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Quantum White Noise with Singular Non-Linear Interaction

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Abstract

A model of a system driven by quantum white noise with singular quadratic self–interaction is considered and an exact solution for the evolution operator is found. It is shown that the renormalized square of the squeezed classical white noise is equivalent to the quantum Poisson process. A convenient regularization of singular quantum differential equations for the evolution operator is suggested. We describe how equations driven by nonlinear functionals of white noise can be derived in nonlinear quantum optics by using the stochastic limit.

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Quantum white noise has emerged in quantum optics [1,2] and it has been widely studied in quantum theory [1,2], in infinite dimensional analysis [3] and in quantum probability [4,5]. Ordinary white noise differential equations describe quantum fluctuations in quantum optics, in laser theory, in atomic physics, in the theory of quantum measurement and in other topics and they are *linear* with respect to white noise in the sense that the typical equation for the evolution operator U_t has the form [1,2,4]

$$\frac{dU_t}{dt} = -i(F_t b_t^+ + F_t^+ b_t)U_t \tag{1}$$

Here F_t is an operator describing the system (for example, atom) and b_t and b_t^+ are quantum white noise operators,

$$[b_t, b_s^+] = \delta(t-s) \ , \quad [b_t, b_s] = 0$$

In this note we attempt to consider white noise with *nonlinear* interaction. The motivation for such a consideration is that if fluctuations in a system are rather large then they can produce a white noise with nonlinear interaction. It is not clear apriori what it means to have white noise with nonlinear interaction because we want the evolution operator U_t to be a bona fide unitary operator and not a distribution. Therefore we first consider a simple exactly solvable model with such interaction and then discuss how one can derive equations driven by nonlinear white noise in nonlinear quantum optics.

The first step to study such noises is to investigate quantum white noise with quadratic interaction. In this note we shall consider a model with the following equation for the evolution operator U_t:

$$\frac{dU_t}{dt} = -i[\omega_t b_t^+ b_t + g_t(b_t^{+2} + b_t^2) + c_t]U_t \tag{2}$$

where ω_t, g_t and c_t are functions of time t. We shall demonstrate that the singularity of the Hamiltonian imposes a restriction to these functions which does not arise in the case of regular Hamiltonians.

The consideration of models of quantum white noise with a non–linear singular interaction like (2) was out of reach in the known approach [6] to classical and quantum stochastic calculus. It became possible only recently with the development of the new white noise approach to quantum stochastic calculus [4,5] in which methods of renormalization theory have been used. Actually already equation (1) requires a regularization as it will be discussed below.

As any model with quadratic interaction the model (2) is exactly integrable in some sense. What makes the consideration of the model interesting is the singular character of the interaction which involves products of operators b_t at the same time, like b_t^2, and it is not clear a priori that such products have a meaning at all. In fact the model (2) shows certain surprising properties. Even after having given a meaning to equation (2) one cannot naively interpret the formal Hamiltonian in (2) as a usual self–adjoint operator.

For example the model (2) with $\omega_t = 0$ and $g_t =$ constant has been considered in [7] and it has been shown that, even after renormalization the solution of (2) is not unitary. The results of the present note show that for a special region of values of the parameters ω_t, g_t and c_t the unitarity of the solution can be guaranteed and in fact one has an explicit and simple form for it. Unitarity of the solution is proved for all values of the parameters below a certain threshold. Strangely enough this threshold corresponds exactly to the square of *classical* white noise. It seems interesting to have an exactly solvable dynamical model driven by a non–linear white noise term because it can be instructive for the consideration of more realistic models.

We solve equation (2) by using a Bogoliubov transformation, that is we look for a real function θ_t such that, defining the operators a_t and a_t^+ by

$$b_t = a_t ch\theta_t - a_t^+ sh\theta_t \ , \quad b_t^+ = a_t^+ ch\theta_t - a_t sh\theta_t \tag{3}$$

one has

$$\omega_t b_t^+ b_t + g_t(b_t^{+2} + b_t^2) = \Omega_t a_t^+ a_t + \kappa_t \delta(0) \tag{4}$$

for appropriate choice of Ω_t and κ_t. In (4) a formal infinity appears as the δ–function in zero $\delta(0)$. We remove it by the renormalization, that is we choose the function c_t in (1) as $c_t = -\kappa_t \delta(0)$ so that one has

$$\omega_t b_t^+ b_t + g_t(b_t^{+2} + b_t^2) + c_t = \Omega_t a_t^+ a_t$$

and the renormalization constant c_t does not alter the dynamics.

