathcal{P}_\kappa\lambda$ define $g_a \in \Pi\rho_\xi$ by $g_a(\xi) = \sup(a \cap \rho_\xi)$ and let $\pi(a) =$ least $\alpha \in \lambda^+$ such that $g_a \leq^* f_\alpha$. Let $\mathcal{F}$ be a filter on $\mathcal{P}_\kappa\lambda$. Let $\theta >> \lambda$, and assume $\mathcal{F}_\theta$ is a filter on $\mathcal{P}_\kappa H_\theta$ projecting to $\mathcal{F}$. Let

$$E = \{ b \prec H_\theta \mid b \in \mathcal{P}_\kappa H_\theta \;\&\; \langle f_\alpha : \alpha \in \lambda^+ \rangle \in b \;\&\; \mathrm{cof}(\lambda) \subseteq b \;\&\; \langle \rho_\xi : \xi < \mathrm{cof}(\lambda) \rangle \in b \}.$$

Claim 1. If $b \in E$ then $\sup(b \cap \lambda^+) \leq \pi(b \cap \lambda)$.

Suppose not. Let $b \in E$ with $\sup(b \cap \lambda^+) > \pi(b \cap \lambda) =_{\text{def}} \xi$. Say $\beta \in b \cap \lambda^+$ with $\beta > \xi$. But $f_\beta{}^* > f_\xi{}^* \geq g_{b \cap \lambda}$. Therefore $\exists \eta < \text{cof}(\lambda)$ such that $f_\beta(\eta) > f_\alpha(\eta) \geq g_{b\cap\lambda}(\eta)$. But $\text{cof}(\lambda) \subseteq b$, $\beta \in b$, and $\langle f_\alpha : \alpha \in \lambda^+ \rangle \in b$. Therefore $f_\beta(\eta) \in b \cap \rho_\eta$. But $g_{b\cap\lambda}(\eta) = \sup(b \cap \rho_\eta)$. Contradiction

Let $T \in \mathcal{F}_\theta$ and define $S_T(a) = \sup\{ \sup(b \cap \lambda^+) \mid b \cap \lambda = a \;\&\; b \in T \}$. Note that S_T is defined on a set in $\mathcal{F}$ (the projection of T), if $T \subseteq T'$ then $S_T(a) \leq S_{T'}(a)$, and if $T \subseteq E$ then $S_T(a) \leq \pi(a)$.

Claim 2. Given $\beta \in \lambda^+$ and $T \subseteq E$ with $T \in \mathcal{F}_\theta$, on an $\mathcal{F}$ measure one set we have $\beta < S_T(a) \leq \pi(a) < \lambda^+$.

We already have that S_T is defined on an $\mathcal{F}$ measure one set and $S_T(a) \leq \pi(a) < \lambda^+$. Let $\beta \in \lambda^+$ and let $T' = \{ b \in T \mid \beta \in b \}$. Then $T' \in \mathcal{F}_\theta$ and (on the projection of T') $S_{T'}(a) > \beta$ and $S_{T'}(a) \leq S_T(a)$.

Claim 3. Assume $f : \mathcal{P}_\kappa\lambda \to \lambda^+$ is such that $\Vdash_{\mathcal{F}^+}[f] = \sup j''\lambda^+$. Then there is a $T \in \mathcal{F}_\theta$ such that $(\forall T' \subseteq T)$ $T' \in \mathcal{F}_\theta$ on a set in $\mathcal{F}$, $S_{T'}(a) = f(a)$.

On an $\mathcal{F}_\theta$ measure one set $f(b \cap \lambda) \geq \sup(b \cap \lambda^+)$ (If not, then there is a $S \in \mathcal{F}_\theta^+$ such that $f(b \cap \lambda) < \sup(b \cap \lambda^+)$, so on some $S' \in \mathcal{F}_\theta^+$, $f(b \cap \lambda) < \eta$ ($\eta \in \lambda^+$ is fixed). Projecting to $\mathcal{F}$ we get $\bar{S} \in \mathcal{F}^+$ such that $f(b) < \eta$ on $\bar{S}$. Contradiction.)

