Partition properties of subsets of $P_\kappa\lambda$

Properties of Ideals on $P_\kappa\lambda$

Shioya, Masahiro

数理解析研究所講究録　数理数学研究所

1999-04

http://hdl.handle.net/2433/62997

Departmental Bulletin Paper

publisher

Kyoto University
Partition properties of subsets of $\mathcal{P}_{\kappa}\lambda$

MASAHIRO SHIOYA

Abstract. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. The following partition property is shown to be consistent relative to supercompactness: For any $f : \bigcup_{n<\omega}[X]^{\subset}_n \rightarrow \gamma$ with $X \subset \mathcal{P}_{\kappa}\lambda$ unbounded and $1 < \gamma < \kappa$ there is an unbounded $Y \subset X$ with $|f[Y]^{\subset}_n| = 1$ for any $n < \omega$.

Let $\kappa$ be a regular cardinal $> \omega$, $\lambda$ a cardinal $\geq \kappa$ and $F$ a filter on $\mathcal{P}_{\kappa}\lambda$. Partition properties of the form $\mathcal{P}_{\kappa}\lambda \rightarrow (F^+)_{2}^{2}$ (see below for the definition) were introduced by Jech [Je] and successfully used to characterize supercompactness: Menas [Me] proved $\mathcal{P}_{\kappa}\lambda \rightarrow (C_{\kappa\lambda}^{+})_{2}^{2}$ for $\kappa 2^{\lambda^{<\kappa}}$-supercompact via a normal ultrafilter on $\mathcal{P}_{\kappa}\lambda$ with the partition property. As noted by Kamo [Kam], Menas' argument can be modified to derive $\mathcal{P}_{\kappa}\lambda \rightarrow (C_{\kappa\lambda}^{+})_{2}^{2}$ directly from $\lambda$-supercompactness of $\kappa$. For the converse direction Di Prisco-Zwicker [DZ] and others refined the global result of Magidor [Mag]: $\lambda$-supercompactness of $\kappa$ follows from $\mathcal{P}_{\kappa}2^{\lambda^{<\kappa}} \rightarrow (C_{\kappa2^{\lambda^{<\kappa}}}^{+})_{2}^{2}$.

Johnson [Jo] studied properties of the form $X \rightarrow (F^+)_{2}^{2}$ for $X \in F^+$, which means that for any $f : [X]^{\subset}_{\omega} \rightarrow 2$ there is $Y \in F^+$ with $Y \subset X$ and $|f[Y]^{\subset}_{\omega}| = 1$. In this note we are concerned with the case where $F$ is canonically defined, in particular $\mathcal{F}_{\kappa\lambda}$, the minimal fine filter on $\mathcal{P}_{\kappa}\lambda$.

We generally follow the terminology of Kanamori [Kan] with the following exception: For a cardinal $\mu \geq \omega$ we set $[X]^{\mu} = \{x \subset X : |x| = \mu\}$, $[X]^{<\mu} = \{x \subset X : |x| < \mu\}$.

1991 Mathematics Subject Classification. 03E05, 03E55.

Part of this work was done during the author's stay at Boston University as one of the Japanese Overseas Research Fellows. He gratefully acknowledges Prof. Kanamori's hospitality. He also wishes to thank Prof. Abe for helpful conversations.

Typeset by A\LaTeX
$|x| < \mu$ and $\lim A = \{\alpha < \mu : \sup(A \cap \alpha) = \alpha > 0\}$ for $A \subset \mu$. By $F^+ \to (F^+)_2^3$ we mean $X \to (F^+)_2^3$ for any $X \in F^+$. We understand $\bigcup a \subseteq \bigcap b$ when the union $a \cup b$ of $a \in [P_{\kappa}\lambda]^m_\subset$ and $b \in [P_{\kappa}\lambda]^n_\subset$ with $m, n < \omega$ is formed.

Abe [A1] proved $\mathcal{F}_{\kappa\lambda}^+ \not\leftrightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^3$ under $\lambda^{<\kappa} = 2^\lambda$. Matet [Mat] used Laver’s idea (see [JS]) to get the same conclusion from an opposite assumption:

**Proposition 1.** Assume $\lambda^\kappa = \lambda$. Then $\mathcal{F}_{\kappa\lambda}^+ \not\leftrightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^3$.

