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Partition properties on $\mathcal{P}_\kappa\lambda$

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Abstract

M. Magidor [9] showed that if part*(κ,λ) holds then κ is λ-ineffable. In [6], we showed that, under some additional assumption, the reverse implication also holds. The main purpose of this paper is to improve this result. We will prove that if κ is $\lambda^{<\kappa}$-ineffable then part*(κ,λ) holds. Also, using a similar technic in the proof of the above result, we prove that if κ is λ-supercompact then there is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ with the partition property. This result is some variation of a Menas's result: If κ is $2^{\lambda^{<\kappa}}$-supercompact, then there are $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters on $\mathcal{P}_\kappa\lambda$ with the partition property.

1 Introduction

There are several combinatorial properties related with supercompactness. Partition properties and ineffabilities are some of these properties. In fact, Menas [11, Theorem 3] proved that if κ is $2^{\lambda^{<\kappa}}$-supercompact then there are $2^{2^{\lambda^{<\kappa}}}$ many normal ultrafilters on $\mathcal{P}_\kappa\lambda$ with the partition property, and Magidor [9] proved that if part*(κ,λ) holds then κ is λ-ineffable and that if κ is λ-ineffable, for all $\lambda\geq\kappa$ then κ is supercompact. In the above results, it is natural to ask whether λ-ineffability imply the partition

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property. In [6], we proved that, under some additional assumption of κ, if κ is $\lambda^{<\kappa}$-ineffable then part$^*(\kappa, \lambda)$ holds. The main purpose of this paper is to improve this result. We prove

Theorem 4.3 If κ is $\lambda^{<\kappa}$-ineffable, then part$^*(\kappa, \lambda)$ holds.

In the proof of Theorem 4.3, we bollow an idea of how to use the notion of presubtlity which is appeared in Kunen and Pelletier's paper [8]. (The notion of presubtlity does not appear in this paper.) Also, using a similar technic in the proof of Theorem 4.3, we prove

Theorem 5.1 If κ is λ-supercompact then there is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ with the partition property.

This is a variation of the above Menas's result.

The paper consists five sections. In the next section, we give some notations and definitions which we will be used the following sections. Section 3 will be devoted to give some lemmas which will be used to prove Theorem 4.3. Theorem 4.3 will be proved in section 4. Theorem 5.1 will be proved in section 5.

2 Notations and definitions

We use standard $\mathcal{P}_\kappa\lambda$-combinatorial terminologies (e.g., see [7]). Throughout this paper, κ denotes a regular uncountable cardinal. Let $\mathcal{I}$ be an ideal on a set S. $\mathcal{I}^*$ denotes the dual filter and $\mathcal{I}^+$ denotes the set $\mathcal{P}(S) \setminus \mathcal{I}$. For any subset $S' \subset S$, $\mathcal{I}{\restriction}S'$ denotes $\mathcal{I} \cap \mathcal{P}(S')$. For any function $f : S \to T$, $f_*(\mathcal{I})$ denotes the ideal $\{ X \subset T \mid f^{-1}X \in \mathcal{I} \}$ on T.

Let A be a set such that $\kappa \subset A$. $\mathcal{P}_\kappa A$ denotes the set $\{ x \subset A \mid |x| < \kappa \}$. For each $x \in \mathcal{P}_\kappa A$, Q_x denotes the set $\mathcal{P}_{|x \cap \kappa|}x$. For any $x, y \in \mathcal{P}_\kappa A$, $x \prec y$ means that $x \in Q_y$. Let Y be a subset of $\mathcal{P}_\kappa A$. Y is said to be <u>unbounded</u>, if for any $x \in \mathcal{P}_\kappa A$ there exists a $y \in Y$ such that $x \subset y$. Y is called a <u>club</u>, if Y is unbounded and closed under

$\subset$-increasing chains with length $< \kappa$. Y is said to be stationary, if $X \cap C \neq \phi$, for any club $C \subset \mathcal{P}_\kappa A$. Let us denote by $\mathrm{NS}_{\kappa,A}$ the set of all non-stationary subsets of $\mathcal{P}_\kappa A$.

A function $f : Y \to A$ is said to be regressive, if $f(x) \in x$ holds, for all $x \in Y$. An ideal $\mathcal{I}$ on $\mathcal{P}_\kappa A$ is normal, if it contains all bounded subsets and, for any $X \in \mathcal{I}^+$, and any regressive function $f : X \to A$, there exists an $a \in A$ such that $f^{-1}\{\,a\,\} \in \mathcal{I}^+$. It is known [11] that $\mathrm{NS}_{\kappa,A}$ is the smallest normal ideal on $\mathcal{P}_\kappa A$.