So let $T \in \mathcal{F}_\theta$ such that $T \subseteq E$ and $(\forall b \in T)$ $f(b \cap \lambda) \geq \sup(b \cap \lambda^+)$. Therefore, on an $\mathcal{F}$ measure one set, $f(a) \geq S_T(a)$. Suppose on $S \in \mathcal{F}^+$ we have $f(a) > S_T(a)$. Then since $\Vdash[f] = \sup j''\lambda^+$, there exists $\bar{S} \subseteq S$ and $\eta < \lambda^+$ such that on $\bar{S}$, $S_T(a) \leq \eta$. This contradicts Claim 2. Finally, assume $T' \subseteq T$. Then on $\mathcal{F}$ measure one set $S_{T'}(a) \leq S_T(a) = f(a)$. Again by Claim 2, $S_{T'}(a) = f(a)$ on an $\mathcal{F}$ measure one set.

Claim 4. Assume $\rho < \kappa$ is regular, $\rho \neq \text{cof}(\lambda)$, $T \subseteq \mathcal{P}_\kappa H_\theta$ is stationary, $T \subseteq E$, and $\forall a \in T$, a is IA (internally approachable) of length ρ (this means there is an increasing, continuous sequence $\langle a_\xi : \xi < \rho \rangle$ where each $a_\xi \in E$, $\forall \rho' < \rho$ $\langle a_\xi : \xi < \rho' \rangle \in a$, and $a = \bigcup_{\xi<\rho} a_\xi$ — see [FMS88]). Let $\bar{T}$ be the projection of T to $\mathcal{P}_\kappa\lambda$. Then for all $a \in \bar{T}$ $S_T(a) = \pi(a)$ and $\text{cof}(\pi(a)) = \rho$.

The idea for the proof of Claim 4 comes from [FM97]. Let $b \in T$, and $\langle b_\alpha : \alpha < \rho \rangle$ be a witness to IA of length ρ. We may assume $(\forall \alpha \in \rho)$ $b_\alpha \in b_{\alpha+1}$. Let $a = b \cap \lambda$. It is enough to see that $\sup(b \cap \lambda^+) = \pi(a)$. (Note that $\text{cof}(\sup(b \cap \lambda^+)) = \rho$.) Given $\alpha < \rho$

we have $(\forall f \in b_\alpha)\ f < g_{b_\alpha}$ (everywhere) and since $g_{b_\alpha} \in b_{\alpha+1}$, there is $\gamma_\alpha \in b_{\alpha+1}$ such that $g_{b_\alpha} \leq^* f_{\gamma_\alpha}$. By Claim 1, $\pi(a) \geq \sup(b \cap \lambda^+)$. So let $\delta = \sup(b \cap \lambda^+)$ and we will show $g_b \leq^* f_\delta$. For all $\alpha < \rho$, $g_{b_\alpha} \leq^* f_{\gamma_\alpha} <^* f_\delta$. Since $\rho \neq \mathrm{cof}(\lambda)$, $\exists A \subseteq \rho$ unbounded and $\nu < \mathrm{cof}(\lambda)$ such that $\forall \alpha \in A$ and $\forall \xi \in (\nu, \mathrm{cof}(\lambda))$ $g_{b_\alpha}(\xi) < f_\delta(\xi)$. But $g_b(\xi) = \sup_{\alpha \in A} g_{b_\alpha}(\xi)$ and so $g_b(\xi) \leq f_\delta(\xi)$.

Let $\rho < \kappa$ be regular, $\rho \neq \mathrm{cof}(\lambda)$.
Let $T = \{\, b \in E \mid b \text{ is IA of length } \rho \,\}$.

Claim 5. T is stationary.

