TITLE:
ON THE VANISHING OF IWASAWA INVARIANTS OF ABSOLUTELY ABELIAN $p$-EXTENSIONS
(Algebraic Number Theory and Related Topics)

AUTHOR(S):
Yamamoto, Gen

CITATION:

ISSUE DATE:
1999-04

URL:
http://hdl.handle.net/2433/63020

RIGHT:
ON THE VANISHING OF IWASAWA INVARIANTS OF ABSOLUTELY ABELIAN $p$-EXTENSIONS

GEN YAMAMOTO (山本 晴)

ABSTRACT. Let $p$ be any odd prime. We determine all absolutely abelian $p$-extension fields such that Iwasawa $\lambda_p$, $\mu_p$ and $\nu_p$-invariants of the cyclotomic $\mathbb{Z}_p$-extension are zero, in terms of congruent conditions, $p$-th power residues, and genus fields.

1. INTRODUCTION

Let $p$ be a prime and $\mathbb{Z}_p$ the ring of $p$-adic integers. Let $k$ be a finite extension of the rational number field $\mathbb{Q}$, $k_\infty$ a $\mathbb{Z}_p$-extension of $k$, $k_n$ the $n$-th layer of $k_\infty/k$, and $A_n$ the $p$-Sylow subgroup of the ideal class group of $k_n$. Iwasawa proved the well-known theorem about the order $\#A_n$ of $A_n$ that there exist integers $\lambda = \lambda(k_\infty/k) \geq 0$, $\mu = \mu(k_\infty/k) \geq 0$, $\nu = \nu(k_\infty/k)$, and $n_0 \geq 0$ such that

$$\#A_n = p^{\lambda n + \mu p^n + \nu}$$

for all $n \geq n_0$. These integers $\lambda = \lambda(k_\infty/k)$, $\mu = \mu(k_\infty/k)$ and $\nu = \nu(k_\infty/k)$ are called Iwasawa invariants of $k_\infty/k$ for $p$. If $k_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $k$, we write $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ for the above invariants, respectively.

In [7], Greenberg conjectured that if $k$ is a totally real, $\lambda_p(k) = \mu_p(k) = 0$. We call this conjecture Greenberg conjecture. For Iwasawa $\lambda_p$, $\mu_p$-invariants of abelian $p$-extension fields of $\mathbb{Q}$, there are results by Greenberg ([7], V), Iwasawa([9]), Fukuda, Komatsu, Ozaki and Taya([6]), Fukuda([4]), and the author([12]), etc. On the other hand, Ferrero and Washington have shown that $\mu_p(k) = 0$ for any abelian extension field $k$ of $\mathbb{Q}$.

In this paper, we will consider a stronger condition than Greenberg conjecture that $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ and determine all absolutely abelian $p$-extensions $k$, i.e. $k$ is an abelian extension of the rational number field $\mathbb{Q}$, with $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ for an odd prime $p$, using the results of G. Cornell and M. Rosen([1])

2. MAIN THEOREM

Throughout this section, we fix an odd prime $p$. For an absolutely abelian $p$-extension field $k$, let $f_k$ be its conductor, i.e. $f_k$ is the minimum positive integer with $k \subseteq \mathbb{Q}(\zeta_{f_k})$. Then, it follows easily that $f_k = p^ap_1 \cdots p_t$, where $a$ is a non-negative integer and $p_1, \cdots, p_t$ are distinct primes which are congruent to 1 modulo $p$. We denote $k_G$ by the genus field of $k$. So $k_G$ is the maximal unramified abelian extension of $k$ such that $k_G/\mathbb{Q}$ is an abelian extension. In general, if $k/\mathbb{Q}$ is an abelian extension of odd degree, then
it has shown by Leopoldt that

$$[k_G : k] = \frac{e_1 e_2 \cdots e_t}{[k : Q]},$$

where $e_1, \cdots, e_t$ are ramification indices of primes which ramify in $k/Q$. Hence in our case, $k_G$ is also an abelian $p$-extension of Q. For instance we denote by $(\cdot)_p$ the $p$-th power residue symbol, i.e., for integers $x, y$, $(\frac{x}{y})_p = 1$ if and only if $x$ is the $p$-th power modulo $y$.