**Proof.** First set $\mathcal{P}_{\kappa\lambda} = \{x_\xi : \xi < \lambda\}$ and $[\mathcal{P}_{\kappa\lambda}]^\kappa = \{Y_\alpha : \alpha < \lambda\}$. By induction on $\xi < \lambda$ construct $x_\xi \subset z_\xi \in \mathcal{P}_{\kappa\lambda}$ mutually distinct and $y_\xi^{\alpha i} \in Y_\alpha$ with $y_\xi^{\alpha i} \subset z_\xi$ mutually distinct for $\alpha \in z_\xi$ and $i < 2$ as follows: At stage $\xi < \lambda$ by induction on $n < \omega$ take $z_\xi n \in \mathcal{P}_{\kappa\lambda}$ and $y_\xi^{\alpha i} \in Y_\alpha$ for $\alpha \in z_\xi n$ and $i < 2$ so that $x_\xi \subset z_\xi 0 \not\subseteq \bigcup_{\zeta < \xi} z_\zeta$ and $z_\xi n \cup \{y_\xi^{\alpha i} : \alpha \in z_\xi n \land i < 2\} \subset z_\xi n+1$. Finally set $z_\xi = \bigcup_{n < \omega} z_\xi n$. We claim that $f$ defined by $f(\{y_\xi^{\alpha i}, z_\xi\}) = i$ witnesses $\{z_\xi : \xi < \lambda\} \not\leftrightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^3$.

Fix an unbounded set $X \subset \{z_\xi : \xi < \lambda\}$. We show $f^u[X]_2^3 = 2$. Take $\alpha < \lambda$ with $Y_\alpha \in [X]^\kappa$, and $\xi < \lambda$ with $\alpha \in z_\xi \in X$. Then $f(\{y_\xi^{\alpha i}, z_\xi\}) = i$ for $i < 2$ by definition, as desired. \(\square\)

The above proof yields in fact for any $\gamma < \kappa$ $f : [X]_2^3 \to \gamma$ with $X \in \mathcal{F}_{\kappa\lambda}^+$ and $f^u[Y]_2^3 = \gamma$ for any $Y \in \mathcal{F}_{\kappa\lambda}^+$ with $Y \subset X$.

It is natural to ask, as did Abe [A1], if $\mathcal{F}_{\kappa\lambda}^+ \not\leftrightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^3$ holds in general. His answer [A2] to the analogous question would make it more interesting: $C_{\kappa\lambda}^+ \not\leftrightarrow (C_{\kappa\lambda}^+)_2^3$.

Appealing more directly to Magidor’s idea [Mag], we give a canonical witness to Abe’s observation:

**Proposition 2.** Let $\mu < \kappa$ be regular. Then $\{x \in \mathcal{P}_{\kappa\lambda} : \text{cf } (x \cap \kappa) = \mu\} \not\leftrightarrow (C_{\kappa\lambda}^+)_2^3$.

**Proof.** Set $S = \{x \in \mathcal{P}_{\kappa\lambda} : \text{cf } (x \cap \kappa) = \mu\}$ and for $x \in S$ fix an unbounded set $c_x \subset$
$x \cap \kappa$ of order type $\mu$. For $\{x, y\} \in [S]^{2}_{\subset}$ let $f(\{x, y\})$ be 0 when $\min(c_x \Delta c_y) \in c_x$, and 1 otherwise. Fix a stationary set $T \subset S$. We show $f''[T]^{2}_{\subset} = 2$.

First we have $\gamma < \kappa$ such that for any $w \in P_{\kappa}\lambda$ there are $w \subset x, y \in T$ with $\gamma \in c_x - c_y$. Suppose to the contrary that we have $g : \kappa \rightarrow P_{\kappa}\lambda$ such that for any $\gamma < \kappa$ and $g(\gamma) \subset x, y \in T, \gamma \in c_x$ iff $\gamma \in c_y$. Take $x, y \in C(g) \cap T$ with $x \cap \kappa < y \cap \kappa$ by the stationarity of $\{z \cap \kappa : z \in C(g) \cap T\}$. Then $c_x = c_y \cap x \cap \kappa$ has order type $\mu$, contradicting the choice of $c_y$.

Now let $\gamma < \kappa$ be the minimal as above. Then for $\alpha < \gamma$ we have $w_\alpha \in P_{\kappa}\lambda$ such that for any $w_\alpha \subset x, y \in T, \alpha \in c_x$ iff $\alpha \in c_y$. Set $w = \bigcup_{\alpha < \gamma} w_\alpha \in P_{\kappa}\lambda$. Take $w \subset x \subset y \subset z$ from $T$ with $\gamma \in c_x \cap c_z - c_y$. Then $\min(c_x \Delta c_y) = \min(c_y \Delta c_z) = \gamma$ by $w_\alpha \subset x \subset y \subset z$ for any $\alpha < \gamma$, and hence $f(\{x, y\}) = 0$ and $f(\{y, z\}) = 1$ by definition, as desired. \(\square\)

The rest of the paper is devoted to a negative answer to Abe's question in the strong sense. We refer to Baumgartner's expository paper [B] for the rudiments of iterated forcings. We are indebted for the definition of the poset $Q_f$ below to Galvin (see [JS]), who proved under $\text{MA}_{\omega_1}$ that for any $f : [X]^{m}_{\subset} \rightarrow n$ with $X \subset [\omega_1]^{<\omega}$ unbounded and $1 < m, n < \omega$ there is an unbounded $Y \subset X$ with $|f''[Y]^{m}_{\subset}| = 1$.

