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Splitting $\mathcal{P}_\kappa\lambda$ into maximally many stationary sets

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ABSTRACT. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. We show that $\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.

0. INTRODUCTION

Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. Solovay's classical result for κ [So] led Menas [Me] to conjecture that a stationary subset of $\mathcal{P}_\kappa\lambda$ would split into $\lambda^{<\kappa}$ stationary sets. Unfortunately his conjecture fails when $2^{<\kappa} > \kappa^+$: While $\mathcal{P}_\kappa\kappa^+$ carries a stationary set of size κ^+ (see [BT]), the conjecture implies that the size is $(\kappa^+)^{<\kappa}$ as well.

What about splitting a stationary set S into $\min\{|S \cap C| : C \text{ is club}\}$ many sets? Gitik's answer [G] was again negative: Relative to supercompactness, it is consistent that some stationary subset of $\mathcal{P}_\kappa\kappa^+$ splits into at most κ stationary sets.

Now it seems natural to ask the same question as above for a canonical stationary set. Let us concentrate on the case where the canonical set is $\mathcal{P}_\kappa\lambda$ itself. When $\kappa = \omega_1$, we have a satisfactory answer by the works of Baumgartner-Taylor [BT] (the case $\lambda \leq 2^\omega$) and Donder-Matet [DM] (otherwise): $\mathcal{P}_{\omega_1}\lambda$ splits into λ^ω stationary sets. In fact the latter proved the diamond principle for $\mathcal{P}_\kappa\lambda$ when $\lambda > 2^{<\kappa}$.

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In this paper we are mainly concerned with the general type of result as follows (see [Ka]): $\mathcal{P}_\kappa\lambda$ splits into λ stationary sets. As suggested above, we should first measure the minimum size of a club subset of $\mathcal{P}_\kappa\lambda$. Elaborating his earlier result [BT], Baumgartner [B] has already shown that it is at least λ^ω. This and the following result of Magidor [Mag] imply that λ^ω is the critical number for our specific splitting problem: If there is no ω_1-Erdős cardinal in the Dodd-Jensen core model, $\mathcal{P}_\kappa\lambda$ carries a club set of size λ^ω when $\operatorname{cf}\lambda \geq \kappa$, and of size $\max\{\lambda^\omega, \lambda^+\}$ otherwise.

Unifying three of the results above, we establish the desired splitting:

Theorem 1. *$\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.*

We also realize the splitting suggested in the latter case of Magidor's theorem:

Theorem 2. *$\mathcal{P}_\kappa\lambda$ splits into λ^+ stationary sets when* $\operatorname{cf}\lambda < \kappa$.

1. Preliminaries

Our notation should be standard. Kanamori's book [Ka] is an excellent source for background material. We reserve κ for a regular cardinal $> \omega$, λ for a cardinal $> \kappa$ and μ, ν for a cardinal $\geq \omega$. When $\mu < \kappa$ is regular, S^μ_κ (resp. $S^{<\mu}_\kappa$, $S^{\geq\mu}_\kappa$) denotes the set of limit ordinals $< \kappa$ of cofinality μ (resp. $< \mu$, $\geq \mu$). For a set X of ordinals let $\lim X$ be the set $\{\gamma < \sup X : \sup(X \cap \gamma) = \gamma > 0\}$ of limit points of X and $\operatorname{cl}_f X$ the closure of X under $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, i.e. the minimal set $Y \supset X$ with $\bigcup f``Y^{<\omega} \subset Y$. Unless otherwise stated, we understand that a set of ordinals is listed in increasing order and a splitting of a stationary set is mutually disjoint.

Thoughout the paper we freely use Solovay's theorem [So] mentioned earlier:

Theorem. *A stationary subset of κ splits into κ stationary sets.*

We need of a version of Shelah's club guessing sequence (see [Ko]). Let us sketch a proof due to Hirata [H]:

Theorem. *Let $\mu < \kappa < \lambda$ be all regular and $S \subset S^{\mu}_{\lambda} \cap \lim S^{\geq\kappa}_{\lambda}$ stationary. Then there is a sequence $\langle c_\gamma : \gamma \in S\rangle$ such that $c_\gamma \subset S^{\geq\kappa}_{\lambda}$ is unbounded in γ and of order type μ for any $\gamma \in S$ and $\{\gamma \in S : c_\gamma \subset C\}$ is stationary for any club set $C \subset \lambda$.*

Proof. First for $\beta \in \lim\lambda$ fix an unbounded set $d_\beta \subset \beta$ of order type $\mathrm{cf}\,\beta$. For $\gamma \in S$ and a club set $D \subset \lim\lambda$ set $x^D_\gamma = \bigcup_{n<\omega} x^D_{\gamma,n} - \{0\}$, where $x^D_{\gamma,n}$ is defined inductively by $x^D_{\gamma,0} = \{\sup(D \cap \alpha) : \alpha \in d_\gamma\}$ and $x^D_{\gamma,n+1} = \{\sup(D\cap\alpha) : \exists\beta \in x^D_{\gamma,n} \cap S^{<\kappa}_{\lambda}(\alpha \in d_\beta)\}$. Note that $x^D_\gamma \subset D$ since D is closed, and $|x^D_{\gamma,n}| < \kappa$ by induction on $n < \omega$. First we find a club set $D \subset \lambda$ such that $\{\gamma \in S : x^D_\gamma \subset C\}$ is stationary for any club set $C \subset \lambda$.