Our main theorem gives a necessary and sufficient condition for $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ in terms of $p$-th power residue symbol, congruent conditions and genus fields:

**Theorem 1.** Let $k$ be an abelian $p$-extension of $Q$, and $f_k = p^{a}p_{1} \cdots p_{t}$ the prime decomposition of its conductor, where primes $p_1, \cdots, p_t$ are distinct. If

$$\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0,$$

then $t \leq 2$. Conversely, in each case of $t = 0$ or $1$ or $2$, the followings are a necessary and sufficient condition of (1):

**In case of $t = 0$ :** (1) holds.

**In case of $t = 1$ :** (1) is equivalent to $k_1 = k_{1,G}$ and,

$$\left( \frac{p}{p_1} \right)_p \neq 1 \text{ or } p_1 \not\equiv 1 \text{ (mod } p^2).$$

**In case of $t = 2$ :** (1) is equivalent to $k_1 = k_{1,G}$, and for $(i, j) = (1, 2)$ or $(2, 1),

$$\left( \frac{p}{p_i} \right)_p \neq 1, \left( \frac{p_i}{p_j} \right)_p \neq 1, p_j \not\equiv 1 \text{ (mod } p^2),$$

and, there exist $x, y, z \in F_p$ such that

$$\left( \frac{p_i p^x}{p_i} \right)_p = 1, \left( \frac{p_i^y}{p_j} \right)_p = 1, p_i p_j^z \equiv 1 \text{ (mod } p^2), \text{ and } xyz \not\equiv -1 \text{ in } F_p$$

In case of $t = 2$, the conditions in Theorem 1 are complicated. So we will give an example. We consider the case $p = 3, p_1 = 7$ and $p_2 = 19$. We denote $k_7$ (resp. $k_{19}$) by the subfield of $Q(\zeta_7)$ (resp. $Q(\zeta_{19})$) with degree 3 over $Q$. As for the condition $k_1 = k_{1,G}$, there exists a field $F$ such that $k_7 \subset F \subset k_7 k_{19}$ and $F \neq k_7 k_{19}, k_7 Q_1$, where $Q_1$ is the first layer of cyclotomic $Z_3$-extension of $Q$. Then $k_7 k_{19} Q_1 / F$ is a nontrivial unramified extension and $k_7 k_{19} Q_1$ is abelian, hence $F \subset k_7 k_{19} Q_1 \subset F_G$. But, for $F_1 = k_7 k_{19} Q_1$, it follows easily that $F_1 = F_{1,G}$. If we restrict the case $p$ is unramified in $k$, i.e. $a = 0$, then, the statement $k_1 = k_{1,G}$ can be simplified to $k = k_G$ because $k_1 = k Q_1$. This restriction is not so strong: In general, for an absolutely abelian $p$-extension field $k$, there exists an absolutely abelian extension field $k'$ such that $p$ is unramified in $k'$ and $k_{\infty} = k'_{\infty}$. Note that if $k$ is the maximal subfield of $Q(\zeta_m)$ ($m = p^a p_1 \cdots p_t$ as above) which is abelian $p$-extension of $Q$, then $k = k_G$.

We continue to examine the above example. If we put $(i, j) = (1, 2)$, then $p_2 = 19 \equiv 1 \text{ (mod } 3^2)$, so the condition (3) is not satisfied. But if we put $(i, j) = (2, 1)$, then we can
verify that $p_i = 19$ and $p_j = 7$ satisfy the conditions (3) and (4). Hence, for example, if $K$ is the maximal subfield of $Q(\zeta_{7,19})$ which is 3-extension of $Q$, then $K$ satisfies the conditions of Theorem 1. Therefore we get

$$\lambda_p(K) = \mu_p(K) = \nu_p(K) = 0.$$ 

As for Greenberg conjecture, we can also get the following: In general, it is known that if $L \subseteq M$ then $\lambda_p(L) \leq \lambda_p(M)$ and $\mu_p(L) \leq \mu_p(M)$ for number fields $L, M$. Hence for any subfield $k$ of $Q(\zeta_{7,19})$ which is 3-extension of $Q$, i.e. $k \subseteq K$, then $\lambda_p(k) = \mu_p(k) = 0$. This consideration is generalized as follows:

**Corollary 2.** Let $m = p^a p_1 \cdots p_i$ satisfy the condition (2) or (3), (4). Then for any subfield $k$ of $Q(\zeta_m)$ which is $p$-extension of $Q$, Greenberg conjecture for $k$ and $p$ is valid.