Assume for the moment that $\kappa$ is a compact cardinal and $\lambda \leq 2^{\kappa}$. Fix a coloring $f : \bigcup_{n<\omega} [S]^{n}_{\subset} \rightarrow \gamma$ with $S \subset P_{\kappa}\lambda$ unbounded and $1 < \gamma < \kappa$. We define a poset $Q_f$ and establish its basic properties.

Fix a fine ultrafilter $U$ on $S$ and define inductively a $\kappa$-complete ultrafilter $U_n$ on $[S]^{n}_{\subset}$ by $U_0 = \{\emptyset\}$ and $U_{n+1} = \{X : \{x : \{a : \{x\} \cup a \in X\} \in U_n\} \in U\}$. For $n < \omega$ let $\beta_n$ be the unique $\beta < \gamma$ with $\{a \in [S]^{n}_{\subset} : f(a) = \beta\} \in U_n$. Let
\[ Q_f = \{ p \in [S]^{<\kappa} : \forall m, n < \omega \forall a \in [p]^{m} (\{ b \in [S]^{n} : f(a \cup b) = \beta_{m+n} \} \in U_n) \} \] and \( q \leq p \) iff \( q \supseteq p \) and \( y \not\subset x \) for any \( x \in p \) and \( y \in q - p \).

First for a generic filter \( G \subset Q_f, \cup G \) is unbounded in \( P_{\kappa}\lambda \) by the density of \( \{ q \in Q_f : \exists y \in q(x \subset y) \} \) for any \( x \in \mathcal{P}_{\kappa}\lambda \), and homogeneous for \( f : f^{\#}[\cup G]^{n} = \{ \beta_{n} \} \) for any \( n < \omega \).

Next \( Q_f \) is \( \kappa \)-centered closed (hence in particular \( \kappa \)-directed closed): A centered subset \( D \) of \( Q_f \) of size \( < \kappa \) has a lower bound \( \cup D \).

Finally we show that \( Q_f \) is \( \kappa \)-linked. Fix an injection \( \pi : \mathcal{P}_{\kappa}\lambda \rightarrow \kappa^2 \) with \( \alpha < \kappa \) set \( Q_{f,A} = \{ p \in Q_f : \{ \pi(x)|\alpha : x \in p \} = A \} \) is injective}. Then \( Q_f = \bigcup \{ Q_{f,A} : \exists \alpha < \kappa(A \subset \alpha^{2}) \} \) by inaccessibility of \( \kappa \). To see linkedness of \( Q_{f,A} \), fix \( p, q \in Q_{f,A} \). Then \( x \nsubseteq y \) for any \( x \in p - q \) and \( y \in q \): Otherwise we would have \( x = z \) for some \( x \in p - q, y \in q \) with \( x \subset y \) and \( z \in q \) with \( \pi(x)|\alpha = \pi(z)|\alpha \). Similarly \( y \nsubseteq x \) for any \( x \in p \) and \( y \in q - p \). Thus \( p \cup q \leq p, q \), as desired.

A minor modification of the original proof [B] for \( \kappa = \omega_1 \) yields the following

**Lemma.** Assume \( 2^{<\kappa} = \kappa \). Let \( (P_{\alpha}, \dot{Q}_{\alpha} : \alpha < \beta) \) be a \( \alpha < \kappa \)-support iteration such that \( \models_{\alpha} \) "\( \dot{Q}_{\alpha} \) is \( \kappa \)-linked and \( \kappa \)-centered closed" for any \( \alpha < \beta \). Then \( P_{\beta} \) satisfies \( \kappa^+ \)-c.c.

