数理解析研究所講究録
1099 巻 1999 年 36-52

A Variant of Itô-Clark Type Formula in Historical Stochastic Analysis*

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§1. Introduction

We consider a version of Itô-Clark type stochastic integration formula (e.g. [U95, p.92]) in the theory of historical superprocesses. The key idea of demonstration of the Itô-Clark formula is to derive a variant of Evans-Perkins type stochastic integration by parts with respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two reasons why this type of integration by parts formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In fact the establishment of the formula asserts that a product of historical functionals of a specific class and stochastic integral relative to the orthogonal martingale measure in the Walsh sense [W86] is, in its mathematical expectation form, equivalent to a certain expression of integration that is involved with stochastic integral with respect to a Dawson-Perkins historical process [DP91] associated with a reference Hunt process. In addition, it also allows us to interpret that the formula is nothing but a variant of stochastic integration by parts in an abstract level, that is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any L^2 functional of superprocess may be represented as a constant C_0 plus a stochastic integral with respect to the associated orthogonal martingale measure M (e.g. [EP94]). Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional of the

*Research supported in part by JMESC Grant-in-Aids SR(C) 07640280 and CR(A) 09304022.

so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95]. Based upon the previous results (1994), they derived partial analogue of the Itô-Wiener chaos expansion in superprocess setting by taking advantage of the "stochastic integral formula" in question.

Lastly we shall give a rough idea of what the integration formula is like, but in the form as simple as possible. First of all, let us consider the functional $F(H)$ of a historical process H with branching mechanism Φ for a real valued function F on $C([0,\infty);M_F(D))$ with the space D of E-valued cadlag paths. Actually, this F should lie in a suitable admissible subspace $U(M(D))$ of $C(C([0,\infty);M_F(D));\mathbf{R})$. Next consider a stochastic integral $J(\Xi;M)$ $=\int\int \Xi(s,y)dM$ of a bounded predictable function Ξ relative to the orthogonal martingale measure M in the Walsh (1986) sense. Then we make a product $F(H)\cdot J(\Xi;M)$. On the other hand, consider the integral of another type $J(F,\Xi;H) = \int\int I[F]\Xi(s,y)dH_s ds$ for some predictable function $I[F]$ which is determined by the functional $F(H)$ given. Thus we attain the integration formula if we take the mathematical expectation of both terms, i.e., $\mathbf{E}[F(H)\cdot J(\Xi;M)] = \mathbf{E}[J(F,\Xi;H)]$.

§2. Notation and Preliminaries

Let $C = C^d = C([0,\infty),\mathbf{R}^d)$ denote the space of $\mathbf{R}^d$-valued continuous paths on $\mathbf{R}_+ = [0,\infty)$ with the compact-open topology. $\mathcal{C} = \mathcal{B}(C)$ is its Borel σ-field and

$$\mathcal{C}_t = \mathcal{B}_t(C) = \sigma(y(s),\ s\le t)$$

denotes its canonical filtration. For $y,w\in C^d$ and $s\ge 0$, we define the stopped path by $y^s(t) = y(t\wedge s)$ and let

$$y/s/w = \begin{cases} y(t), & \text{for}\quad t<s,\\ w(t-s), & \text{for}\quad t\ge s.\end{cases} \tag{1}$$

$M_F(C)$ is the space of finite measures on C with the topology of weak convergence and we define

$$M_F(C)^t := \left\{m\in M_F(C);\ y=y^t,\ m-a.s.\ y\right\},\qquad t\ge 0.$$

If P_x denotes Wiener measure on $(C,\mathcal{B}(C))$ starting at x, $\tau\ge 0$, and $m\in M_F(C)^\tau$, define $P_{\tau,m}\in M_F(C)$ by

$$P_{\tau,m}(A) := \int_C P_{y(\tau)}(\{w;\ y/\tau/w\in A\})dm(y).$$

Let

$$\Omega_H[\tau,\infty) := \left\{H\in C([\tau,\infty),M_F(C));\ H_t\in M_F(C)^t,\ \forall t\ge\tau\right\},$$

and put $\Omega_H := \Omega_H[0,\infty)$. We write $\mathcal{H}$ for the totality of Borel sets of Ω_H. We use the notation $H_t(\omega) = \omega(t)$ for $\omega\in\Omega_H$ as for the canonical realization of historical process.