For each function $\tau : \mathcal{P}_\kappa A \to \mathcal{P}_\kappa A$, $\mathrm{cl}(\tau)$ denotes the set $\{\, x \in \mathcal{P}_\kappa A \mid \forall t \in Q_x \ (\ \tau(t) \in Q_x\ )\,\}$. For each $\tau : A \times A \to \mathcal{P}_\kappa A$, $\mathrm{cl}(\tau)$ denotes the set $\{\, x \in \mathcal{P}_\kappa A \mid \forall \alpha, \beta \in x \ (\ \tau(\alpha, \beta) \subset x\ )\,\}$. It is known [10] that, for any $X \subset \mathcal{P}_\kappa A$, X contains a club if and only if there exists a $\tau : A \times A \to \mathcal{P}_\kappa A$ such that $\mathrm{cl}(\tau) \subset X$. For any $B \supset A$, the function $p : \mathcal{P}_\kappa B \to \mathcal{P}_\kappa A$ which is defined by $p(x) = x \cap A$ is called the projection.

Let $Y \subset \mathcal{P}_\kappa A$. $[Y]^2$ denotes the set $\{\,(x, y) \in Y \times Y \mid x \subset y \text{ and } x \neq y\,\}$. For any function $f : [Y]^2 \to 2$, a subset H of Y is said to be homogeneous for f, if $|f``[H]^2| = 1$. We say that Y has the partition property, if for any $f : [Y]^2 \to 2$, there exists a stationary subset H of Y such that H is homogeneous for f. Y is said to be ineffable (almost ineffable), if for any $s_x \subset x$ (for $x \in Y$), there exists an $S \subset A$ such that $\{\, x \in Y \mid s_x = S \cap x\,\}$ is stationary (unbounded). Set

$\mathrm{NP}_{\kappa,A} = \{\, X \subset \mathcal{P}_\kappa A \mid X \text{ does not have the partition property}\,\}$,

$\mathrm{NIn}_{\kappa,A} = \{\, X \subset \mathcal{P}_\kappa A \mid X \text{ is not ineffable}\,\}$, and

$\mathrm{NAIn}_{\kappa,A} = \{\, X \subset \mathcal{P}_\kappa A \mid X \text{ is not almost ineffable}\,\}$.

It is known [3] that $\mathrm{NP}_{\kappa,A}$, $\mathrm{NIn}_{\kappa,A}$, and $\mathrm{NAIn}_{\kappa,A}$ are normal ideals on $\mathcal{P}_\kappa A$.

We say that $\mathrm{part}^*(\kappa, A)$ holds, if $\mathrm{NP}_{\kappa,A}$ is a proper ideal and that κ is A-ineffable, if $\mathrm{NIn}_{\kappa,A}$ is a proper ideal, and that κ is almost A-ineffable, if $\mathrm{NAIn}_{\kappa,A}$ is a proper ideal. It is known that, for any $B \supset A$, $\mathrm{NIn}_{\kappa,A} \subset p_*(\mathrm{NIn}_{\kappa,B})$, where p denotes the projection from $\mathcal{P}_\kappa B$ to $\mathcal{P}_\kappa A$.

Let U be an ultrafilter on $\mathcal{P}_\kappa A$. We say that U is normal, if the dual ideal of U is a normal ideal. U has the partition property, if for any $X \in U$ and any $f : [X]^2 \to 2$, there exists $Y \in U$ such that $Y \subset X$ and Y is homogeneous for f.

3 Several Lemmas

In this section, we will state some lemmas which will be used in the next section. From now on, λ denotes a cardinal greater than or equal to κ.

3.1 The λ-Shelah property

The λ-Shelah property was first introduced by Carr [3]. A subset $X \subset \mathcal{P}_\kappa\lambda$ has the Shelah property, if for any $f_x : x \to x$ (for $x \in X$), there exists a function $f : \lambda \to \lambda$ such that

$$\forall x \in \mathcal{P}_\kappa\lambda\ \exists y \in X\ (\ x \subset y \text{ and } f_x = f{\restriction}x\ ).$$

Set $\mathrm{NSh}_{\kappa,\lambda} = \{\, X \subset \mathcal{P}_\kappa\lambda \mid X \text{ does not have the Shelah property}\,\}$. It is known that $\mathrm{NSh}_{\kappa,\lambda}$ is a normal ideal on $\mathcal{P}_\kappa\lambda$ and $\mathrm{NSh}_{\kappa,\lambda} \subset \mathrm{NAIn}_{\kappa,\lambda}$. We say that κ is λ-Shelah, if $\mathrm{NSh}_{\kappa,\lambda}$ is a proper ideal.