3. The results of G. Cornell and M. Rosen

In this section, we review briefly part of [1]. Let $K/Q$ be an abelian $p$-extension, $p$ a prime. In the 1950's, A. Fröhlich determined all such fields with class number prime to $p$ (cf. [2]). In [1], G. Cornell and M. Rosen reconsidered this problem in the case where $p$ is an odd prime, and reduced the problem to the case when $Gal(K/Q)$ is an elementary abelian $p$-group, i.e. $Gal(K/Q) \cong (\mathbb{Z}/p\mathbb{Z})^m$ for some integer $m$.

We suppose that $p$ is an odd prime and $Gal(K/Q)$ is an abelian $p$-group. Then the genus field $K_G$ of $K$ is also abelian $p$-extension. If $p$ does not divide the class number $h_K$ of $K$, then $K$ does not have any non-trivial unramified abelian $p$-extension by class field theory, hence $K_G = K$. In the following we will assume $K_G = K$. Further, we consider the central $p$-class field $K_C$ of $K$, i.e. $K_C$ is the maximal $p$-extension of $K$ such that $K_C/K$ is abelian and unramified, $K_C/Q$ is Galois and $Gal(K_C/K)$ is in the center of $Gal(K_C/Q)$. Since a $p$-group must have a lower central series that terminates in the identity, one sees that $p \nmid h_K$ if and only if $K_C = K$. So we are interested in which case $K_C = K$. This can be reduced the case when $Gal(K/Q)$ is an elementary abelian $p$-group by the following result:

**Lemma 3 ([1] Theorem 1).** Let $K/Q$ be an abelian $p$-extension with $K_G = K$. Let $k$ be the maximal intermediate extension between $Q$ and $K$ such that $Gal(k/Q)$ is an elementary abelian $p$-group. Then $p$-rank of $Gal(K_G/K)$ is equal to the $p$-rank of $Gal(k_G/k)$.

In the case $Gal(K/Q)$ is an elementary abelian $p$-group, by the results of Furuta and Tate, we have the following lemma:

**Lemma 4 ([1] Section 1).** Let $K$ be an absolutely abelian $p$-extension such that $Gal(K/Q)$ is an elementary abelian $p$-group and $K_G = K$. Then, we have

$$Gal(K_C/K) \cong \text{Coker}(\oplus_{i=1}^n \wedge^2(G_i) \rightarrow \wedge^2(G)),$$

where $G_i$'s are the decomposition groups of primes ramified in $K/Q$ and $G = Gal(K/Q)$. 
We will assume $\text{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^m$. Let $p_1, \ldots, p_i$ be the primes ramified in $K$ and $h_K$ the class number of $K$. From genus theory, it follows that if $h_K$ is not divisible by $p$, then $t = m$. Also it follows that if $m \geq 4$ then $p$ divides $h_K$ by Lemma 4. So, we assume $t = m$ and $m = 2$ or 3. (In case of $t = m = 1$, $p \nmid h_K$. cf. [8].)

Lemma 5 ([1] Proposition 2). Suppose $m = 2$ and $p_i \neq p$ for $i = 1, 2$. Then $p| h_K$ if and only if $(\frac{p}{p_1})_p = 1$ and $(\frac{p}{p_2})_p = 1$.

Next, we consider the case where one of the ramified primes is $p$. Suppose $m = 2$ and $p$ and $p_1$ are the only primes ramified in $K$. Then we can easily $K = k(p_1)\mathbb{Q}_1$ and $p_1 \equiv 1 \pmod{p}$, where $k(p_1)$ is the unique subfield of $\mathbb{Q}(\zeta_{p_1})$ which is cyclic over $\mathbb{Q}$ of degree $p$, $\zeta_{p_1}$ is a primitive $p_1$-th root of unity, and $\mathbb{Q}_1$ is the first layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Lemma 6 ([1] Proposition 3). Suppose $m = 2$ and $p$ and $p_1$ are the only primes ramified in $K$. Then $p| h_K$ if and only if $(\frac{p}{p_1})_p = 1$ and $p_1 \equiv 1 \pmod{p^2}$.

Suppose $t = m = 3$ and $p_1, p_2$ and $p_3$ all the primes ramified in $K$. We put $D_{p_i}$ the decomposition field of $p_i (i = 1, 2, 3)$ in $K$. In [1], the following simple result is given:

Lemma 7 ([1] Theorem 2). Suppose $t = m = 3$. Following statements (a) and (b) are equivalent:

(a) $h_K$ is not divisible by $p$,

(b) $[D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = [D_{p_3} : \mathbb{Q}] = p$ and $D_{p_1}D_{p_2}D_{p_3} = K$.

In the next section, we shall prove Theorem 1, using these results.