Notice that (3) implies that

$$[a_t, a_s^+] = \delta(t-s) \ , \quad [a_t, a_s] = 0$$

so the operators a_t, a_s^+ define a *squeezed white noise.* One easily proves that such a real function θ_t must satisfy the equation

$$\frac{g_t}{\omega_t} = \frac{sh\theta_t ch\theta_t}{sh^2\theta_t + ch^2\theta_t} \tag{5}$$

Therefore the only restriciton on the real functions g_t and ω_t is that $|g_t/\omega_t| < 1/2$. Under this condition one deduces the expression of Ω_t and κ_t

$$\Omega_t = \frac{\omega_t}{sh^2\theta_t + ch^2\theta_t} \tag{6}$$

$$\kappa_t = -\frac{\omega_t sh^2\theta_t}{sh^2\theta_t + ch^2\theta_t}$$

Let T be the formal unitary operator of the Bogoliubov transformation

$$T^+ b_t T = a_t \ , \quad T^+ b_t^+ T = a_t^+$$

The Bogoliubov transformation (3) is not unitary represented in the original Fock space $\mathcal{H}_b$ for b–particles with vacuum ψ_b. The operator T^+ acts actually from $\mathcal{H}_b$ to another Fock space $\mathcal{H}_a$ for a–particles with the vacuum $\psi_a = T^+\psi_b$ (see for example [8]). One has the relation

$$(T^+\psi_b, a_{\tau_1}\dots a_{\tau_m}U_t a^+_{t_1}\dots a^+_{t_n}T^+\psi_b) = (\psi_a, a_{\tau_1}...a_{\tau_m}U_t a^+_{t_1}\dots a^+_{t_n}\psi_a) \qquad (7)$$

In this relation in the left hand side one has the inner product in the Fock space of b–particles $\mathcal{H}_b$ with operators a_t, a^+_t being expressed in terms of b_t and b^+_t by formulas (3) and with U_t satisfying equation (1). In the left hand side of (7) there are meaningless operators T that requires a regularization.

However the right hand side of the relation (7) is well defined. In the right hand side one has the inner product in the Fock space $\mathcal{H}_a$ of a–particles with U_t satisfying the following equation

$$\frac{dU_t}{dt} = -i\Omega_t a^+_t a_t U_t \qquad (8)$$

So we reduce the solution of equation (1) to the solution of equation (8) with Ω_t given by (6). Eq. (8) defines the squeezed quantum Poisson process.

Now let us solve equation (8). It is not in the normal form and we have to use a regularization. We choose the following convenient one:

$$a^+_t a_t U_t = \lim_{\varepsilon\to 0}\frac{1}{2}a^+_t(a_{t-\varepsilon} + a_{t+\varepsilon})U_t \qquad (9)$$

¿From (8) one gets the following equation for the normal simbol $\tilde{U}_t = \tilde{U}_t(\alpha^+,\alpha)$ of the operator U_t (about normal simbols see, for example [9])

$$\frac{d\tilde{U}_t}{dt} = -i\lim_{\varepsilon\to 0}\Omega_t\alpha^+_t\left(\alpha_t + \frac{1}{2}\left(\frac{\delta}{\delta\alpha^+_{t-\varepsilon}} + \frac{\delta}{\delta\alpha_{t+\varepsilon}}\right)\right)\tilde{U}_t$$

the solution of which is

$$\tilde{U}_t = \exp\left\{-i\int_0^t \sigma_\tau\alpha^+_\tau\alpha_\tau d\tau\right\} \qquad (10)$$

where

$$\sigma_t = \frac{\Omega_t}{1+\frac{i}{2}\Omega_t} \qquad (11)$$

The operator U_t with the normal simbol (10), (11) is unitary if Ω_t is a real function. The normal simbol (10) corresponds to the following stochastic differential equation in the sense of [6]

$$dU_t = -idN(\sigma_t)U_t \qquad (12)$$

The regularization we have used is not unique. A more general regularization is

$$a_t U_t = \lim_{\varepsilon\to 0}[ca_{t-\varepsilon} + (1-c)a_{t+\varepsilon}]U_t$$

where c is an arbitrary complex constant. Then instead of (11) one gets

$$\sigma_t = \frac{\Omega_t}{1 + ic\Omega_t}$$

In this case the operator U_t is unitary if $c = \frac{1}{2} + ix$ where x is an arbitrary real number.