The following two lemmas are due to Carr [2, 3].

Lemma 3.1 (Carr [2]) $\{\, x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa \textit{ is an inaccesible cardinal}\,\} \in \mathrm{NSh}^*_{\kappa,\lambda}$. □

Lemma 3.2 (Carr [3]) *If κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact.* □

Let S be an infinite set. A function F from S^ω to S is called an ω-Jonsson function for S, if for any $T \subset S$ with cardinality $|S|$, it holds that $F``T^\omega = S$. Concerning this, Erdös-Hajnal (e.g., see [7, Theorem 23.13]) proved

Lemma 3.3 (Erdös-Hajnal) *For any infinite set S, there exists an ω-Jonsson function for S.* □

Furthermore, we need

Lemma 3.4 (Johnson [4]) *Let $\delta \leq \lambda$ and F an ω-Jonsson function for δ. Then, it holds that*

$$\{ x \in \mathcal{P}_\kappa\lambda \mid F{\restriction}(x \cap \delta)^\omega \textit{ is an } \omega\textit{-Jonsson function for } x \cap \delta \} \in \mathrm{NSh}^*_{\kappa,\lambda}. \qquad \square$$

3.2 How reflects cardinals on a certain set in $\mathrm{NIn}_{\kappa,\lambda}$

In this subsection, we begin with an easy lemma. We left a proof to the reader.

Lemma 3.5 *If δ is a cardinal $\leq \lambda$, then it holds that*

$$\{ x \in \mathcal{P}_\kappa\lambda \mid \mathrm{ot}(x \cap \delta) \textit{ is a cardinal} \} \in \mathrm{NIn}^*_{\kappa,\lambda}. \qquad \square$$

Lemma 3.6 *If γ is a strong limit cardinal $\leq \lambda$, then it holds that*

$$\{ x \in \mathcal{P}_\kappa\lambda \mid |x \cap \gamma| \textit{ is a strong limit cardinal} \} \in \mathrm{NIn}^*_{\kappa,\lambda}.$$

Proof To get a contradiction, assume that

$$X = \{ x \in \mathcal{P}_\kappa\lambda \mid |x \cap \gamma| \text{ is not a limit cardinal} \} \in \mathrm{NIn}^+_{\kappa,\lambda}.$$

For each $x \in X$, take $\alpha_x \in x \cap \gamma$ such that $|x \cap \gamma| \leq 2^{|x \cap \alpha_x|}$. Since $\mathrm{NIn}_{\kappa,\lambda}$ is normal, there is an $\alpha < \gamma$ such that

$$X' = \{ x \in X \mid \alpha_x = \alpha \} \in \mathrm{NIn}^+_{\kappa,\lambda}.$$

For each $x \in X'$, take an injection $f_x : x \cap \gamma \to \mathcal{P}(x \cap \alpha)$ and set

$$a_x = \{ (\xi, \eta) \in x \times x \mid \xi < \gamma \text{ and } f_x(\xi) = \eta \}.$$

Since $X' \in \mathrm{NIn}^+_{\kappa,\lambda}$, there exists an $A \subset \gamma \times \alpha$ such that

$$Y = \{ x \in X' \mid a_x = A \cap x \times x \} \in \mathrm{NS}^+_{\kappa,\lambda}.$$

Define $f : \gamma \to \mathcal{P}(\alpha)$ by

$$f(\xi) = \{ \eta < \alpha \mid (\xi, \eta) \in A \}.$$

Since Y is unbounded, it holds that f is an injection. This contradicts that γ is a strong limit cardinal. $\square$

The following lemma is due to Abe [1].

Lemma 3.7 (Abe [1]) *Let γ, δ be cardinals such that $2^\gamma = \delta \leq \lambda$. Then, it holds that*

$$\{ x \in \mathcal{P}_\kappa\lambda \mid 2^{|x\cap\gamma|} = |x \cap \delta| \} \in \mathrm{NSh}^*_{\kappa,\lambda}.$$

A similar argment gives a proof of the next lemma. We left a proof to the reader.

Lemma 3.8 *If $\gamma \leq \lambda \leq 2^\gamma$, then it holds that*

$$\{ x \in \mathcal{P}_\kappa\lambda \mid |x| \leq 2^{|x\cap\gamma|} \} \in \mathrm{NIn}^*_{\kappa,\lambda}. \qquad \square$$

3.3 $\mathcal{P}_\kappa\lambda$ vasus $\mathcal{P}_\kappa\lambda^{<\kappa}$

Let $\theta = \lambda^{<\kappa}$ and $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ the projection. Take a bijection $h : \mathcal{P}_\kappa\lambda \to \theta$ and define $q : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ by

$$q(y) = \bigcup h^{-1}y, \text{ for all } y \in \mathcal{P}_\kappa\theta.$$

The lemmas of this subsection were appeared in [5, 6] implicitly. Set

$$Y_0 = \{ y \in \mathcal{P}_\kappa\theta \mid p(y) = q(y) \text{ and } h``Q_{p(y)} = y \}.$$

Lemma 3.9 $Y_0 \in \mathrm{NIn}^*_{\kappa,\theta}$.