Let $f\colon H_\theta^{<\omega} \to H_\theta$. Let $b_0 \in E \cap cl_f$. If $\xi < \rho$ is limit let $b_\xi = \bigcup_{\varepsilon<\xi} b_\varepsilon$. Given b_ξ, let $b_{\xi+1} \in E \cap cl_f$ such that $b_\xi \cup \{\langle b_\varepsilon : \varepsilon \leq \xi\rangle\} \in b_{\xi+1}$. So $b = \bigcup_{\varepsilon<\rho} b_\varepsilon \in E \cap cl_f$. To see b is IA of length ρ we just need $\forall \xi < \rho\ \langle b_\alpha : \alpha \in \xi\rangle \in b$. But $\langle b_\alpha : \alpha \in \xi\rangle \in b_{\xi+1} \subseteq b$.

Finally, let $\mathcal{F}_\theta = \mathrm{CF} \restriction T$. $\mathcal{F}$ is gotten by projection. We will show that $\mathcal{F}$ is not saturated, and therefore by Corollary (1.2) the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.

For a contradiction, assume $\mathcal{F}$ is saturated. So there is an $f\colon \mathcal{P}_\kappa\lambda \to \lambda^+$ such that $\Vdash [f] = \sup j''\lambda^+$, and on a set in $\mathcal{F}$, $\mathrm{cof}(f(a)) > \rho$ (otherwise we could force to have $\mathrm{cof}([f]) \leq \rho$ in the ultrapower—so this collapses λ^+).

By Claim 3, $\exists R \in \mathcal{F}_\theta$ such that for any $R' \subseteq R$ $(R' \in \mathcal{F}_\theta)$ on a set in $\mathcal{F}$, $S_{R'}(a) = f(a)$. So on a set in $\mathcal{F}$, $S_{R\cap T}(a) = f(a)$. But $R \cap T$ is a set as in Claim 4. Hence on a set in $\mathcal{F}$ (the projection of $R \cap T$) $S_{R\cap T}(a) = \pi(a)$ and $\mathrm{cof}(\pi(a)) = \rho$. Therefore on a set in $\mathcal{F}$, $\mathrm{cof}(f(a)) = \rho$. This contradiction completes the proof. □

Question. In the above proof, $\mathcal{F}$ is the projection of $\mathrm{CF} \restriction T$. Is $\mathcal{F}$ the club filter restricted to a stationary set?

We conclude this section with three previously known theorems.

Theorem 2.11. ([GS97]) *For all $\kappa > \aleph_1$, the club filter on κ is not saturated. In fact, for any regular ρ with $\rho^+ < \kappa$,* $\mathrm{CF} \restriction \{\, \alpha < \kappa \mid \mathrm{cof}(\alpha) = \rho \,\}$ *is not saturated.*

Corollary 2.12. *For all regular κ and all regular $\lambda \geq \aleph_2$, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Proof. Define $g : \mathcal{P}_\kappa\lambda \to \lambda$ by $g(a) = \sup(a)$. Suppose $S \subseteq \lambda$ is stationary and $(\forall \alpha \in S)\ \mathrm{cof}(\alpha) < \kappa$. Then $g^{-1}(S)$ is stationary (let $f\colon \lambda^{<\omega} \to \lambda$ and choose $\alpha \in S$ such that α is closed under f. Now build $a \in \mathcal{P}_\kappa\lambda \cap \mathrm{cl}_f$ such that $\sup(a) = \alpha$). Also, if $S \subseteq \mathcal{P}_\kappa\lambda$ is stationary, then $g''S \subseteq \lambda$ is stationary (if $f\colon \lambda^{<\omega} \to \lambda$, define $h(\alpha) = \mathrm{cl}_f(\alpha + 1)$.

If $a \in S \cap \mathrm{cl}_h$, then $\sup(a)$ is closed under f. The result now follows from Theorem 2.11. □

Theorem 2.13. [DM93] *If $\lambda > 2^{<\kappa}$ then $\diamond_{\kappa,\lambda}$ holds. Hence the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Theorem 2.14. [BT82] *For any $\lambda > \aleph_1$, $\mathcal{P}_{\aleph_1}\lambda$ can be split into 2^ω many disjoint stationary sets.*

Remark. Piecing everything together, we have the following partial results towards the theorem of Foreman and Magidor: The club filter on $\mathcal{P}_\kappa\lambda$ is not saturated unless

1. $\kappa = \lambda = \aleph_1$ (consistent).
2. $\kappa = \aleph_1$, $\lambda = 2^\omega$ is singular.
3. κ is limit and $\mathrm{cof}(\lambda) = \kappa$ and $2^{<\kappa} \geq \lambda$.