Otherwise we would have inductively a descending sequence $\langle C_\xi : \xi < \kappa\rangle$ of club subsets of $\lim\lambda$ such that $C_{\xi+1} \cap \{\gamma \in S : x^{C_\xi}_\gamma \subset C_{\xi+1}\} = \emptyset$ for any $\xi < \kappa$. Fix $\gamma \in S \cap \bigcap_{\xi<\kappa} C_\xi$. Then we have inductively $\{\xi_n : n < \omega\} \subset \kappa$ such that $x^{C_\xi}_{\gamma,n} = x^{C_{\xi_n}}_{\gamma,n}$ for any $\xi_n \leq \xi < \kappa$, since the map $\xi \mapsto \sup(C_\xi \cap \alpha)$ is decreasing for any $\alpha < \lambda$ and $|x^{C_{\xi_n}}_{\gamma,n}| < \kappa$ by the note above. Set $\xi = \sup_{n<\omega} \xi_n < \kappa$. Then $x^{C_\xi}_\gamma = x^{C_{\xi+1}}_\gamma \subset C_{\xi+1}$ by the note above. This contradicts $C_{\xi+1} \cap \{\gamma \in S : x^{C_\xi}_\gamma \subset C_{\xi+1}\} = \emptyset$.

Now fix a club set $D \subset \lambda$ as above. Then $S^* = \{\gamma \in S \cap \lim D : x^D_\gamma \subset \lim D\}$ is stationary by the claim above. Fix $\gamma \in S^*$. We have $x^D_\gamma - \lim x^D_\gamma \subset S^{\geq\kappa}_{\lambda}$, since $\beta \in x^D_{\gamma,n} \cap S^{<\kappa}_{\lambda}$ implies $\beta \in \lim x^D_{\gamma,n+1}$ by $\beta \in \lim D$. Also $x^D_\gamma - \lim x^D_\gamma$ is unbounded in γ, since $x^D_{\gamma,0}$ is unbounded in γ by $\gamma \in \lim D$.

Finally we get the desired sequence by taking an unbounded subset of $x^D_\gamma - \lim x^D_\gamma$

of order type μ as c_γ for $\gamma \in S^*$. □

In fact we use only the sequence of the form $\langle c_\gamma : \gamma \in S^\omega_\lambda \rangle$ and do not appeal to the clause $c_\gamma \subset S^{\geq\kappa}_\lambda$. The second result we quote from Shelah's pcf theory is a scale on a singular cardinal [Sh] (see also [BMag]):

Theorem. *Let λ be singular. Then there are an unbounded set $\{\lambda_\xi : \xi < \operatorname{cf}\lambda\} \subset \lambda$ of regular cardinals and $\{f_\gamma : \gamma < \lambda^+\} \subset \prod_{\xi<\operatorname{cf}\lambda} \lambda_\xi$ such that $f_\beta \leq^* f_\gamma$ for any $\beta < \gamma < \lambda^+$ and for any $g \in \prod_{\xi<\operatorname{cf}\lambda} \lambda_\xi$ there is $\gamma < \lambda^+$ with $g \leq^* f_\gamma$.*

Here $\leq^*$ denotes the eventual dominance: $f \leq^* g$ iff $\{\xi < \operatorname{cf}\lambda : f(\xi) \leq g(\xi)\}$ is cobounded. The later development of the theory as presented in [Ko] yields a more transparent proof of this deep result.

2. Main Theorems

This section is devoted to establishing Theorems 1 and 2.

Our proof of Theorem 1 consists of two major parts. For the first part we are strongly indebted to Todorčević [T2], who reproved Gitik's answer [G] to Abraham's question [AS] and claimed that his method would yield the Baumgartner-Taylor result as well via the following: Let $\langle c_\gamma : \gamma \in S^\omega_{\omega_2} \rangle$ be a club guessing sequence with $c_\gamma = \{\gamma_n : n < \omega\}$. Then $\{x \in \mathcal{P}_{\omega_1}\omega_2 : \exists\gamma \in S^\omega_{\omega_2}(\sup x = \gamma \wedge \{n < \omega : x \cap (\gamma_{n+1} - \gamma_n) \neq \emptyset\} = r)\}$ is stationary for any $r \in [\omega]^\omega$.

Let λ be regular. We endow $[\lambda]^{<\omega}$ with the tree ordering $\leq\, = \{(a,b) : a$ is an initial segment of $b\}$. Let T be a subtree of $[\lambda]^{<\omega}$, i.e. a subset of $[\lambda]^{<\omega}$ closed under initial segments. Set $[T] = \{B \in [\lambda]^\omega : \forall\beta \in B(B \cap \beta \in T)\}$, the set of infinite branches through T, and $T^a = \{b \in [\lambda]^{<\omega} : a \leq a \cup b \in T\}$, the tree above

$a \in [\lambda]^{<\omega}$. We call $T \neq \emptyset$ stationary if the set of immediate successors of $a \in T$ $\mathrm{suc}_T(a) = \{\alpha < \lambda : a \leq a \cup \{\alpha\} \in T\}$ is always stationary, and $g : T \to \lambda$ regressive when $g(a) \leq g(b) \in \min b \cup \{0\}$ for any $a \leq b \in T$.