4. PROOF OF THEOREM 1

Notations are as in previous section.

Firstly, we suppose $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$. Clearly, this condition is equivalent to $A(k_n) = 0$ for any sufficiently large $n$. Then, $k_n$ satisfies $k_n = k_{n,G}$ and $k_1 = k_{1,G}$, because all ramified primes are totally ramified in $k_n/k_1$. Since $k_n$ is also an abelian $p$-extension of $\mathbb{Q}$, we can apply the results of Cornell-Rosen:

Let $L$ be the maximal subfield of $k_n$ such that $\text{Gal}(L/\mathbb{Q})$ is an elementary abelian extension of $\mathbb{Q}$. Since $k_n = k_{n,G}$, $\text{Gal}(k_n/\mathbb{Q})$ is the direct sum of the inertia groups of primes ramified in $k_n/\mathbb{Q}$, hence it follows that $L = k(p_1) \cdots k(p_i)\mathbb{Q}_1$. By Lemma 3, $A(k_n) = 0$ is equivalent to $p \nmid h_L$. Note that if $t \geq 3$ then we always have $p|h_L$ as in the previous section. Hence we may examine in each case of $t = 0$ or 1 or 2.

If $t = 0$ then $L = \mathbb{Q}_1$, hence it is well known that $A(L) = A(\mathbb{Q}_1) = 0$ (cf. [8]).

If $t = 1$ then $L = k(p_1)\mathbb{Q}_1$. By lemma 6, we get the statement in Theorem 1.

In the following we assume that $t = 2$. In this case, $L = k(p_1)k(p_2)\mathbb{Q}_1$. Let $G_p, G_{p_i}(i = 1, 2)$ be the decomposition groups for $p, p_i$ in $\text{Gal}(L/\mathbb{Q})$ and let $D_p, D_{p_i}$ be the fixed field of $G_p, G_{p_i}$, respectively. We note that $D_p \subset k(p_1)k(p_2)\mathbb{Q}_1$ and $D_{p_i} \subset k(p_1)\mathbb{Q}_1$.

Now, from our assumption $p \nmid h_L$, we have $[D_p : \mathbb{Q}] = [D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p$ and $D_pD_{p_1}D_{p_2} = L$ by Lemma 7. Here, we assume that either $(\frac{p}{p_1})_p = 1$ or $(\frac{p}{p_2})_p = 1$ or
p_2 \equiv 1 \pmod{p^2} \text{ holds, and either } \left( \frac{p}{p_2} \right)_p = 1 \text{ or } \left( \frac{p_2}{p_1} \right)_p = 1 \text{ or } p_1 \equiv 1 \pmod{p^2}. \text{ This is equivalent to}

\[ D_p = k(p_1) \text{ or } D_{p_1} = k(p_2) \text{ or } D_{p_2} = \mathbb{Q}_1 \text{ for } (i,j) = (1,2) \text{ and } (2,1), \]

Because \([D_p : \mathbb{Q}] = [D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p\).

If \(D_p = k(p_1)\), then \(D_{p_2} \neq k(p_1)\) because \(D_pD_{p_1}D_{p_2} = L\). Hence by (5) (put \((i,j) = (2,1)\)), we have \(D_{p_1} \subseteq k(p_1)\mathbb{Q}_1 = D_pD_{p_1}\), which contradicts \(D_pD_{p_1}D_{p_2} = L\). In the same way, if \(D_p = k(p_2)\), then \(D_{p_1} \neq k(p_2)\) and we have \(D_{p_2} = \mathbb{Q}_1\) by (5), which contradicts. Thus, it follows that the assumption (5) cause contradiction. Therefore, for \((i,j) = (1,2)\) or \((2,1)\), \(\left( \frac{p}{p_i} \right)_p \neq 1, \left( \frac{p_1}{p_j} \right)_p \neq 1, \text{ and } p_j \neq 1 \pmod{p^2}\).