3. Cardinal preserving to pre-saturation

A filter $\mathcal{F}$ on $\mathcal{P}(\lambda)$ is *weakly pre-saturated* if $\mathcal{F}$ is precipitous and $\Vdash_{\mathcal{F}^+}$ " λ^+ is preserved". The filter $\mathcal{F}$ is called *cardinal preserving* if $\Vdash_{\mathcal{F}^+}$ " λ^+ is preserved". If $|\mathcal{F}^+| = \lambda^+$, then pre-saturated, weakly pre-saturated and cardinal preserving are all equivalent. It is not known if they are equivalent in general.

We use a number of known combinatorial principles to get that the club filter cannot have these strong properties. For the case λ regular, the solution is complete—the club filter on $\mathcal{P}_\kappa\lambda$ is not cardinal preserving unless $\kappa = \aleph_1$ or $\kappa = \lambda$ is weakly inaccessible (and both these cases are consistent).

Definition 3.1. *$Sh(\lambda)$ means for any $\mathbb{P} \in V$, if $V^{\mathbb{P}} \models \mathrm{cof}(\lambda) \neq \mathrm{cof}(|\lambda|)$, then $V^{\mathbb{P}}$ collapses λ^+_V.*

Definition 3.2. *$AD(\lambda)$ means $\exists\langle a_\alpha : \alpha \in \lambda^+\rangle$ such that each a_α is an unbounded subset of λ and $\forall\alpha \in \lambda^+ \, \exists f_\alpha \colon \alpha \to \lambda$ such that $\beta_1 < \beta_2 < \alpha$ implies $[a_{\beta_1} \setminus f_\alpha(\beta_1)] \cap [a_{\beta_2} \setminus f_\alpha(\beta_2)] = \emptyset$.*

Definition 3.3. *Suppose λ is singular. A* scale *on λ is an increasing sequence of regular cardinal $\langle\rho_\xi : \xi \in \mathrm{cof}(\lambda)\rangle$ cofinal in λ, and a sequence $\langle f_\alpha : \alpha \in \lambda^+\rangle$ such that for each α, $f_\alpha \in \Pi_{\xi\in\mathrm{cof}(\lambda)}\rho_\xi$, $\alpha < \alpha'$ implies $f_\alpha <^* f_{\alpha'}$, and $\forall f \in \Pi_{\xi\in\mathrm{cof}(\lambda)}\rho_\xi \; \exists\alpha \in \lambda^+$ such that $f <^* f_\alpha$. We will assume $\langle\rho_\alpha : \alpha \in \mathrm{cof}(\lambda)\rangle$ is discontinuous everywhere and $\forall\alpha \in \lambda^+ \; \forall\xi \in \mathrm{cof}(\lambda) \; f_\alpha(\xi) > \sup\{ \rho_{\xi'} \mid \xi' < \xi \}$. An ordinal γ is* good *for $\langle f_\alpha : \alpha \in \lambda^+\rangle$ if $\exists A \subseteq \gamma$ unbounded and $\sigma < \mathrm{cof}(\lambda)$ such that $\forall\alpha < \alpha'$ from A and $\nu \in (\sigma, \mathrm{cof}(\lambda))$ $f_\alpha(\nu) < f_{\alpha'}(\nu)$. The scale is* good *if $\exists$ club $C \subseteq \lambda^+$ such that $\forall\alpha \in C$ if $\mathrm{cof}(\alpha) > \mathrm{cof}(\lambda)$, then α is good for the scale. $GS(\lambda)$ means there is a good scale on λ.*

Remarks. 1. λ regular implies AD(λ).