Let us start with a tree version of the regressive function lemma:

Lemma. *Let $g : T \to \lambda$ be regressive with T a stationary subtree of $[S^{\kappa}_{\lambda}]^{<\omega}$. Then for some stationary subtree T^* of T $g``T^*$ is bounded in λ.*

Proof. For $\gamma < \lambda$ set $T_\gamma = \{a \in T : g(a) < \gamma\}$, a subtree of T by order preservation of g. First we find $\gamma < \lambda$ with $[T_\gamma] \cap [C]^\omega \neq \emptyset$ for any club set $C \subset \lambda$.

Suppose to the contrary that for $\gamma < \lambda$ we have a club set $C_\gamma \subset \lambda$ with $[T_\gamma] \cap [C_\gamma]^\omega = \emptyset$. Take inductively $B \in [T] \cap [\Delta_{\gamma<\lambda} C_\gamma]^\omega$ by stationarity of T. Take $\alpha < \min B$ with $B \in [T_\alpha]$ by $\mathrm{cf} \min B = \kappa > \omega$ and regressiveness of g. Then $B \in [C_\alpha]^\omega$ by $B \in [\Delta_{\gamma<\lambda} C_\gamma]^\omega$. This contradicts $[T_\alpha] \cap [C_\alpha]^\omega = \emptyset$ by the choice of C_α.

Fix $\gamma < \lambda$ as above. Set $T^* = \{a \in T_\gamma : \forall b \leq a \forall C \subset \lambda \text{ club } ([{T_\gamma}^b] \cap [C]^\omega \neq \emptyset)\}$, a subtree of T. Note that $\emptyset \in T^*$ by the choice of γ. We claim that T^* is stationary as desired.

Suppose to the contrary $D \cap \mathrm{suc}_{T^*}(a) = \emptyset$ for some $a \in T^*$ and some club set $D \subset \lambda$. Then for $\alpha \in D$ we have a club set $C_\alpha \subset \lambda$ with $[{T_\gamma}^{a\cup\{\alpha\}}] \cap [C_\alpha]^\omega = \emptyset$ by $a \in T^*$ and $a \cup \{\alpha\} \notin T^*$. Thus $C = D \cap \Delta_{\alpha \in D} C_\alpha$ is club in λ. Take $B \in [{T_\gamma}^a] \cap [C]^\omega$ by $a \in T^*$. Set $\beta = \min B$. Then $B - \{\beta\} \in [{T_\gamma}^{a\cup\{\beta\}}]$ by $B \in [{T_\gamma}^a]$, and $B - \{\beta\} \in [C_\beta]^\omega$ by $B \in [C]^\omega$. This contradicts $[{T_\gamma}^{a\cup\{\beta\}}] \cap [C_\beta]^\omega = \emptyset$ by $\beta \in D$ and the choice of C_β. $\square$

For the following lemma we fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_\lambda \rangle$ with

$c_\gamma = \{\gamma_n : n < \omega\}$.

Main Lemma 1. *Let $S_n \subset S^\kappa_\lambda$ be stationary for $n < \omega$. Then $\{x \in \mathcal{P}_\kappa\lambda : \exists\gamma \in S^\omega_\lambda(\sup x = \gamma \wedge \forall n < \omega(\min(x - \gamma_n) \in S_n))\}$ is stationary.*

Proof. Fix $f : \lambda^{<\omega} \to P_\kappa\lambda$. Set $T = \{a : \forall n < |a|(\text{the } n\text{th element of } a \text{ is in } S_n)\}$, a stationary subtree of $[S^\kappa_\lambda]^{<\omega}$. We build inductively a stationary subtree T_n of T and $h_n : T_n \cap [\lambda]^n \to \lambda$ so that $T_{n+1} \subset T_n$, $T_{n+1} \cap [\lambda]^n = T_n \cap [\lambda]^n$ and $\mathrm{cl}_f(a \cup B) \cap \min B \subset h_n(a)$ for any $a \in T_{n+1} \cap [\lambda]^n$ and $B \in [{T_{n+1}}^a]$.

First set $T_0 = T$. Next suppose that T_n is defined. Fix $a \in T_n \cap [\lambda]^n$. Then the map $g_a : b \mapsto \sup(\mathrm{cl}_f(a \cup b) \cap \min b)$ is regressive on ${T_n}^a$ by $\mathrm{cf}\min b = \kappa$. By the lemma above we have a stationary subtree T_a of ${T_n}^a$ and $h_n(a) < \lambda$ with $g_a``T_a \subset h_n(a)$. Then $T_{n+1} = (T_n \cap [\lambda]^{<n}) \cup \{a \cup b : a \in T_n \cap [\lambda]^n \wedge b \in T_a\}$ is the desired stationary subtree of T_n: Fix $a \in T_{n+1} \cap [\lambda]^n$ and $B \in [{T_{n+1}}^a]$. Then $\mathrm{cl}_f(a \cup B) \cap \min B = \bigcup_{\beta \in B} \mathrm{cl}_f(a \cup (B \cap \beta)) \cap \min B \subset \bigcup_{\beta \in B} g_a(B \cap \beta) \subset h_n(a)$.