Proof It suffices to show that

(1) $\{ y \in \mathcal{P}_\kappa\theta \mid p(y) \subset q(y) \} \in \mathrm{NIn}^*_{\kappa,\theta}$,

(2) $\{ y \in \mathcal{P}_\kappa\theta \mid q(y) \subset p(y) \} \in \mathrm{NIn}^*_{\kappa,\theta}$,

(3) $\{ y \in \mathcal{P}_\kappa\theta \mid h``Q_{p(y)} \subset y \} \in \mathrm{NIn}^*_{\kappa,\theta}$,

(4) $\{ y \in \mathcal{P}_\kappa\theta \mid y \subset h``Q_{p(y)} \} \in \mathrm{NIn}^*_{\kappa,\theta}$.

These are proved by similar argments. We only deal (3). To get a contradiction, assume that

$$Y = \{ y \in \mathcal{P}_\kappa\theta \mid h``Q_{p(y)} \not\subset y \} \in \mathrm{NIn}^+_{\kappa,\theta}.$$

For each $y \in Y$, Take $t_y \in Q_{p(y)}$ such that $h(t_y) \notin y$. Since $Y \in \mathrm{NIn}^+_{\kappa,\theta}$, there exists $T \subset \lambda$ such that

$$Z = \{ y \in Y \mid T \cap y = t_y \} \in \mathrm{NS}^+_{\kappa,\theta}.$$

Claim 1 $|T| < \kappa$.

Proof of Claim 1 Suppose not. Take an injection $f : \kappa \to T$ and set

$$C = \{\, y \in \mathcal{P}_\kappa\theta \mid f``(y \cap \kappa) \subset y \,\}.$$

Since C is a club of $\mathcal{P}_\kappa\theta$, there exists a $y \in C \cap Z$. Since $y \in C$, we have that $|y \cap \kappa| \leq |y \cap T|$. Since $y \in Z$, we have that $|y \cap T| = |t_y| < |y \cap \kappa|$. This is a desired contradiction. QED of Claim 1

By Claim 1, set $\alpha = h(T)$. Since Z is unbounded, we can take a $y \in Z$ such that $T \cup \{\, \alpha \,\} \subset y$. Then, $t_y = T$ and $h(t_y) = \alpha \in y$. This contradicts the choise of t_y. □

Lemma 3.10 *Let $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$. Then, for any $a_x \subset Q_x$ $(x \in X)$, there exists an $A \subset \mathcal{P}_\kappa\lambda$ such that*

$$\forall \tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda \; \exists x \in X \cap \mathrm{cl}(\tau) \; (\, a_x = A \cap Q_x \,).$$

Proof Set $Y = p^{-1}X \cap Y_0$. By Lemma 3.9, $Y \in \mathrm{NIn}^+_{\kappa,\theta}$. For each $y \in Y$, set $b_y = h``a_{p(y)}$. Then, since $y \in Y_0$, it holds that

$$b_y \subset y, \text{ for all } y \in Y,$$

So, there exists a $B \subset \theta$ such that

$$Y' = \{\, y \in Y \mid b_y = B \cap y \,\} \in \mathrm{NS}^+_{\kappa,\theta}.$$

Set $A = h^{-1}B$. We claim that A is as required. To show this, let $\tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$. Define $\tau' = h\tau h^{-1} : \theta \to \theta$. Since $Y' \in \mathrm{NS}^+_{\kappa,\theta}$, there exists a $y \in Y' \cap \mathrm{cl}(\tau')$. Then, it is easy to check that $p(y) \in X \cap \mathrm{cl}(\tau)$ and $a_{p(y)} = A \cap Q_{p(y)}$. □

Lemma 3.11 *Suppose that* $\mathrm{part}^*(\kappa, \lambda)$ *fails. Then, it holds that*

$$\{\, x \in \mathcal{P}_\kappa\lambda \mid \mathrm{part}^*(x \cap \kappa, x) \textit{ fails} \,\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^*.$$

Proof To get a contradiction, assume that

$$X = \{\, x \in \mathcal{P}_\kappa\lambda \mid \mathrm{part}^*(x \cap \kappa, x) \text{ holds} \,\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^+.$$