2. AD(λ) implies Sh(λ). [She82]
3. GS(λ) and λ singular implies Sh(λ). [Cum97]
4. $\square^*_\lambda$ implies AD(λ). [CFM]
5. It is not known if $\exists\lambda\neg$Sh(λ) is consistent (it is consistent to have $\exists\lambda[\neg$AD(λ) and $\neg$GS(λ)]).
6. Shelah has proved that there is a scale for all singular λ and that the set of good points is stationary for all scales ([HJS86]; also see [Cum97]). Shelah also gives an example of a model with no good scale ([HJS86]). Another example of a model with no good scale is given by Foreman and Magidor in [FM97], where they show a version of Chang's Conjecture, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, implies there is no good scale on $\aleph_\omega$.

The proof of the following theorem is essentially the same as Theorem 2.3.

Theorem 3.4. *Assume Sh(λ) and $\mathcal{F}$ is a pre-saturated filter on $\mathcal{P}(\lambda)$. Then $S_\lambda \in \mathcal{F}$.*

Theorem 3.5. *Suppose $\mathcal{F}$ is a cardinal preserving filter on $\mathcal{P}(\lambda)$ and AD(λ). Then $S_\lambda \in \mathcal{F}$.*

Proof. We will use Shelah's method of proof of Theorem 2.1 (page 440, [She82]). Let $\langle a_\alpha : \alpha \in \lambda^+\rangle$, $\langle f_\alpha : \alpha \in \lambda^+\rangle$ witness $AD(\lambda)$. Suppose $S_\lambda \notin \mathcal{F}$. Let $G \subseteq \mathcal{F}^+$ be generic with $\mathcal{P}(\lambda) \setminus S_\lambda \in G$. So we get $j : V \to (M, E) \subseteq V[G]$ with $\lambda^+ \subseteq M$ (we collapse the well-founded part of M), and $\mathcal{P}^V(\lambda) \subseteq M$, and $M \models \mathrm{cof}(\lambda) \neq \mathrm{cof}(|\lambda|)$. Work in M: we write $\lambda = \bigcup_{\alpha \in \mathrm{cof}(|\lambda|)} A_\alpha$ where the A_α's are increasing, continuous and $|A_\alpha| < |\lambda|$. So if $a \subseteq \lambda$ is unbounded, then $\exists \alpha < \mathrm{cof}(|\lambda|)$ such that $a \cap A_\alpha$ is unbounded in λ. Now work in $V[G]$: we have $\forall \alpha \in \lambda^+\ \exists \beta \in \mathrm{cof}(|\lambda|)^M$ such that $a_\alpha \cap A_\beta$ is unbounded in λ. So there is a fixed β_0 and an unbounded $\mathcal{A} \subseteq \lambda^+$ such that $(\forall \alpha \in \mathcal{A})\ a_\alpha \cap A_{\beta_0}$ is unbounded in λ. Let $\alpha_0 \in \mathcal{A}$ be such that $\mathcal{A} \cap \alpha_0$ has order type λ. Note that $\langle a_\alpha : \alpha \in \alpha_0\rangle, A_{\beta_0}$, and f_{α_0} are all in M. Now work in M: The set

$$\{\ (a_\alpha \cap A_{\beta_0}) \setminus f_{\alpha_0}(\alpha) \mid \alpha < \alpha_0 \ \&\ a_\alpha \cap A_{\beta_0} \text{ is unbounded in } \lambda\ \}$$

is a family of $|\lambda|$ many non-empty pairwise disjoint subsets of A_{β_0}. But $|A_{\beta_0}| < |\lambda|$, contradiction. $\square$

As in section 2, these two theorems have the following three corollary's:

Corollary 3.6. *Assume AD(λ) [Sh(λ)], $\kappa = \rho^+$, $\mathrm{cof}(\lambda) < \kappa$, and $\mathrm{cof}(\lambda) \neq \mathrm{cof}(\rho)$. Then there is no cardinal preserving [pre-saturated] filter on $\mathcal{P}_\kappa\lambda$.*