Now set $T^* = \bigcap_{n<\omega} T_n$, a stationary subtree of T, and $h = \bigcup_{n<\omega} h_n : T^* \to \lambda$. Then $C = \{\gamma < \lambda : \mathrm{cl}_f\gamma = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(h(a) < \gamma \wedge \gamma \in \lim \mathrm{suc}_{T^*}(a))\}$ contains a club set. Fix $\gamma \in S^\omega_\lambda \cap C$ with $c_\gamma = \{\gamma_n : n < \omega\} \subset C$. Take inductively $B = \{\beta_n : n < \omega\} \in [T^*]$ so that $\gamma_n < \beta_n < \gamma_{n+1}$ by $\gamma_{n+1} \in C$ and the inductive hypothesis $\{\beta_i : i < n\} \in T^* \cap [\gamma_n]^{<\omega}$. Then $\mathrm{cl}_f B$ is as desired: First we have $\sup \mathrm{cl}_f B = \gamma$, since $\sup B = \gamma$ and $\mathrm{cl}_f B \subset \mathrm{cl}_f \gamma = \gamma$ by $\gamma \in C$. Next $\min(\mathrm{cl}_f B - \gamma_n) = \beta_n$, since $\mathrm{cl}_f B \cap \beta_n \subset h_n(B \cap \beta_n) = h(B \cap \beta_n) < \gamma_n$ by $\gamma_n \in C$ and $B \cap \beta_n \in T^* \cap [\gamma_n]^{<\omega}$. □

The following lemma is due to Foreman-Magidor [FM], who introduce the notion of mutual stationarity and show that the club filter on $\mathcal{P}_{\omega_1}\lambda$ is not $\lambda^{\mathrm{cf}\,\lambda}$-saturated when λ is singular.

Let $\mathrm{cf}\,\lambda = \omega$ and $\{\lambda_n : n < \omega\} = \{\kappa_i : i < \omega\} \subset \lambda$ an unbounded set of regular cardinals $> \kappa$ such that $\lambda_n < \lambda_{n+1}$ and $\{i < \omega : \kappa_i = \lambda_n\}$ is infinite for any $n < \omega$. Let W be the tree $\bigcup_{m<\omega} \prod_{i<m} \kappa_i$ ordered by inclusion. For a subtree T of W set $[T] = \{B \in \prod_{i<\omega} \kappa_i : \forall m < \omega(B|m \in T)\}$, the set of infinite branches through T, and $\mathrm{suc}_T(s) = \{\alpha : s * \langle\alpha\rangle \in T\}$, the set of immediate successors of $s \in T$.

Main Lemma 2. *Let $S_n \subset S^{\omega}_{\lambda_n}$ be stationary for $n < \omega$. Then $\{x \in \mathcal{P}_\kappa\lambda : \forall n < \omega(\sup(x \cap \lambda_n) \in S_n)\}$ is stationary.*

Proof. Fix $f : \lambda^{<\omega} \to P_\kappa\lambda$. We build inductively a subtree T_n of W so that $T_{n+1} \subset T_n$, $\sup(\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_{n-1}) \in S_{n-1}$ for any $B \in [T_n]$ and for any $s \in T_n$ $\mathrm{suc}_{T_n}(s)$ is a singleton if $\kappa_{|s|} < \lambda_n$, and is unbounded in $\kappa_{|s|}$ otherwise.

First set $T_0 = W$. Next suppose that T_n is defined. For $\gamma < \lambda_n$ we call a subtree $U \neq \emptyset$ of W cobounded below γ if for any $s \in U$ $\mathrm{suc}_U(s)$ is $\kappa_{|s|}$ if $\kappa_{|s|} < \lambda_n$, and is cobounded in γ (resp. $\kappa_{|s|}$) if $\kappa_{|s|} = \lambda_n$ (resp. $\kappa_{|s|} > \lambda_n$). We claim that $C = \{\gamma < \lambda_n : \forall U \text{ cobounded below } \gamma \exists B \in [T_n] \cap [U](\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \subset \gamma)\}$ contains a club set.

Suppose to the contrary that we have a stationary set $S \subset \lambda$ and for $\gamma \in S$ a subtree U_γ of W cobounded below γ with $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \not\subset \gamma$ for any $B \in [T_n] \cap [U_\gamma]$. Build inductively a subtree T of T_n so that $\mathrm{suc}_T(s)$ is $\mathrm{suc}_{T_n}(s)$ if $\kappa_{|s|} \leq \lambda_n$, and is $\{\alpha\}$ with $s * \langle\alpha\rangle \in \bigcap\{U_\gamma : s \in U_\gamma\}$ otherwise. Note that the map $s \mapsto s|\{i : \kappa_i = \lambda_n\}$ is injective on $\{s \in T : \kappa_{|s|} = \lambda_n\}$. Hence $D = \{\gamma < \lambda_n : \forall s \in T((\kappa_{|s|} = \lambda_n \wedge s``\{i : \kappa_i = \lambda_n\} \subset \gamma) \Rightarrow (\mathrm{cl}_f \mathrm{ran}\, s \cap \lambda_n \subset \gamma \wedge \gamma \in \lim \mathrm{suc}_T(s)))\}$ contains a club set. Fix $\gamma \in S \cap D$. Take inductively $B \in [T] \cap [U_\gamma]$ as follows: Suppose that $s \in T \cap U_\gamma$ is defined. Then $\mathrm{suc}_T(s) \cap \mathrm{suc}_{U_\gamma}(s) \neq \emptyset$, since $\mathrm{suc}_{U_\gamma}(s) = \kappa_{|s|}$ when $\kappa_{|s|} < \lambda_n$, since