Without loss of generality, we may assume \((i,j) = (1,2)\). Since \(\left( \frac{p}{p_1} \right)_p \neq 1\), \(p\) is inert in \(k(p_1)\). Hence \(\sigma = \left( \frac{k(p_1)/\mathbb{Q}}{p} \right)_p \neq 1\), where \(\left( \frac{k(p_1)/\mathbb{Q}}{p} \right)\) is the Artin symbol, and \(\sigma\) generates \(\text{Gal}(k(p_1)/\mathbb{Q})\): \(< \sigma > = \text{Gal}(k(p_1)/\mathbb{Q})\). We often regard \(< \sigma > = \text{Gal}(k(p_1)k(p_2)/k(p_2))\) or \(\text{Gal}(L/k(p_2)\mathbb{Q})\) in the natural way. Similarly, we put \(\tau = \left( \frac{k(p_2)/\mathbb{Q}}{p_1} \right)_p\) and \(\eta = \left( \frac{\mathbb{Q}_1/\mathbb{Q}}{p_2} \right)\), then \(< \tau > = \text{Gal}(k(p_2)/\mathbb{Q})\) and \(< \eta > = \text{Gal}(\mathbb{Q}_1/\mathbb{Q})\).

Since \(\left( \frac{p}{p_1} \right)_p \neq 1\), there exists \(x \in \mathbb{F}_p\) such that \(\left( \frac{p_2p^x}{p_1} \right)_p = 1\). Then

\[
\left( \frac{p_2p^x}{p_1} \right)_p = 1 \iff \left( \frac{k(p_1)/\mathbb{Q}}{p_2p^x} \right)_p = \left( \frac{k(p_1)/\mathbb{Q}}{p_2} \right)_p \left( \frac{k(p_1)/\mathbb{Q}}{p} \right)^x = 1.
\]

Therefore \(\left( \frac{k(p_1)/\mathbb{Q}}{p_2} \right)_p = \sigma^{-x}\). Similarly, we obtain \(y, z \in \mathbb{F}_p\) such that \(\left( \frac{p_1p^y}{p_2} \right)_p = 1\) and \(p_1p_2^x \equiv 1 \pmod{p^2}\), and hence \(\left( \frac{k(p_1)/\mathbb{Q}}{p_2} \right)_p = \tau^{-y}\) and \(\left( \frac{\mathbb{Q}_1/\mathbb{Q}}{p_1} \right)_p = \eta^{-z}\).

Since \(\left( \frac{k(p_1)k(p_2)/\mathbb{Q}}{p} \right)_p = \left( \frac{k(p_1)/\mathbb{Q}}{p} \right)\left( \frac{k(p_2)/\mathbb{Q}}{p} \right) = \sigma\tau^{-y}\), \(D_p\) is the fix field of \(< \sigma\tau^{-y} >\) in \(k(p_1)k(p_2)\). Therefore, when we consider \(G_p\) in \(\text{Gal}(L/\mathbb{Q})\),

\(G_p =< \eta, \sigma\tau^{-y} >\).

And similarly,

\(G_{p_1} =< \sigma, \tau\eta^{-z} >\),

and

\(G_{p_2} =< \tau, \eta\sigma^{-z} >\),

in \(\text{Gal}(L/\mathbb{Q})\).

By a direct computation, we have,

\(G_p \cap G_{p_1} =< \sigma\tau^{-y}\eta^{yz} >\).

Hence,

\(G_p \cap G_{p_1} \cap G_{p_2} =< \sigma\tau^{-y}\eta^{yz} > \cap < \tau, \eta\sigma^{-z} >\)

\[\begin{cases} 
\{1\}, & \text{if } xyz \neq -1, \\
< \sigma\tau^{-y}\eta^{yz} >, & \text{if } xyz = -1.
\end{cases}\]

But, our assumption \(D_pD_{p_1}D_{p_2} = L\) implies \(G_p \cap G_{p_1} \cap G_{p_2} = \{1\}\). Hence \(xyz \neq -1\).
Conversely, we assume $k$ satisfies the conditions of Theorem 1 in case of $t = 2$. Since $k_1 = k_{1,G}$, it follows easily that $L = k(p_1)k(p_2) \mathbb{Q}_1$ is the maximal intermediate extension between $\mathbb{Q}$ and $k_n(n \geq 1)$ such that $\text{Gal}(L/\mathbb{Q})$ is an elementary abelian $p$-group. Without loss of generality, we may assume $(i, j) = (1, 2)$. Since $\text{Gal}(k(p_1)k(p_2)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $p$ is unramified in $k(p_1)k(p_2)$, $p$ must decompose in $k(p_1)k(p_2)$. But the condition $(\ell_{p_1})p \neq 1$ implies $p$ is inert in $k(p_1) \subset k(p_1)k(p_2)$, hence we obtain $[D_p : \mathbb{Q}] = p$. Similarly, $(\ell_{p_2})p \neq 1$ and $p_2 \not\equiv 1 \pmod{p^2}$ imply $[D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p$. Therefore, as in the above computation of $G_p, G_{p_1}$, we have $D_pD_{p_1}D_{p_2} = L$, by $xyz \neq -1$. □

5. REMARKS

The condition of Theorem 1 in [12] means $xyz = 0$ which is a special case of $xyz \neq -1$. Hence, our Corollary 2 contains some known results and there exist infinitely many fields satisfying the conditions of Theorem 1 (cf. [12]).