Fix $0 \le t_1 < \cdots < t_n$ and $\psi \in C_b^2(\mathbf{R}^{nd})$. For $y \in C$ we set

$$\begin{aligned} \bar{y}(t) &= (y(t\wedge t_1), \cdots, y(t \wedge t_n)), \\ \bar{\psi}(y) &\equiv \bar{\psi}(t_1, \cdots, t_n)(y) = \psi(y(t_1), \cdots, y(t_n)), \end{aligned}$$

and $\tilde{\psi}(t,y) = \bar{\psi}(y^t)$. ψ_i (resp. ψ_{ij}) stands for the first (resp. second) order partials $\partial_i \psi$ (resp. $\partial^2_{ij}\psi$) of ψ. $\nabla\bar{\psi} : [0,\infty) \times C \to \mathbf{R}^d$ is the $(\mathcal{C}_t)$-predictable process whose j-th component at (t,y) is given by

$$\sum_{i=0}^{n-1} \mathbf{I}(t < t_{i+1}) \psi_{id+j}(\bar{y}(t)).$$

While, for $1 \le i,j \le d$, $\bar{\psi}_{ij} : [0,\infty) \times C \to \mathbf{R}$ is the $(\mathcal{C}_t)$-predictable process defined by

$$\bar{\psi}_{ij}(t,y) := \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \mathbf{I}(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i}\partial_{ld+j}(\bar{y}(t)).$$

Let us define the domains

$$\begin{aligned} D_0 &:= \bigcup_{n=1}^{\infty} \left\{ \bar{\psi}(t_1, \cdots, t_n);\ 0 \le t_1 < \cdots < t_n,\ \psi \in C_0^\infty(\mathbf{R}^{nd}) \right\} \bigcup \{1\}, \\ \tilde{D}_0 &:= \left\{ \tilde{\psi};\ \tilde{\psi}(t,y) = \bar{\psi}(y^t) \quad \text{for some} \quad \bar{\psi} \in D_0 \right\}. \end{aligned}$$

Let $\bar{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge\tau}, \mathbf{P})$ be a filtered probability space and let $(\omega, y) = (\omega, y_1, \cdots, y_d)$ denote sample points in $\hat{\Omega} = \Omega \times C^d$. Here $\tau \ge 0$ is fixed. When f is a function on $[\tau,\infty)$ $\times$ $\hat{\Omega}$ taking values in a normed linear space $(E, \|\ \|)$, then a bounded $(\mathcal{F}_t)$-stopping time T is a reducing time for if and only if

$$\mathbf{I}(\tau < t \le T)\, \|f(t,\omega,y)\|$$

is uniformly bounded. In addition we say that a sequence $\{T_n\}$ reduces f if and only if each T_n reduces f and $T_n \nearrow \infty$ holds $\mathbf{P}$-a.s. We say that f is locally bounded if such a sequence $\{T_n\}$ exists. We assume that

(LB) $\gamma \in [0,\infty), a \in S^d, b \in \mathbf{R}^d$ and $g \in \mathbf{R}$ are $(\hat{\mathcal{F}}_t^*)$-predictable processes on $[\tau,\infty) \times \hat{\Omega}$ such that $\Lambda = (\gamma, a, b, g\gamma^{-1}\, \mathbf{I}(\hat{g} \ne))$ is locally bounded.

Notice that the above assumption implies that g is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For $\tau \ge 0$ and $m \in M_F(C)^\tau$, we define

$$A_{\tau,m}\tilde{\psi}(t,y) \equiv A(\bar{\psi})(t,y) := \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t,,y)\bar{\psi}_{ij}(t,y) + b(t,\omega,y)\cdot\nabla\bar{\psi}(t,y) + g(t,\omega,y)\bar{\psi}(y^t)$$

for $\bar{\psi} \in D_0$. We write $\langle \mu, f\rangle$ or sometimes $\mu(f)$ for the integral $\int f d\mu$ when μ is a measure and f is a suitable μ-integrable function. Suggested by [DkTn98], we may define