Corollary 3.7. *Assume AD(λ) [Sh(λ)], $\kappa = \rho^+ \geq \aleph_2$ and* $\mathrm{cof}(\lambda) \geq \kappa$. *Then the club filter on $\mathcal{P}_\kappa\lambda$ is not cardinal preserving [pre-saturated].*

Corollary 3.8. *Assume AD(λ) [Sh(λ)], κ is a regular limit cardinal and* $\mathrm{cof}(\lambda) \neq \kappa$. *Then the club filter on $\mathcal{P}_\kappa\lambda$ is not cardinal preserving [pre-saturated].*

Theorem 3.9. *Assume* $\mathrm{cof}(\lambda) < \kappa$ *and there is a good scale on λ. Then there is no weakly pre-saturated filter on $\mathcal{P}_\kappa\lambda$.*

Proof. Suppose not. So there is $j : V \to M \subseteq V[G]$ such that λ_V^+ is still a cardinal of $V[G]$, M is well-founded, $\mathcal{P}^V(\lambda) \subseteq M$, and $\mathrm{cp}(j) = \kappa$ with $j(\kappa) > \lambda$. Let $\langle f_\alpha : \alpha \in \lambda^+\rangle$ be a good scale on λ. So there is a club $C \subseteq \lambda^+$ such that $\alpha \in C$ and $\mathrm{cof}(\alpha) > \mathrm{cof}(\lambda)$ implies α is good for $\langle f_\alpha : \alpha \in \lambda^+\rangle$. Let $\rho = \sup j''\lambda^+$ Note that $\rho < j(\lambda^+)$ (see [BM97]) and so $\rho \in j(C)$. Since $V[G] \models \mathrm{cof}(\rho) = \lambda^+$, $M \models \mathrm{cof}(\rho) \geq \lambda^+ > \mathrm{cof}(\lambda)$. so in M, there is an $A \subseteq \rho$ such that $\sup(A) = \rho$ and $\exists \sigma < \mathrm{cof}(\lambda)$ such that $\alpha_1 < \alpha_2$ from A and $\nu \in (\sigma, \mathrm{cof}(\lambda))$ implies $j(f)_{\alpha_1}(\nu) < j(f)_{\alpha_2}(\nu)$. Now work in $V[G]$ and repeat an argument from [Cum97]. For each α in λ^+ choose $\beta_\alpha < \delta_\alpha$ from A and $\gamma_\alpha \in \lambda^+$ such that $\beta_\alpha < j(\gamma_\alpha) < \delta_\alpha$. Do this so $\alpha_1 < \alpha_2$ implies $\delta_{\alpha_1} < \beta_{\alpha_2}$ and $\sup\{ \beta_\alpha \mid \alpha \in \lambda^+ \} = \rho$. For each $\alpha \in \lambda^+$ $\exists \sigma_\alpha < \mathrm{cof}(\lambda)$ such that $j(f)_{\beta_\alpha} < j(f)_{j(\gamma_\alpha)} < j(f)_{\delta_\alpha}$ beyond σ_α. Since λ^+ is regular there is an unbounded $B \subseteq \lambda^+$ and fixed σ_1 such that $\forall \alpha \in B$ $\sigma_\alpha = \sigma_1$. Let $\bar{\sigma} = \max(\sigma, \sigma_1)$. But then if $\alpha_1 < \alpha_2$ are from B, then $f_{\gamma_{\alpha_1}}(\bar{\sigma} + 1) < f_{\gamma_{\alpha_2}}(\bar{\sigma} + 1)$. Hence λ^+ must be collapsed in $V[G]$. □

Precipitousness is ruled out under certain conditions by the following theorem of Matsubara and Shioya.

Theorem 3.10. [MS] *If $\lambda^{<\kappa} = 2^\lambda$ and $2^{<\kappa} < 2^\lambda$, then the club filter on $\mathcal{P}_\kappa\lambda$ is nowhere precipitous.*

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