Let $X' = \{\, x \in X \mid x \cap \kappa \text{ is inaccesible} \,\}$. By Lemma 3.1, $X' \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$. Since part*$(\kappa,\lambda)$ fails, there exists a function $f : [\mathcal{P}_\kappa\lambda]^2 \to 2$ such that

$$\forall H \in \mathrm{NS}^+_{\kappa,\lambda} \text{ (} H \text{ is not homogeneous for } f \text{).}$$

For each $x \in X'$, take $H_x \in \mathrm{NS}^+_{\kappa\cap x,x}$ and $e_x < 2$ such that $f``[H_x]^2 = \{ e_x \}$. By Lemma 3.10, there exists an $H \subset \mathcal{P}_\kappa\lambda$ and $e < 2$ such that

(*) $\forall \tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda \; \exists x \in X' \cap \mathrm{cl}(\tau)$ ($H_x = H \cap Q_x$ and $e_x = e$).

It is easy to check that H is homogeneous for f. It suffices to complete the proof to show that H is stationary. So, let C be a club of $\mathcal{P}_\kappa\lambda$. Take a function $\tau : \mathcal{P}_\kappa\lambda \to C$ such that $x \subset \tau(x)$. Then, by (*), there exists an $x \in X' \cap \mathrm{cl}(\tau)$ such that $H_x = H \cap Q_x$. Since $x \in \mathrm{cl}(\tau)$, it holds that $C \cap Q_x$ is club in Q_x. So, it holds that $\phi \neq H_x \cap C \cap Q_x \subset H \cap C$. □

4 Proof of Main Theorem

In this section, we prove the main theorem. In the proof, we will use some known results. The next lemma is well-known. But, I don't know who established it.

Lemma 4.1 (folklore) *If κ is $<\lambda$-supercompact and λ is θ-supercompact, then κ is θ-supercompact.* □

The next lemma was appeared in [5].

Lemma 4.2 $\{ x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa$ *is almost x-ineffable* $\} \in p_*(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}})^*$,

where p denotes the projection from $\mathcal{P}_\kappa\lambda^{<\kappa}$ to $\mathcal{P}_\kappa\lambda$. □

Theorem 4.3 *If κ is $\lambda^{<\kappa}$-ineffable, then* part*(κ,λ) *holds.*

Proof To get a contradiction, assume that κ is $\lambda^{<\kappa}$-ineffable and part*(κ,λ) fails. Let $\theta = \lambda^{<\kappa}$, $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ the projection, and δ the largest strong limit cardinal $\leq \lambda$. Define δ_i (for $i < \omega$) by $\delta_0 = \delta$ and $\delta_{i+1} = 2^{\delta_i}$. Let $n < \omega$ be such that $\delta_n \leq \lambda < \delta_{n+1}$. Take ω-Jonsson functions F and F_i (for $i \leq n$) for λ and δ_i (for $i \leq n$), respectively.

Let X be the set of all $x \in \mathcal{P}_\kappa\lambda$ which satisfies

(1) $x \cap \kappa$ is an inaccessible cardinal,

(2) $x \cap \delta$ is a strong limit cardinal cardinal,

(3) $\mathrm{ot}(x \cap \delta_{i+1}) = 2^{\mathrm{ot}(x \cap \delta_i)}$, for all $i < n$ and $\mathrm{ot}(x) \leq 2^{\mathrm{ot}(x \cap \delta_n)}$,

(4) $F{\restriction}x$ and $F_i{\restriction}(x \cap \delta_i)^\omega$ are ω-Jonsson functions for x and $x \cap \delta_i$, for $i \leq n$, respectively,

(5) $x \cap \kappa$ is almost x-ineffable,

(6) $\mathrm{part}^*(x \cap \kappa, x)$ fails.

By Lemmas 3.1, 3.4, 3.6, 3.7, 3.8, 3.11, and 4.2, it holds that $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^*$. By this, since κ is θ-ineffable, we have that $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$. Note that, by Lemma 3.2 and (5), it holds that

(7) $x \cap \kappa$ is $x \cap \alpha$-supercompact, for all $\alpha \in x \cap [\kappa, \delta)$, for all $x \in X$.

Claim 2 $\forall (x, y) \in [X]^2$ (*if* $x \cap \delta_n \neq y \cap \delta_n$ *then* $x \prec y$).

Proof of Claim 2 To get a contradiction, assume that there exists $(x, y) \in [X]^2$ such that

$$x \cap \delta_n \neq y \cap \delta_n \text{ and not } x \prec y.$$

By (3), it holds that $x \cap \delta \neq y \cap \delta$. Since $F{\restriction}(y \cap \delta)^\omega$ is ω-Jonsson, it holds that $|x \cap \delta| < |y \cap \delta|$. Since $|y \cap \kappa|$ and $|y \cap \delta|$ is strong limit cardinals, we have that

(8) $2^{|x|} < |y \cap \delta|$ and $y \cap \kappa \leq |x \cap \delta|$.