Definition. (cf. [P95], §2) A predictable process $K = \{K_t, t \geq \tau\}$ on $\bar{\Omega}$ with sample paths a.s. in $\Omega_H[\tau, \infty)$ is a generalized $\{\gamma,\ a, b,\ g\}$-historical process (GHP) (or $(A,\ -\gamma\lambda^2/2)$-historical process) if and only if $K_t \in M_F(C)^t$ for all $t \geq \tau$, a.s. and $\mathbf{P}[K_\tau(1)] < \infty$, and if there exists a probability measure $\mathcal{P}$ on $\Omega_H[\tau, \infty)$ such that it satisfies the martingale problem (MP) with initial data $\{\tau, m\}$ and $\{\gamma, a, b, g\}$: for $\forall\ \bar{\psi} \in D_0$,

$$Z_t(\bar{\psi}) = \langle K_t, \bar{\psi}\rangle - \langle m, \bar{\psi}\rangle - \int_\tau^t \langle K_s, A(\bar{\psi})(s)\rangle ds, \quad t \geq \tau, \tag{2}$$

is a continuous $(\mathcal{F}_t)$-local martingale satisfying $Z_\tau(\bar{\psi}) = 0$ and

$$\langle Z(\bar{\psi})\rangle_t = \int_\tau^t \int \gamma(s, \omega, y)\bar{\psi}(y)^2 K_s(dy)ds, \qquad \forall t \geq \tau, \quad a.s.$$

Remark. The existence and uniqueness of the law of K is essentially due to [F88] (cf. [DIP89]).

Set $T_s = [s, \infty)$, and in particular $T_0 = [\tau, \infty)$. Define $C(M_F(C)) := C(T_0; M_F(C))$, and we write $C(t) = (\tau, t] \times C$ for the integral domain. When $\mathcal{F}$ is the σ-field or the usual filtration, then $f \in \mathcal{F}$ indicates that the function f is $\mathcal{F}$-measurable and $\mathcal{P}(\mathcal{F})$ is the totality of $(\mathcal{F})$-predictable functions, and $b\mathcal{P}(\mathcal{F})$ denotes the whole space of functions that are all bounded elements of $\mathcal{P}(\mathcal{F})$. We use the symbol $U(M_F(C))$ for an admissible subset of the space $C(C(M_F(C)); \mathbf{R})$; more precisely $U(M_F(C))$ is the totality of real valued continuous functions F on $C(M_F(C))$ such that for some compactly supported finite measure $L(dt)$ on T_0, the estimate

$$|\Delta F(h, g)| \leq \int_{T_0} g(t, C)L(dt)$$

holds for all $h, g \in C(M_F(C))$, where we define $\Delta F(x, y) := F(x + y) - F(x)$.

§3. Predictable Representation Property

Let $\{T_N\}$ be a reducing sequence. Take a sequence $\{\bar{\psi}_n\}$, $\bar{\psi}_n \in D_0$ such that $\bar{\psi}_n$ converges bounded pointwise (*bp* for short) to ψ, namely,

$$\bar{\psi}_n \to \psi, \quad bp \quad (n \to \infty).$$

An application of dominated convergence theroem together with the local boundedness of γ implies that

$$\langle Z(\bar{\psi}_n - \bar{\psi}_m)\rangle_t \to 0 \quad \text{as} \quad n, m \to \infty$$

for $\forall\ t \geq \tau$, a.s. Therefore we obtain

Proposition 1. *There is an a.s. continuous adapted process $\{Z_t(\psi);\ t \geq \tau\}$ such that*

$$\sup_{\tau \leq t \leq N} \left|Z_t(\bar{\psi}_n) - Z_t(\psi)\right| \to 0$$

holds in probability (*w.r.t.* $\mathbf{P}$) *as* $n \to \infty$ *for* $\forall N > \tau$.

To proceed our discussion, we need the following lemmas.

Lemma 1. (cf. Corollary 2.2, p.11, [P95]) *Let* T *be a reducing time for* (γ, g). *Then we have*

(a) $0 < \mathbf{P}[K_T(1)] \le \mathbf{P}[\sup_{\tau \le t \le T} |K_t(1)| + \langle Z(1)\rangle_T] < \infty$.