$\mathrm{suc}_{U_\gamma}(s)$ is cobounded in γ and $\mathrm{suc}_T(s)$ is unbounded in γ by $\gamma \in D$, $s \in T$ and $s``\{i : \kappa_i = \lambda_n\} \subset \gamma$ when $\kappa_{|s|} = \lambda_n$, and by $s \in U_\gamma$ and the choice of $\mathrm{suc}_T(s)$ when $\kappa_{|s|} > \lambda_n$. Thus $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n = \bigcup\{\mathrm{cl}_f B``i \cap \lambda_n : \kappa_i = \lambda_n\} \subset \gamma$ by $\gamma \in D$ and $B|i \in T$. This contradicts $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \not\subset \gamma$ by $\gamma \in S$ and the choice of U_γ.

Fix $\gamma \in S_n \cap C$. Set $T^* = \{s \in T_n : \forall t \leq s \forall U \ni t$ cobounded below $\gamma \exists B \in [T_n] \cap [U](t \subset B \wedge \mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \subset \gamma)\}$, a subtree of T_n. Note that $\emptyset \in T^*$ by $\gamma \in C$. Fix $s \in T^*$. We claim that $\mathrm{suc}_{T^*}(s)$ is a singleton if $\kappa_{|s|} < \lambda_n$, and is unbounded in γ (resp. $\kappa_{|s|}$) if $\kappa_{|s|} = \lambda_n$ (resp. $\kappa_{|s|} > \lambda_n$). We show the case $\kappa_{|s|} = \lambda_n$. The case $\kappa_{|s|} > \lambda_n$ (resp. $\kappa_{|s|} < \lambda_n$) is given by a similar (resp. simpler) argument.

Suppose to the contrary that $A = \gamma - \mathrm{suc}_{T^*}(s)$ is cobounded. Then for $\alpha \in A$ we have a subtree $U_\alpha \ni s * \langle\alpha\rangle$ of W cobounded below γ such that $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \not\subset \gamma$ for any $s * \langle\alpha\rangle \subset B \in [T_n] \cap [U_\alpha]$ by $s \in T^*$ and $s * \langle\alpha\rangle \notin T^*$. Fix a subtree U of W cobounded below γ with $\{t \in U : s < t\} = \bigcup_{\alpha \in A}\{t \in U_\alpha : s * \langle\alpha\rangle \leq t\}$. Take $s \subset B \in [T_n] \cap [U]$ with $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \subset \gamma$ by $s \in T^*$, and then $\alpha \in A$ with $s * \langle\alpha\rangle \subset B \in [U_\alpha]$ by the minimal choice of U. This contradicts $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n \not\subset \gamma$ by $s * \langle\alpha\rangle \subset B \in [T_n] \cap [U_\alpha]$ and the choice of U_α.

Now fix an unbounded set $\{\gamma_i : i < \omega\} \subset \gamma$. Build inductively a subtree T_{n+1} of T^* so that $\mathrm{suc}_{T_{n+1}}(s)$ is $\mathrm{suc}_{T^*}(s)$ if $\kappa_{|s|} \neq \lambda_n$, and is $\{\alpha\}$ with $\gamma_m < \alpha < \gamma$ otherwise, where $m = |\{i < |s| : \kappa_i = \lambda_n\}|$. Then T_{n+1} is as desired: Fix $B \in [T_{n+1}]$. Then $\sup(\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n) = \gamma$, since $\sup\{B(i) : \kappa_i = \lambda_n\} = \gamma$ and $\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n = \bigcup_{i<\omega} \mathrm{cl}_f B``i \cap \lambda_n \subset \gamma$ by $B|i \in T^*$.

Finally $\bigcap_{n<\omega} T_n$ has a unique branch B and $\sup(\mathrm{cl}_f \mathrm{ran}\, B \cap \lambda_n) \in S_n$ for any $n < \omega$ as desired. $\square$

We are ready to prove the main result of this paper:

Theorem 1. *$\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.*

Proof. When $\lambda \leq \mu^\omega$ for some regular cardinal $\kappa < \mu \leq \lambda$, fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_\mu \rangle$ with $c_\gamma = \{\gamma_n : n < \omega\}$ and split S^κ_μ into stationary sets $\{S_\xi : \xi < \mu\}$. Then for $p : \omega \to \mu$ $\{x \in \mathcal{P}_\kappa\lambda : \exists\gamma \in S^\omega_\mu(\sup(x \cap \mu) = \gamma \wedge \forall n < \omega(\min(x - \gamma_n) \in S_{p(n)}))\}$ is stationary by Main Lemma 1 and mutually disjoint.

When $\operatorname{cf}\lambda = \omega$, fix an unbounded set $\{\lambda_n : n < \omega\} \subset \lambda$ of regular cardinals $> \kappa$. Then $|\prod_{n<\omega} \lambda_n| = \lambda^\omega$. For $n < \omega$ split $S^\omega_{\lambda_n}$ into stationary sets $\{S_{n\xi} : \xi < \lambda_n\}$. Then for $p \in \prod_{n<\omega} \lambda_n$ $\{x \in \mathcal{P}_\kappa\lambda : \forall n < \omega(\sup(x \cap \lambda_n) \in S_{np(n)})\}$ is stationary by Main Lemma 2 and mutually disjoint.