We will show elsewhere that the choice of different regularizations corresponds, in the probabilistic language, to different notions of stochastic integration.

Finally by using (10)–(11) and performing the Gaussian funcitonal integrals for the normal simbol one gets the following expression for correlators

$$(\psi_a, \exp\left\{\int f_1(\tau) a_\tau d\tau\right\} U_t \exp\left\{\int f_2(\tau) a_\tau^+ d\tau\right\} \psi_a) =$$

$$= \exp\left\{\int f_1(\tau) \frac{1 + \frac{i}{2}\Omega_\tau}{1 - \frac{i}{2}\Omega_\tau} f_2(\tau) d\tau\right\} \tag{13}$$

where f_1 and f_2 are arbitrary functions.

Note that by using the similar regularization for equation (1)

$$U_t = 1 - i \lim_{\varepsilon \to 0} \int_0^t [F_\tau b_t^+ + F_\tau^+ (c b_{\tau-\varepsilon} + (1-c) b_{\tau+\varepsilon})] U_\tau d\tau$$

one can write it in the normal form as

$$\frac{dU_t}{dt} = -i(F_t b_t^+ U_t + F_t^+ U_t b_t - icF_t^+ F_t U_t) \tag{14}$$

Now let us comment about the unitarity condition in the model (2). The operator U_t (8) is unitary in the Hilbert space $\mathcal{H}_a$ for any real function Ω_t. However to come to this U_t we have used the Bogoliubov transformation with the restriction $|g_t/\omega_t| < 1/2$ to the parameters in the original model (2). In the limit $g_t/\omega_t \to 1/2$ there is no dynamics because one gets $\Omega_t \to 0$ and $U_t \to 1$. The critical case $g_t/\omega_t = 1/2$ corresponds exactly to the equation

$$dU_t/dt = -ig_t : (b_t^+ + b_t)^2 : U_t \tag{15}$$

driven by the renormalized square of the *classical* white noise $w_t = b_t^+ + b_t$. So we get that eq. (15) leads only to the trivial evolution operator $U_t = 1$. To obtain a nontrivial evolution one has to make a renormalization of the parameter g_t. Let us consider the following model

$$\frac{dU_t}{dt} = -i\frac{f_t}{\epsilon}[2b_t^+ b_t + (1 - \frac{\epsilon^2}{2})(b_t^{+2} + b_t^2) + c_{\epsilon,t}]U_t \tag{16}$$

Eq. (16) is a regularization of the square of the classical white noise because in the formal limit $\epsilon \to 0$ one gets eq.(15) with $g_t = f_t/\epsilon$. In the limit $\epsilon \to 0$ by using formulas (5) and (6) one gets $\Omega_t \to 2f_t$ and therefore the model (16) is equivalent in this limit to the model describing the quantum Poisson process

$$\frac{dU_t}{dt} = -i2f_t a_t^+ a_t U_t$$

So we have demonstrated that the model with the renormalized square of the squeezed classical white noise has a meaning and it defines the quantum Poisson process.

It would be interesting to study also the case when $|g_t/\omega_t| > 1/2$. Here we would like to mention a possible relation of this question with a non–associative Ito algebra of stochastic differentials introduced in [4]. The linear stochastic differentials are defined as

$$dB_t = b_t dt, \quad dB_t^+ = b_t^+ dt \tag{17}$$

and they satisfy the quantum Ito multiplication rule $dB_t dB_t^+ = dt$ that can be derived from (17) by using the formal identity

$$\delta(0) dt = 1 \tag{18}$$

and also $b_t^+ b_t (dt)^2 = 0$. Eq. (12) in these notations takes the form of the quantum stochastic differential equation [6]

$$dU_t = -i(F_t dB_t^+ U_t + F_t^+ U_t dB_t - icF_t^+ F_t dt U_t)$$

To write equation (2) as the quantum stochastic differential equation we have to introduce nonlinear stochastic differentials

$$dB_t^{(m,n)} = b_t^{+m} b_t^n dt$$

By using (18) and a renormalization prescription one can get the following non-associative generalization of the Ito multiplication rule

$$dB_t^{(m,n)} dB_t^{(k,l)} = nk dB_t^{(m+k-1,n+l1)} \tag{19}$$

It is an important open problem to study the relation of the algebra (19) with the unitarity of the evolution operator in the model (2).