(b) *If* $\mathbf{P}[K_\tau(1)^p] < \infty$ *for* $p \in \mathbf{N}$, *then*

$$\mathbf{P}\left\{\left(\sup_{\tau \le t \le T} |K_t(1)|\right)^p + \langle Z(1)\rangle_T^{\frac{p}{2}}\right\} < \infty.$$

Lemma 2. (cf. [EP94, p.123]) D_0 *is dense in* $b\mathcal{B}(C)$ *relative to the bounded pointwise convergence topology.*

We may use Lemma 1 to obtain

$$\sup_{\tau \le t \le T_N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \to 0 \quad \text{in} \quad L^2$$

as $n \to \infty$, for $\forall N \in \mathbf{N}$. Clearly $Z_t(\psi)$ is a continuous $(\mathcal{F}_t)$-local martingale whose quadratic variation process is given by

$$\langle Z(\psi)\rangle_t = \int_\tau^t \int_C \gamma(s,\omega,y)\psi(y)^2 K_s(dy)ds. \tag{3}$$

By virtue of Lemma 2, it is a routine work to show that this Z_t extends to an orthogonal martingale measure

$$\{Z_t(\psi);\ \ t \ge \tau,\ \psi \in b\mathcal{B}(C)\ \}.$$

Consequently, the mapping $t \mapsto Z_t(\psi)$ is a continuous local martingale satisfying Eq.(3) for eaxh $\psi \in b\mathcal{B}(C)$, and $\psi \mapsto Z_{t\wedge T_N}(\psi)$ is an L^2-valued measure on $\mathcal{B}(C)$ for each $t \ge \tau$, $N \in \mathbf{N}$. By a trivial localization argument, we may define the stochastic integral

$$Z_t(\psi) = \int_\tau^t \int \psi(s,\omega,y)dM(s,y) \tag{4}$$

($\exists$ an orthogonal martingale measure $M = M^K$ in the sense of Walsh [W86, Chapter 2]) such that

$$\langle Z(\psi)\rangle_t = \int_\tau^t \langle K_s, \gamma(s,\omega)\psi(s,\omega)^2\rangle ds, \tag{5}$$

$\forall t \ge \tau$, a.s., as long as ψ belongs to $L^2_{loc}(K, \mathbf{P})$. Here $L^2_{loc}(K, \mathbf{P})$ denotes the L^2 space of $(\mathcal{F}_t \times C)_{t \ge \tau}$-predictable functions f and

$$\int_\tau^t \int \gamma(s,y)f(s,y)^2 K_s(dy)ds < \infty$$

for $\forall t \ge \tau$, $\mathbf{P}$-a.s.

We write $f \in L^2(K, \mathbf{P})$ (resp. $L^2_\infty(K, \mathbf{P})$) if, in addition,

$$\mathbf{P}\left\{\int_\tau^t \int \gamma(s,\omega,y) f(s,\omega,y)^2 K_s(dy)ds\right\} < \infty, \qquad \forall t > 0,$$

respectively,

$$\mathbf{P}\left\{\int_\tau^\infty \int \gamma(s,\omega,y) f(s,\omega,y)^2 K_s(dy)ds\right\} < \infty.$$

Theorem 1. (Predictable Representation Property) *If* $V \in L^2(\Omega, \mathcal{F}, \mathbf{P})$, *then there is an* f *in* $L^2_\infty(K, \mathbf{P})$ *such that*

$$V = \mathbf{P}[V] + \int_\tau^\infty \int f(s,\omega,y) dM^K(s,y), \qquad \mathbf{P} - a.s. \tag{6}$$

Proof. It is sufficient to verify (6) for the particular case where V is a square integrable martingale M_t. Then Jacod's general theory (cf. Theorem 2 and Proposition 2 of [J77]) provides with a stochastic integral representation of M_t. For the rest, it goes almost similarly as in the proof of Theorem 1.1 [EP94, p.124].