Otherwise we have $\omega < \operatorname{cf}\lambda < \lambda$ and $\alpha^\omega < \lambda$ for any $\alpha < \lambda$, and hence $\lambda^\omega = \lambda$. For completeness we provide a proof implicit in [T1]. First we claim that $\{x \in \mathcal{P}_\kappa\lambda : \sup(x \cap \mu) \in S \wedge \sup(x \cap \nu) \in S'\}$ is stationary for any regular cardinals $\kappa \leq \mu < \nu < \lambda$ and stationary sets $S \subset S^\omega_\mu$ and $S' \subset S^\omega_\nu$. Fix $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$. Take $\beta \in S'$ with $\operatorname{cl}_f\beta \cap \nu = \beta$, and an unbounded set $b \subset \beta$ of size ω, and then $\alpha \in S$ with $\operatorname{cl}_f(\alpha \cup b) \cap \mu = \alpha$, and an unbounded set $a \subset \alpha$ of size ω. Then $\sup(\operatorname{cl}_f(a \cup b) \cap \mu) = \alpha$ and $\sup(\operatorname{cl}_f(a \cup b) \cap \nu) = \beta$ as desired. Now set $\mu = \max\{\kappa, \operatorname{cf}\lambda\} < \lambda$ and split S^ω_μ into stationary sets $\{S_\xi : \xi < \operatorname{cf}\lambda\}$. Also fix an unbounded set $\{\lambda_\xi : \xi < \operatorname{cf}\lambda\} \subset \lambda$ of regular cardinals $> \mu$ and for $\xi < \operatorname{cf}\lambda$ split $S^\omega_{\lambda_\xi}$ into stationary sets $\{S_{\xi\zeta} : \zeta < \lambda_\xi\}$. Then for $(\xi\zeta) \in \sum_{\xi<\operatorname{cf}\lambda} \lambda_\xi$ $\{x \in \mathcal{P}_\kappa\lambda : \sup(x \cap \mu) \in S_\xi \wedge \sup(x \cap \lambda_\xi) \in S_{\xi\zeta}\}$ is stationary by the claim above and mutually disjoint. $\square$

Our second result is inspired by Burke's theorem [BMat] that the club filter on $\mathcal{P}_\kappa\lambda$ is not λ^+-saturated when $\kappa > \omega_1$ and $\operatorname{cf}\lambda < \kappa$:

Theorem 2. *$\mathcal{P}_\kappa\lambda$ splits into λ^+ stationary sets when* $\operatorname{cf}\lambda < \kappa$.

Proof. The case $\operatorname{cf}\lambda = \omega$ follows from Theorem 1.

Otherwise fix a scale $\{f_\gamma : \gamma < \lambda^+\} \subset \prod_{\xi < \operatorname{cf}\lambda} \lambda_\xi$ with $\lambda_0 > \kappa$. Define $\rho : \mathcal{P}_\kappa\lambda \to \lambda^+$ by $\rho(x) = \min\{\gamma < \lambda^+ : \langle \sup(x \cap \lambda_\xi) : \xi < \operatorname{cf}\lambda\rangle \leq^* f_\gamma\}$. We show that $\rho^{-1}S$ is stationary in $\mathcal{P}_\kappa\lambda$ for any stationary set $S \subset S^\omega_{\lambda^+}$.

Fix a club set $C \subset \mathcal{P}_\kappa\lambda$. Construct $\{x_a : a \in [\lambda^+]^{<\omega}\} \subset C$ by induction on $|a|$ so that $\operatorname{ran} f_{\max a} \subset x_a \subset x_b$ for any $a \subset b \in [\lambda^+]^{<\omega}$ by $\operatorname{cf}\lambda < \kappa$. Take $\gamma \in S$ with $\rho(x_a) < \gamma$ for any $a \in [\gamma]^{<\omega}$, and an unbounded set $B \subset \gamma$ of order type ω. Set $x = \bigcup_{\beta \in B} x_{B\cap\beta} \in C$. We claim that $\rho(x) = \gamma$ as desired.

First we have $\rho(x) \geq \gamma$, since for any $\beta \in B$ $\rho(x) \geq \rho(x_{B\cap\beta}) \geq \max(B \cap \beta)$ by $\operatorname{ran} f_{\max(B\cap\beta)} \subset x_{B\cap\beta}$. Next $\langle \sup(x \cap \lambda_\xi) : \xi < \operatorname{cf}\lambda\rangle = \langle \sup_{\beta\in B} \sup(x_{B\cap\beta} \cap \lambda_\xi) : \xi < \operatorname{cf}\lambda\rangle \leq^* f_\gamma$, since $\operatorname{cf}\lambda > \omega$ and for any $\beta \in B$ $\langle \sup(x_{B\cap\beta} \cap \lambda_\xi) : \xi < \operatorname{cf}\lambda\rangle \leq^* f_\gamma$ by $\rho(x_{B\cap\beta}) < \gamma$.