**Proof.** Fix \( X \in [P_{\beta}]^{\kappa^+} \). For \( \alpha < \beta \) let \( \models_{\alpha} \) "\( \dot{Q}_{\alpha} = \bigcup_{\gamma < \kappa} \dot{Q}_{\alpha\gamma} \) with \( \dot{Q}_{\alpha\gamma} \) linked for any \( \gamma < \kappa \)." For \( p \in X \) by induction on \( \xi < \kappa \) take \( p_{\xi} \leq p, \alpha^{\xi}_{\xi} \in \text{supp}(p_{\xi}) \) and \( \gamma^{\xi}_{\xi} < \kappa \) so that \( p_{\xi} \leq p_{\xi} \) and \( p_{\xi+1}|\alpha^{\xi}_{\xi} \models_{\alpha^{\xi}_{\xi}} \) "\( p_{\xi}(\alpha^{\xi}_{\xi}) = \dot{Q}_{\alpha^{\xi}_{\xi}\gamma^{\xi}_{\xi}} \)" for any \( \xi < \zeta < \kappa \), and \( \{ \xi < \kappa : \alpha^{\xi}_{\xi} = \alpha \} \) is unbounded for any \( \alpha \in \bigcup_{\xi < \kappa} \text{supp}(p_{\xi}) \). Take \( Y \in [X]^{\kappa^+} \) and \( \delta < \kappa \) so that \( \delta \in \bigcap_{\xi < \kappa} \lim\{ \xi < \kappa : \alpha^{\xi}_{\xi} = \alpha \} \) for any
$p \in Y$. Next take $Z \in [Y]^{\kappa^+}$ so that $\{\alpha^p_\xi : \xi < \delta \in Z\}$ forms a $\Delta$-system with root $d \in [\beta]^{<\kappa}$. Finally take $W \in [Z]^{\kappa^+}$ and $H \in [\delta \times d \times \kappa]^{<\kappa}$ so that

$H = \{(\xi, \alpha^p_\xi, \gamma^p_\xi) : \xi < \delta \land \alpha^p_\xi \in d\}$ for any $p \in W$.

To see that $W$ is linked, fix $p, q \in W$. Inductively we construct a lower bound $r$ of $\{p_\xi, q_\xi : \xi < \delta\} \subseteq P_\beta$ with support $s = \bigcup_{\xi < \delta} \text{supp}(p_\xi) \cup \bigcup_{\xi < \delta} \text{supp}(q_\xi)$. At stage $\alpha \in s$ it suffices to show $r|\alpha \Vdash_{\alpha} \{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subseteq \dot{Q}_\alpha$ is centered.

When $\alpha \in d = \bigcup_{\xi < \delta} \text{supp}(p_\xi) \cap \bigcup_{\xi < \delta} \text{supp}(q_\xi)$, for unboundedly many $\xi < \delta$

$\alpha = \alpha^p_\xi$ by the choice of $\delta$, and hence $r|\alpha \leq p_{\xi+1}|\alpha, q_{\xi+1}|\alpha$ forces $p_\xi(\alpha), q_\xi(\alpha) \in \dot{Q}_{\alpha\gamma}$, where $(\xi, \alpha, \gamma) \in H$, as desired. Otherwise the claim follows, since $r|\alpha \Vdash_{\alpha} \{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subseteq \dot{Q}_\alpha$ is descending. \hfill \Box

We are now ready to prove our main result. By $F^+ \rightarrow (F^+)^\gamma_\omega$ we mean that for any $f : \bigcup_{n<\omega}[X]^n_{\leq} \rightarrow \gamma$ with $X \in F^+$ there is $Y \in F^+$ with $Y \subseteq X$ and $|f^{-1}[Y]^n_{\leq}| = 1$ for any $n < \omega$. Note that $\mathcal{F}_{\kappa\lambda}^+ \rightarrow (\mathcal{F}_{\kappa\lambda}^+)^\gamma_\omega$ iff $\kappa$ is Ramsey for any $1 < \gamma < \kappa$.

**Theorem.** Let $\kappa$ be a supercompact cardinal and $\lambda$ a cardinal $> \kappa$. Then there is a $\kappa^+-c.c.$ poset forcing supercompactness of $\kappa$ and $\mathcal{F}_{\kappa\lambda}^+ \rightarrow (\mathcal{F}_{\kappa\lambda}^+)^\gamma_\omega$ for any $1 < \gamma < \kappa$.

**Proof.** First we force with the Laver poset $[L]$ for $\kappa$ and then add $\lambda$ Cohen subsets of $\kappa$ to ensure supercompactness of $\kappa$ and $\lambda \leq 2^\kappa$ in the further extensions. Next we perform a $\kappa$-support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < 2^{\lambda^{<\kappa}} \rangle$ with $\Vdash_{\alpha} \dot{Q}_\alpha = Q_f$ for some canonical $P_\alpha$-name $\dot{f}$ for a coloring. The standard inductive argument, together with $\kappa$-closure and $\kappa^+-c.c.$ of $P_\alpha$ shows that $P_\alpha$ is of size $\leq 2^{\lambda^{<\kappa}}$, and so is the set of canonical $P_\alpha$-names for colorings for any $\alpha < 2^{\lambda^{<\kappa}}$, whose union can be identified with that of canonical $P_{2^\lambda^{<\kappa}}$-names for colorings. Thus the iteration can
be arranged so that a homogeneous set for a coloring in the final model by $P_{2^{\lambda}}<\kappa$
appears in an intermediate model, which remains unbounded by absoluteness of $\mathcal{P}_{\kappa\lambda}$, as desired. □

REFERENCES


INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, 305-8571 JAPAN
E-mail address: shioya@math.tsukuba.ac.jp