§4. Canonical Measure and Campbell Measure

For $y \in D = D(\mathbf{R}_+; \mathbf{R}^d)$, we define $y^{t-}(s)$ as $y(s)$ itself if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s,y)$ is a σ-finite measure on $C(M_F(D))$ such that

$$Q\big(s, y^{s-}; \ \{h \in C(M_F(D)); \ \ \tau \leq \exists t \leq s, \ \ h(t) \neq 0\}\big) = 0,$$

which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ [D93] associated with the law of $K_t = K(t)$ and the path restriction mapping π (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here R is characterized by

$$\log \mathcal{P}_{s,\delta_y}[\exp\langle K_t, -\varphi\rangle] = \int_{M_F(M_F(C))} \left(e^{-\langle \zeta, \varphi\rangle} - 1\right) R(s,t,y; d\zeta)$$

(cf. Lemma 1 in [Dk99c]; see also [DP.91, Proposition 3.3, pp.38-39]). Let F be a real valued Borel function on $C(M_F(C))$. Assume that

$$I^Q_{s,y}[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h,g) Q(s, y^{s-}; dg) \tag{7}$$

is well-defined and bounded below for all $s > \tau$, $y \in C$, and $h \in C(M_F(C))$. For a bounded $(\mathcal{F}_t)$-stopping time T, we define the Campbell measure P_T associated with $K(t)$ by

$$P_T(A \times B) := \mathbf{P}(K(T, A) \cdot \mathbf{I}_B\{K(T)\})/m(C) \tag{8}$$

for any $A\times B \in (C\times\Omega, \mathcal{C}\times\mathcal{F})$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_\tau = m$. Since the mapping $(s,y,\omega) \mapsto I^Q_{s,y}[\Delta F](K(\omega))$ is bounded below and measurable with respect to the product of the predictable σ-field associated with the filtration $(\mathcal{C}_t)$ and the σ-field $\mathcal{F}$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(\mathcal{C}_t\times\mathcal{F}_t)_{t\geq\tau}$-predictable function $Pr[F](s,y,\omega) : (\tau,\infty)\times C\times\Omega \to \mathbf{R}$ such that

$$P_T\{I^Q[\Delta F](T)/(\mathcal{C}\times\mathcal{F})_T\} = Pr[F](T,\omega,y) \tag{9}$$

holds P_T-a.s. for all bounded $(\mathcal{F}_t)$-predictable stopping times $T > s$. It is quite interesting to note that in particular

$$\mathbf{P}\int_C I^Q[\Delta F](T,y)K(T,dy) = \mathbf{P}\int_C Pr[F](T,y)K(T,dy).$$

We shall introduce an approximation map. For each $l \in \mathbf{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j);\ 1\leq j\leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \cdots < t^{(l)}(k[l]) < \infty$,

$$\lim_{l\to\infty}\{\sup_k \Delta t[l;k]\} = 0 \quad\text{and}\quad \lim_{l\to\infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map $W[l]$ from $C(M_F(C))$ into $C(M_F(C))$ is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i+1))\cdot g(t^{(l)}(i)) - Sb(t^{(l)}(i))\cdot g(t^{(l)}(i+1))\}\Delta t[l;i]^{-1}$$

if $t\in[t^{(l)}(i), t^{(l)}(i+1))$, and $:= g(t^{(l)}(k[l]))$ if $t\geq t^{(l)}(k[l])$, for any element g of $C(M_F(C))$ with $Sb(k) = k - t$. Immediately we get

Lemma 3.(cf. Lemma 4 [DK98a]) *Let F be an element of $C(C(M_F(C));\mathbf{R})$. Then for all $g\in C(M_F(C))$*

$$\lim_{l\to\infty}(F\circ W[l])(g) = F(g).$$

§5. Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section. C^t denotes the image of C under the map: $y\mapsto y^t$. We define a measure $K^*[s,t]$ on C^s by $K^*[s,t](F) := K_t(\{y:\ y^s\in F\})$. Then the measure $K^*[s,t]$ is atomic with a finite set of atoms, and we write $L[s,t](\subset C^s)$ for the locations of these atoms. For $s\in(a,b]$, let $\lambda_s[\varphi]$ be the random measure on C that places mass $\varphi(s,y)$ at each point y in $(L[b,c])^s = L[s,c]$. With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure $\mathbf{Q}_N$ on $(\Omega,\mathcal{F})$ such that

$$\left.\frac{d\mathbf{Q}_N}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \exp\Big\{ \int_\tau^{t\wedge T_N}\int g\gamma^{-1}(s)\mathbf{I}(g(s)\neq 0)dM^K(s,y) - \frac{1}{2}\int_\tau^{t\wedge T_N}\int g^2\gamma^{-1}(s)\mathbf{I}(g(s)\neq 0)K_s(dy)ds\Big\}.$$