Now let us discuss how quantum white noises with non–linear singular interactions arise in the stochastic approximation to the usual Hamiltonian quantum systems. Actually this is a rather general effect in theory of quantum fluctuations [5]. In the stochastic limit of quantum theory white noise Hamiltonian equations such as (1) are obtained as scaling limits of usual Hamiltonian equations.

One has to use the formalism of non-linear quantum optics for the consideration of such problems as how is a short pulse of squeezed light generated when an intense laser pulse undergoes parametric downconversion in a traveling-wave configuration inside a non-linear crystal or how does such a squeezed pulse undergo self-phase modulation as it propagates in a non-linear optical fiber [1,10-13]. The Lagrangian describing the propagation of quantum light through a nonlinear medium contains nonlinear terms in electric field E and magnetic field B [10-13]:

$$L = \frac{1}{2}(E^2 - B^2) + \chi_{(2)} E^2 + \chi_{(3)} E^3 + ...$$

where $\chi_{(i)}$ are non–linear optical susceptibilities. Such nonlinear terms lead to various non–linear quantum noises that can be approximated by non–linear quantum *white* noises.

Let us consider a system interacting with quantum field with the evolution equation of the form

$$\frac{dU(t)}{dt} = -i[\lambda^{2-n}\chi A(t)^n + ...]U(t) \tag{20}$$

where

$$A(t) = \int a(k)f(k)e^{i\omega(k)t}dk, \quad [a(k), a^+(p)] = \delta^3(k-p)$$

$f(k)$ is a form–factor, χ is a constant and λ is a small parameter. Here the field operator $A(t)$ can be interpreted as a mode of electromagnetic field in nonlinear quantum optics.

For example in the process of parametric down conversion a photon of frequency 2ω splits into two photons each with frequency ω and in the simple model of parametric amplifier where the pump mode at frequency 2ω is classical and the signal mode at frequency ω is described by the annihilation operator a one has the Hamiltonian of the form [1]

$$H = \omega a^+ a - i\chi(a^2 e^{2i\omega t} - a^{+2}e^{-2i\omega t})$$

In *the stochastic (or t/λ^2–) limit* [14,4,5] one obtains from (14) an equation of the type (2) with a singular interaction driven by a non-linear quantum white noise. Indeed after the rescaling $t \to t/\lambda^2$ one gets equation

$$\frac{dU(t/\lambda^2)}{dt} = -i[\chi\lambda^{-n}A(\frac{t}{\lambda^2})^n + ...]U(t/\lambda^2)$$

which in the limit $\lambda \to 0$ becomes

$$\frac{dU_t}{dt} = -i[\chi b_t^n + ...]U_t$$

because, as shown in [5],

$$\frac{1}{\lambda}A(\frac{t}{\lambda^2}) \to b_t$$

in the sense that all the vacuum correlators of the left hand side converge to the corresponding correlators of the right hand side.

Finally let us briefly comment about possible applications of quantum white noise with nonlinear interaction to non-linear quantum optics and to theory of quantum measurement. The physical meaning of parameters ω_t and g_t depends on the physical model. For instance in the case of a short pulse propagating in a nonlinear medium the parameters in the evolution equation for the white noise are related with the nonlinear optical susceptibilities as it was discussed above. If we take the sum of terms of the form (1) and (2) and F_t are function then such a model also is easily solved by means of non-homogeneous Bogoliubov transformation. However for a realistic model not only F_t should be operators but also the parameters ω_t and g_t in principle should be operators as well. The consideration of such a model is very important but it is out of the scope of this note.

In the modern theory of quantum measurement [1] one considers a system interacting with a linear quantum white noise. For a system interacting with a nonlinear medium one has to consider quantum white noise with nonlinear interaction. One expects that a dynamical model of a system interacting with nonlinear white noise can be interpreted as describing the process of self-measurement by analogy with the self-focusing of a beam.

To conclude, the model (2) with quadratic singular interaction of quantum white noise has been considered in this note. We have demonstrated that the singular equation (2) leads to a well defined unitary evolution operator (10), (11) and we have computed explicitly its matrix elements (see (13)). The model was solved under some restrictions to the parameters ω_t and g_t which deserve a further study. The results obtained in this exactly solved model could be useful for investigation of more realistic and more complicated models with quadratic as well as with higher order singular white noise interactions.

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