By this, and (7), and Lemma 4.1, it holds that

$$x \cap \kappa \text{ is } x\text{-supercompact.}$$

But this contradicts that $\mathrm{part}^*(x \cap \kappa, x)$ fails. QED of Claim 2

We complete the proof by proving that $X \in \mathrm{NP}^+_{\kappa,\lambda}$. The proof is divided into two cases.

Case 1. $\lambda = \delta_n$.

By Claim 2, it holds that

$$\forall (x,y) \in [X]^2 \ (x \prec y).$$

In order to show that $X \in \mathrm{NP}^+_{\kappa,\lambda}$, let $f : [X]^2 \to 2$. Define $a_x \subset Q_x$ (for $x \in X$) by

$$a_x = \{ t \in Q_x \mid t \in X \text{ and } f(t,x) = 0 \}.$$

Then, by Lemma 3.10, there exists an $A \subset \mathcal{P}_\kappa\lambda$ such that

$$\forall \tau : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda \ \exists x \in X \cap \mathrm{cl}(\tau) \ (a_x = A \cap Q_x).$$

Set $X' = \{ x \in X \mid a_x = A \cap Q_x \}$. Note that $X' \in \mathrm{NS}^+_{\kappa,\lambda}$. It is easy to check that

$$\forall (x,y) \in [X' \cap A]^2 \ (f(x,y) = 0) \text{ and } \forall (x,y) \in [X' \setminus A]^2 \ (f(x,y) = 1).$$

So, $X' \cap A$ or $X' \setminus A$ is as required.

Case 2. $\delta_n < \lambda$.

Define $g,\ f_i : \kappa \to \kappa$ (for $i \leq n+1$) by

$$g(\alpha) = \begin{cases} \text{the smallest } \beta \geq \alpha \text{ such that } \alpha \text{ is not } \beta\text{-supercompact}, & \text{if such } \beta \text{ exists} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(\alpha) = \text{the largest strong limit cardinal} \leq g(\alpha),$$

$$f_{i+1}(\alpha) = 2^{f_i(\alpha)}, \text{ for all } \alpha < \kappa \text{ and } i \leq n.$$

For any $x \in X$, since $\mathrm{ot}(x \cap \delta) \leq g(x \cap \kappa) \leq \mathrm{ot}(x)$, it holds that

$$f_0(x \cap \kappa) = \mathrm{ot}(x \cap \delta) \text{ and } f_n(x \cap \kappa) = \mathrm{ot}(x \cap \delta_n) \leq \mathrm{ot}(x) \leq f_{n+1}(x \cap \kappa).$$

For each $\alpha < \kappa$, take $w^\alpha_\xi \subset f_n(\alpha)$ (for $\xi \leq f_{n+1}(\alpha)$) such that

$$\forall \xi < \forall \eta \leq f_{n+1}(\alpha) \ (w^\alpha_\xi \neq w^\alpha_\eta).$$

For each $x \in X$, define π_x and s_x by

$\pi_x : \mathrm{ot}(x \cap \delta_n) \to x \cap \delta_n$ is the order isomorphism, and

$$s_x = \pi_x \text{``} w^{x \cap \kappa}_{\mathrm{ot}(x)} \ (\subset x \cap \delta_n).$$

To show that $X \in \mathrm{NP}^+_{\kappa,\lambda}$, let $f : [X]^2 \to 2$. As in the case 1, set

$$a_x = \{ t \in Q_x \mid t \in X \text{ and } f(t,x) = 0 \}, \text{ for } x \in X.$$

Since $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$, there exist $S \subset \delta_n$ and $A \subset \mathcal{P}_\kappa\lambda$ such that

$$X' = \{ x \in X \mid s_x = S \cap x \text{ and } a_x = A \cap Q_x \} \in \mathrm{NS}^+_{\kappa,\lambda}.$$

Claim 3 $\forall (x,y) \in [X']^2 \; (\; x \cap \delta_n \neq y \cap \delta_n \;)$.