For brevity's sake we rather write $\mathcal{E}(t\wedge T_N)$ than the above. On this account, $K_{\cdot\wedge T_N}$ satisfies the martingale problem (MP)$[\gamma_N, a_N, b_N, 0]$ instead of (MP)$[\gamma, a, b, g]$, where we set $f_N := f\cdot \mathbf{I}(\tau < t \le T_N)$. Moreover, for $s \in (a, b]$, $y \in C^s$, the symbol $\mathcal{M}[s, y]$ denotes the mapping of the set of functions $\{m : (\tau, \infty) \to M_F(C)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s, y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \ge s$.

Let us now introduce assumptions for our principal results.

(A.1) $g : [\tau, \infty) \times \Omega \times C \to \mathbf{R}$ is a $(\mathcal{F}_t \times \mathcal{C}_t)^*$-predictable process such that $g\gamma^{-1}\cdot \mathbf{I}(g \neq 0)$ is locally bounded.

(A.2) For any predictable function f on $[\tau, \infty) \times I \times C^* \times \Omega$, the counting measure n^* satisfies

$$\mathbf{P}\int_{C^*} n^*((s,t] \times I)G_t(dx) = m(C^*)(t-s)$$

where G_t is a marked historical process corresponding to K and N_t is the martingale measure associated with G_t (cf. §7 for details).

(A.3) There exists a random measure Λ_φ on $(\tau, \infty) \times C$ such that

$$\int\int_{C(\infty)} f(s,y)\Lambda_\varphi(ds \otimes dy) = \int_{a+}^{b}\int_C f(s,y)\lambda_s[\varphi](dy)ds$$

holds for any suitable predictable function f.

(A.4) $\Psi(s,y)\mathcal{E}(t\wedge T_N)^{-1}$ is uniformly bounded in s, K_s-a.e. y, $\mathbf{Q}_N$-a.s.

(A.5) There exists some constant C_0 (> 0) such that

$$\int\int_{C(t)} \Psi(s,y)^2\ \mathcal{E}(t\wedge T_N)^{-2}\ \gamma(s,y)\ K_s(dy)ds \le C_0$$

holds $\mathbf{Q}_N$-a.s., for all $t \ge \tau$.

Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

§6. Main Results: Stochastic Integration Formulae

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let K be a predictable measure-valued process whose law is specified by a general martingale problem (MP)$[\tau, K_\tau, \gamma, a, b, g]$.

Theorem 2.(cf. [Dk98b]) *Assume that $\Phi : C(M_F(C)) \to \mathbf{R}$ is a cylinder function with bounded representing function $\varphi : [M(C)]^k \to \mathbf{R}$ and base $\tau < t(1) < \cdots < t(k)$, such that*

$$|\Delta\varphi(\alpha, \beta)| \le c_0 \sum_j \beta_j(C)$$

for some positive constant c_0, for all α, $\beta = (\beta_j) \in [M(C)]^k$. Then for $t > \tau$

$$\mathbf{P}\left\{\Phi(K)\int\int_{C(t)} \Psi(s,y)dM^K(s,y)\right\} = \mathbf{P}\int\int_{C(t)} Pr[\Phi](s,y)\Psi(s,y)\gamma(s,y)K_s(dy)ds$$

holds where Ψ *is a bounded* $(\mathcal{C}_t \times \mathcal{F}_t)_{t\geq\tau}$*-predictable function,* K_t *is a GHP, and* $Pr[\Phi]$ *is a predictable function determined by (9) in accordance with the given* Φ.

Remark 1. The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]).