If $K = k(p_1)k(p_2)$ satisfies the conditions of Theorem 1, then $\lambda_p(k) = \mu_p(k) = 0$ for any field $k \subseteq K$ with $[k : \mathbb{Q}] = p$. This is a result of Fukuda [4]. He has shown this result using a technic of capitulation of ideal class group. The case $xyz = -1$ is a difficult case. But we can get some results:

**Proposition 8.** Notations are as in section 3. Assume that $(\ell_{p_1})p \neq 1, (\ell_{p_2})p \neq 1, \quad \text{and} \quad p_2 \not\equiv 1 \pmod{p^2}$.
Then $\lambda_p(k) = \mu_p(k) = 0$ for the decomposition field $k$ of $p$ in $k(p_1)k(p_2)$.

**Proof.** We apply a result of [6]:

**Lemma 9 ([6] Corollary 3.6).** Let $k$ be a cyclic extension of $\mathbb{Q}$ of degree $p$. Then the following conditions are equivalent:

(a) $\lambda_p(k) = \mu_p(k) = 0$,
(b) For any prime ideal $w$ of $k_\infty$ which is prime to $p$ and ramified in $k_\infty/\mathbb{Q}_\infty$, the order of the ideal class of $w$ is prime to $p$.

If $xyz \neq -1$ then we have $\lambda_p(k) = \mu_p(k) = 0$ by Corollary 2. So we only consider the case $xyz = -1$. In this case we have $k \neq k(p_i)$ $(i = 1, 2)$. It follows easily that $A(k)$, the $p$-part of the ideal class group of $k$, is cyclic of order $p$, and it is generated by products of primes of $k$ above $p$. On the other hand, for $i = 1, 2$, the prime $p_i$ of $k$ above $p_i$ generates $A(k)$, and is inert in $k_\infty/k$. Since the primes of $k$ above $p$ is principal for some $k_n$ by the natural mapping $A(k) \rightarrow A(k_n)$ (cf. [7]), $p_i$ is principal in $k_\infty$.

Since the primes ramified in $k_\infty/\mathbb{Q}_\infty$ are $p_1$ and $p_2$, which is principal in $k_\infty$, we can apply Lemma 9 and obtain $\lambda_p(k) = \mu_p(k) = 0$. □

Recently, Fukuda verified Greenberg conjecture for various cubic cyclic fields $k$ with $f_k = p_1p_2$ and $p = 3$. He gives an example, which is the case $p_1 = 7$ and $p_2 = 223$. Note that there exist two such fields, and these $p_1$ and $p_2$ do not satisfy condition (3) in Theorem 1. He verified $\lambda_3 = \mu_3 = 0$ for one of such fields by using his result concerning, with the unit group of $k$ (cf. [5]).
When $t \geq 3$, i.e. at least 3 primes are ramified in $k/\mathbb{Q}$, there are a few results for Greenberg conjecture. In this case, the $p$-rank of $A(k)$ is greater than 2. Greenberg ([7]) gave the following example, but the proof are omitted in his paper: $p = 3$ and $k$ is an cubic cyclic field with conductor $7 \cdot 13 \cdot 19$ and 3 is inert in $k/\mathbb{Q}$. He mentioned that by "delicate" arguments one can show $\lambda_3(k) = \mu_3(k) = 0$. The author had a chance to contact Prof. Greenberg, and asked him about this example. He kindly taught the author the "delicate" arguments, which is a system to examine relations of the ideal class group of intermediate fields of $k\mathbb{Q}_1$. Applying his idea, we can show the following result:

**Theorem 10** ([13]). Let $p$ be any odd prime. For any integer $0 \leq m \leq p-1$, there exist infinitely many cyclic extension fields $k$ of $\mathbb{Q}$ with $[k : \mathbb{Q}] = p$ such that $p$-rank$A(k) = m$ and $\lambda_p(k) = \mu_p(k) = 0$.

**References**


Department of Mathematical Science, School of Science and Engineering, Waseda University, 3-4-1, Okubo Shinjuku-ku, Tokyo 169-8555, Japan  
E-mail address: 697m5068@mse.waseda.ac.jp