Now split $S^\omega_{\lambda^+}$ into stationary sets $\{S_\alpha : \alpha < \lambda^+\}$. Then for $\alpha < \lambda^+$ $\rho^{-1}S_\alpha$ is stationary in $\mathcal{P}_\kappa\lambda$ by the claim above and mutually disjoint. □

3. Some Remarks

For the moment let us assume that $\mu < \kappa < \lambda$ are all regular and consider the stationary set $S^\mu_{\kappa\lambda} = \{x \in \mathcal{P}_\kappa\lambda : \operatorname{cf}\sup x = \mu\}$. Main Lemma 1 implies that $S^\omega_{\kappa\lambda}$ splits into λ^ω stationary sets. On the other hand Matsubara [Mat] proved that a stationary subset of $S^\mu_{\kappa\lambda}$ splits into λ stationary sets. This is optimal when $\mu > \omega$ and $\lambda < \kappa^{+\omega}$, since Baumgartner [B] shows that $|\{x \in \mathcal{P}_\kappa\lambda : \kappa \leq \forall\nu \leq \lambda(\operatorname{cf}\sup(x \cap \nu) > \omega)\} \cap C| = \lambda$ for some club set $C \subset \mathcal{P}_\kappa\lambda$. In fact the map $x \mapsto \langle \sup(x \cap \nu) : \kappa \leq \nu \leq \lambda\rangle$ is injective on this set. Complementing a result of

Abe [A], we remark that the map $x \mapsto \sup x$ is not injective on $S^{\mu}_{\kappa\lambda} \cap C$ for any club set $C \subset \mathcal{P}_\kappa\lambda$: Fix $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$ generating C. Take $\kappa < \gamma \in S^{\mu}_{\lambda}$ closed under f, an unbounded set $a \subset \gamma$ of size μ and $\alpha \in \gamma - \mathrm{cl}_f a$. Then $\mathrm{cl}_f a \neq \mathrm{cl}_f(a \cup \{\alpha\})$ and $\sup \mathrm{cl}_f a = \sup \mathrm{cl}_f(a \cup \{\alpha\}) = \gamma$ as desired.

The rest of the section is devoted to a detailed proof of the Donder-Matet theorem mentioned earlier.

Let $\mu > \omega$ be regular and $d_\gamma = \{\gamma_n : n < \omega\} \subset \gamma$ unbounded for $\gamma \in S^{\omega}_{\mu}$. The following lemma from [B] (see also [BT]), where it is stated in (harmlessly) inaccurate form, is implicit in Lemma 9.1 of [DM].

Lemma 1. *Let $S \subset S^{\omega}_{\mu}$ be stationary. Then $\{\alpha < \mu : \{\gamma \in S : \alpha \in d_\gamma\}$ is stationary$\}$ is unbounded.*

Proof. Suppose to the contrary that we have $\beta < \mu$ and for $\beta < \alpha < \mu$ a club set $C_\alpha \subset \mu$ with $C_\alpha \cap \{\gamma \in S : \alpha \in d_\gamma\} = \emptyset$. Take $\beta < \gamma \in S \cap \Delta_{\beta<\alpha<\mu} C_\alpha$. Then for any $\beta < \alpha < \gamma$ $\alpha \notin d_\gamma$ by $\gamma \in S \cap C_\alpha$. This contradicts the unboundedness of d_γ in γ. □

We call a subtree $T \neq \emptyset$ of $[\mu]^{<\omega}$ in the sense of Section 2 unbounded (resp. cobounded) if $\mathrm{suc}_T(a)$ is unbounded (resp. cobounded) in μ for any $a \in T$. The following lemma from [RS] (see also [BMag]) would ensure that the map ξ in Lemma 9.2 of [DM] is well-defined (at least in the case we are interested in).

Lemma 2. *Let $g : T \to \nu$ with T an unbounded subtree of $[\mu]^{<\omega}$ and $\nu^\omega < \mu$. Then for some unbounded subtree T^* of T g is constant on $T^* \cap [\mu]^n$ for any $n < \omega$.*

Proof. For $h : \omega \to \nu$ set $T_h = \{a \in T : \forall b \leq a(g(b) = h(|b|))\}$, a subtree of T. First we find $h : \omega \to \nu$ with $[T_h] \cap [U] \neq \emptyset$ for any cobounded subtree U of $[\mu]^{<\omega}$.

Suppose to the contrary that for $h : \omega \to \nu$ we have a cobounded subtree U_h of $[\mu]^{<\omega}$ with $[T_h] \cap [U_h] = \emptyset$. Take inductively $B \in [T] \cap [\bigcap\{U_h : h : \omega \to \nu\}]$ by $\nu^\omega < \mu$. Take $h : \omega \to \nu$ with $B \in [T_h]$. This contradicts $[T_h] \cap [U_h] = \emptyset$.

Now fix $h : \omega \to \nu$ as above. Set $T^* = \{a \in T_h : \forall b \leq a \forall U \ni b$ cobounded $\exists B \in [T_h] \cap [U](b \subset B)\}$, a subtree of T. Note that $\emptyset \in T^*$ by the choice of h. We claim that T^* is unbounded as desired.