Proof of Claim 3 To get a contradiction, assume that

$$(x,y) \in [X']^2 \text{ and } x \cap \delta_n = y \cap \delta_n.$$

Note that $s_x = s_y$. Set $\alpha = x \cap \kappa \; (= y \cap \kappa)$, $\xi = \mathrm{ot}(x)$, $\eta = \mathrm{ot}(y)$. Since $|x| < |y|$, it holds that $\xi < \eta$. Since $x \cap \delta_n = y \cap \delta_n$, it holds that $\pi_x = \pi_y$. By this, since $w^\alpha_\xi \neq w^\alpha_\eta$, we have that

$$s_x = \pi_x``w^\alpha_\xi \neq \pi_y``w^\alpha_\eta = s_y.$$

This is a contradiction. QED of Claim 3

By Claims 2 and 3, it holds that

$$\forall (x,y) \in [X']^2 \; (\; x \prec y \;).$$

So, $X' \cap A$ or $X' \setminus A$ is as a desired homogeneous set for f. □

Corollary 4.4 *Let $\kappa \leq \lambda \leq \mu$. If* $\mathrm{part}^*(\kappa, \mu)$ *holds, then* $\mathrm{part}^*(\kappa, \lambda)$ *holds.*

Proof Assume that $\mathrm{part}^*(\kappa, \mu)$ holds. The case $\lambda = \mu$ is trivial. We assume that $\lambda < \mu$. By a result of Magidor [9], it holds that κ is μ-ineffable. Then, by a result of Johnson [4], it holds that $\lambda^{<\kappa} \leq \lambda^+ \leq \mu$. So, κ is $\lambda^{<\kappa}$-ineffable. So, $\mathrm{part}^*(\kappa, \lambda)$ holds. □

5 Normal ultrafilters with the partition property

Concerning the partition property of a normal ultrafilter on $\mathcal{P}_\kappa\lambda$, Solovay (see Menas [11]) proved the existence of a normal ultrafilter without the partition property under the assumption of that the existence of a certain large cardinal greater than κ. After Solovay established this result, Kunen (see Kunen-Pelletier [8]) improved his results, and proved that the existence of a normal ultrafilter without the partition property implies the existence of a certain large cardinal above κ. On the other hand, Menas [11] proved that there exist $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters with the partition property, under the assumption that κ is $2^{\lambda^{<\kappa}}$ supercompact. In this section, we prove

Theorem 5.1 *If κ is λ-supercompact, then there exists a normal ultarfilter on $\mathcal{P}_\kappa\lambda$ with the partition property.*

The proof will be done by a similar, but different argment as in the proof of Menas. We first reduce this theorem to a certain lemma (Lemma 5.4, below). Let U be a normal ultrafilter on $\mathcal{P}_\kappa\lambda$. Denote by M_U the ultrapower of the universe by U. The following two lemmas are due to Menas [10, 11].

Lemma 5.2 *If κ is λ-supercompact, then there exists a normal ultrafilter U on $\mathcal{P}_\kappa\lambda$ such that*

$$M_U \models \kappa \textit{ is not } \lambda\textit{-supercompact.}$$

Lemma 5.3 *Let U be a normal ultrafilter on $\mathcal{P}_\kappa\lambda$. Then, the following* (a) *and* (b) *are equivalent.*

(a) *U has the partition property.*

(b) *There exists an $X \in U$ such that $\forall (x, y) \in [X]^2$ ($x \prec y$).*

By these results, the next lemma directly follows Theorem 5.1.

Lemma 5.4 *Suppose that*

(1) $M_U \models \kappa$ *is not λ-supercompact.*

Then, there exists an $X \in U$ such that

(2) $x \prec y$, *for all* $(x, y) \in [X]^2$.

Proof Suppose that U is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ which satisfies (1). Let δ be the largest strong limit cardinal $\leq \lambda$. Define δ_i (for $i < \omega$) by $\delta_0 = \delta$, and $\delta_{i+1} = 2^{\delta_i}$. Let $n < \omega$ be such that $\delta_n \leq \lambda < \delta_{n+1}$. Let F and F_i (for $i \leq n$) be ω-Jonnson functions for λ and δ_i's, respectively. Define $X_0 \subset \mathcal{P}_\kappa\lambda$ by, for any $x \in \mathcal{P}_\kappa\lambda$,

$x \in X_0$ if and only if

(3) $x \cap \kappa$ is inaccesible and $x \cap \kappa$ is not x-supercompact,

(4) $\mathrm{ot}(x \cap \delta_i)$ is a cardinal, for $i \leq n$,

(5) $2^{|x \cap \delta_i|} = |x \cap \delta_{i+1}|$, for $i < n$ and $|x| \leq 2^{|x \cap \delta_n|}$,

(6) $F_i \restriction (x \cap \delta_i)^\omega$ is an ω-Jonnson function for $x \cap \delta_i$, for $i \leq n$,

(7) $F \restriction x^\omega$ is an ω-Jonnson function for x.

Note that $X_0 \in U$.