Suppose to the contrary that $A = \mu - \mathrm{suc}_{T^*}(a)$ is cobounded for some $a \in T^*$. Then for $\alpha \in A$ we have a cobounded subtree $U_\alpha \ni a \cup \{\alpha\}$ of $[\mu]^{<\omega}$ such that $a \cup \{\alpha\} \not\subset B$ for any $B \in [T_h] \cap [U_\alpha]$ by $a \in T^*$ and $a \cup \{\alpha\} \notin T^*$. Fix a cobounded subtree U of $[\mu]^{<\omega}$ with $\{b \in U : a < b\} = \bigcup_{\alpha \in A}\{b \in U_\alpha : a \cup \{\alpha\} \leq b\}$. Take $a \subset B \in [T_h] \cap [U]$ by $a \in T^*$, and then $\alpha \in A$ with $a \cup \{\alpha\} \subset B \in [U_\alpha]$ by the minimal choice of U. This contradicts $a \cup \{\alpha\} \not\subset B$ by $B \in [T_h] \cap [U_\alpha]$ and the choice of U_α. □

We are ready to prove the main claim of Proposition 9.6 of [DM]:

Theorem. *Let $\lambda > 2^{<\kappa}$. Then there is a sequence $\langle v_x : x \in \mathcal{P}_\kappa\lambda\rangle$ such that $\{x \in \mathcal{P}_\kappa\lambda : v_x = X \cap x\}$ is stationary for any $X \subset \lambda$.*

Proof. Set $\mu = (2^{<\kappa})^+$ and split S^ω_μ into stationary sets $\{S^w : w \in \mathcal{P}_\kappa\kappa\}$. For $x \in \mathcal{P}_\kappa\lambda$ with $\mathrm{cf}\sup(x \cap \mu) = \omega$ set $v_x = \pi(x)^{-1}w$, where $\sup(x \cap \mu) \in S^w$ and $\pi(x) : x \to \mathrm{ot}\, x$ is the increasing bijection. Fix $X \subset \lambda$. We show that $\{x \in \mathcal{P}_\kappa\lambda : v_x = X \cap x\}$ is stationary.

Fix $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$. We build inductively an unbounded subtree T of $[\mu]^{<\omega}$ and for $a \in T$ a stationary set $S_a \subset S^\omega_\mu$ and an increasing injection $\chi_a : \mathrm{cl}_f a \to \kappa$ so that for any $a \leq b \in T$ $S_b \subset S_a$ and for any $\gamma \in S_a$ $a \subset d_\gamma$ and $\pi(\mathrm{cl}_f d_\gamma)|\mathrm{cl}_f a = \chi_a$.

Note that these conditions imply $\chi_a \subset \chi_b$ for any $a \leq b \in T$.

First set $S_\emptyset = S^\omega_\mu$ and $\chi_\emptyset = \emptyset$. Next suppose that $T \cap [\mu]^n$ and S_a for $a \in T \cap [\mu]^n$ are defined. Fix $a \in T \cap [\mu]^n$. Let $\mathrm{suc}_T(a) = \{\alpha < \mu : \max a < \alpha \wedge \{\gamma \in S_a : \alpha \in d_\gamma\}$ is stationary$\}$, which is unbounded by Lemma 1. Fix $\alpha \in \mathrm{suc}_T(a)$. Take a stationary set $S_{a\cup\{\alpha\}} \subset \{\gamma \in S_a : \alpha \in d_\gamma\}$ and $\chi_{a\cup\{\alpha\}} : \mathrm{cl}_f(a \cup \{\alpha\}) \to \kappa$ so that for any $\gamma \in S_{a\cup\{\alpha\}}$ $\pi(\mathrm{cl}_f d_\gamma)|\mathrm{cl}_f(a \cup \{\alpha\}) = \chi_{a\cup\{\alpha\}}$ by $2^{<\kappa} < \mu$.

By Lemma 2 with $\nu = 2^{<\kappa}$ take an unbounded subtree T^* of T and $\{y_n : n < \omega\}$, $\{z_n : n < \omega\} \subset \mathcal{P}_\kappa\kappa$ so that $\mathrm{ran}\,\chi_a = y_n$ and $\chi_a``(X \cap \mathrm{cl}_f a) = z_n$ for any $a \in T^* \cap [\mu]^n$. Then $C = \{\gamma < \mu : \mathrm{cl}_f\gamma \cap \mu = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(\gamma \in \lim \mathrm{suc}_{T^*}(a))\}$ contains a club set. Set $w = \pi(\bigcup_{n<\omega} y_n)``\bigcup_{n<\omega} z_n \in \mathcal{P}_\kappa\kappa$. Fix $\gamma \in S^w \cap C$. Take inductively $B = \{\beta_n : n < \omega\} \in [T^*]$ so that $\gamma_n < \beta_n < \gamma$ by $\gamma \in C$ and the inductive hypothesis $\{\beta_i : i < n\} \in T^* \cap [\gamma]^{<\omega}$. Then $\mathrm{cl}_f B$ is as desired: First we have $\sup(\mathrm{cl}_f B \cap \mu) = \gamma$, since $\sup B = \gamma$ and $\mathrm{cl}_f B \cap \mu \subset \mathrm{cl}_f\gamma \cap \mu = \gamma$ by $\gamma \in C$. Next $\pi(\mathrm{cl}_f B)``(X \cap \mathrm{cl}_f B) = w$, since $\chi = \bigcup_{\beta \in B} \chi_{B\cap\beta} : \mathrm{cl}_f B \to \bigcup_{n<\omega} y_n$ is an increasing bijection and $\chi``(X \cap \mathrm{cl}_f B) = \bigcup_{n<\omega} z_n$ by the note above. □

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