Claim 4 $\forall (x, y) \in [X_0]^2$ (*if* $x \cap \delta_n \neq y \cap \delta_n$ *then* $x \prec y$).

Proof of Claim 4 To get a contradiction, assume that

$$(x, y) \in [X_0]^2 \text{ and } x \cap \delta_n \neq y \cap \delta_n \text{ and not } x \prec y.$$

Since $y \cap \kappa$ is a strong limit cardinal, it holds that $y \cap \kappa \leq |x \cap \delta_0|$. Since $x \cap \kappa$ is $x \cap \alpha$-supercompact, for all $\alpha \in x \cap [\kappa, \delta)$, we have that

$$x \cap \kappa \text{ is } y \cap \alpha\text{-supercompact, for all } \alpha \in [x \cap \kappa, y \cap \kappa).$$

By this and Lemma 4.1, since $y \cap \kappa$ is $y \cap \alpha$-supercompact, for all $\alpha \in y \cap [\kappa, \delta)$, we have that

$$x \cap \kappa \text{ is } \alpha\text{-supercompact, for all } \alpha \in [x \cap \kappa, \mathrm{ot}(y \cap \delta)).$$

By this, since $x \cap \kappa$ is not x-supercompact, it holds that $|y \cap \delta| \leq |x|$. Since $|y \cap \delta|$ is a strong limit cardinal, we have that $|y \cap \delta| \leq |x \cap \delta|$. By this and (6), $x \cap \delta = y \cap \delta$. This implies that $x \cap \delta_n = y \cap \delta_n$. This contradicts the assumption. QED of Claim 4

By Claim 4, in the case $\lambda = \delta_n$, $X = X_0$ satisfies (2). So, henceforth, we assume that $\delta_n < \lambda$. Define $g : \kappa \to \kappa$ and $f_i : \kappa \to \kappa$ (for $i \leq n+1$) by

$$g(\alpha) = \begin{cases} \text{the least } \beta \geq \alpha \text{ such that } \alpha \text{ is not } \beta\text{-supercompact}, & \text{if such } \beta < \kappa \text{ exists,} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(\alpha) = \text{the largest strongly limit cardinal} \leq g(\alpha),$$

$$f_{i+1}(\alpha) = 2^{f_i(\alpha)}, \text{ for } i \leq n.$$

For each $\alpha < \kappa$, take $\langle s^\alpha_\xi \mid \xi < f_{n+1}(\alpha) \rangle$ such that

$$s^\alpha_\xi \subset f_n(\alpha) \text{ and } s^\alpha_\xi \neq s^\alpha_\eta, \text{ if } \xi \neq \eta.$$

For each $x \in X_0$, define π_x and a_x by

$$\pi_x : \mathrm{ot}(x \cap \delta_n) \to x \cap \delta_n \text{ is the order isomorphism,}$$

$$a_x = \pi_x \text{``} s^{x \cap \kappa}_{\mathrm{ot}(x)}.$$

Since $a_x \subset x \cap \delta_n$, for all $x \in X_0$, there exists an $A \subset \delta_n$ such that

$$X = \{ x \in X_0 \mid a_x = A \cap x \} \in U.$$

We claim that X satisfies (2). To get a contradiction, assume that there exists $(x, y) \in [X]^2$ such that not $x \prec y$. By Claim 4, it holds that $x \cap \delta_n = y \cap \delta_n$. So, we have that $\pi_x = \pi_y$. Set $\alpha = x \cap \kappa$ $(= y \cap \kappa)$, $\xi = \mathrm{ot}(x)$, $\eta = \mathrm{ot}(y)$. Since $\xi \neq \eta$, we have that $s^\alpha_\xi \neq s^\alpha_\eta$. So, $a_x = \pi_x``s^\alpha_\xi = \pi_y``s^\alpha_\xi \neq \pi_y``s^\alpha_\eta = a_y$. But, this contradicts the fact $a_x = A \cap x = A \cap y = a_y$. □

Define the Mitchell ordering $\lhd$ on the set of normal ultrafilters on $\mathcal{P}_\kappa\lambda$ by

$F \lhd U$ if and only if $F \in M_U$.

Similar to measurable cardinals (see Mitchell [12]), $\lhd$ is well-founded ordering and it can be defined

$\mathrm{o}(U) = \sup\{\, \mathrm{o}(F) + 1 \mid F \lhd U \,\}$, for all normal ultrafilter U on $\mathcal{P}_\kappa\lambda$.

Using this, Theorem 5.1 can be restated as:

If $\mathrm{o}(U) = 0$, then U has the partition property.

So, the following question is natural.

Question Suppose that U is a normal ultrafilter which does not have the partition property. How small the value $\mathrm{o}(U)$ is?

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