Theorem 3. (Stochastic Integration By Parts) *Let* $F \in U(M_F(C))$. *If* Ψ *is an element of* $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$, *then for all* $t > s$,

$$\mathbf{P}\left\{F(K)\int\int_{C(t)} \Psi(s,y) \quad dM^K(s,y)\right\}$$
$$= \quad \mathbf{P}\int\int_{C(t)} Pr[F](s,y)\gamma(s,y)\Psi(s,y)K_s(dy)ds. \qquad (10)$$

Remark 2. Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional $F(K)$. As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it(cf. §3, [Dk98a]). We shall refer to this generalization in §8.

Theorem 4. (Itô-Clark Type Formula) *Let* $F \in U(M_F(C))$.

$$F(K) = \mathbf{P}[F(K)] + \int_{\tau+}^{\infty}\int Pr[F](s,y)dM^K(s,y) \qquad (11)$$

where $Pr[F](s,y)$ is a $\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$-measurable version (relative to P_T) of

$$P_T\left[\int_{C(M_F(C))} \Delta F(K,h)Q(s,y^{s-};dh) \;/\; (\mathcal{D}\times\mathcal{F})_T\right].$$

§7. Marked Historical Processes and the Girsanov-Dawson-Perkins Theorem

Set $I = [0,1]$, $E^* = C \times I$ and $C^* = C(\mathbf{R}_+, E^*)$, and let $\mathcal{C}^*$ (resp. $\mathcal{C}_t^*$) be the Borel σ-field (resp. the canonical filtration) of C^*. Put $x = (y,n) \in E^*$. Let G be the corresponding counterpart historical process of K starting at (τ,μ), defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P}^*)$. Suppose that $\varphi : (\tau,\infty) \times C \times \Omega \to I$ be an element of $\mathcal{P}(\mathcal{C}_t \times \mathcal{H}_t)$. Given any cadlag function $n : \mathbf{R}_+ \to I$, we can construct a σ-finite counting measure n^* on $\mathbf{R}_+ \times I$ by assigning an atom of mass one to each point (s,z) such that $n(s) - n(s-) = z \neq 0$. Put

$$A(t,x,\omega) := n^*(\{(s,z) \in [\tau,t) \times I;\ \ \varphi(s,y,\omega) > z\}) \qquad (12)$$

and $B(t,x,\omega) = \mathbf{I}\{A(t,x,\omega) = 0\}$. Then we can define an $M_F(C)$-valued process $K[\varphi](t)$ by

$$K[\varphi;J](t) := \int_{C^*} \mathbf{I}\{J\}(y)B(t,x)G_t(dx). \tag{13}$$

Put

$$I_1(\varphi,N) = \int\int_{C^*(t)} \varphi(s,y)dN(s,x), \quad \text{and} \quad I_2(\varphi,G) = \int\int_{C^*(t)} \gamma(s,y)\varphi(s,y)^2 G_s(dx)ds$$

with $C^*(t) = (\tau, t] \times C^*$. Then we define

$$\Lambda[\varphi](t) := \exp\Big\{I_1(\varphi,N) - \frac{1}{2}I_2(\varphi,G)\Big\}. \tag{14}$$

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$-martingale. The new probability space $(\Omega, \mathcal{H}, \mathbf{P}^*[\varphi])$ is defined by $\mathbf{P}^*[\varphi]\{F\} := \mathbf{P}^*\{F \cdot \Lambda[\varphi](t)\}$ (cf. [Dk98a]) for any $F \in b\mathcal{H}_t$ with

$$\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \tag{15}$$

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

Proposition 2.(cf. Theorem 5.1, p.1798, [EP95]) *The law of $K[\varphi]$ under $\mathbf{P}[\varphi]$ is equivalent to the law of K under $\mathbf{P}$.*

§8. Sketch of Proofs of Main Theorems

§8.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §6, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function L^* on $C(M_F(C))$ by

$$L^*[g] := \int_{T_0} g(t,C)L(dt) = \langle L, g(\cdot, C)\rangle. \tag{16}$$

In connection with the measure L (see §2), we introduce the finite measure $L(l) \equiv L(l,dt)$ which concentrates its mass on $\{t^{(l)}(j);\ 0 \leq j \leq k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, C)\rangle$ for $g \in C(M_F(C))$. Recall that

$$\int g(t,C)\, Q(s,y;dg) = \int \xi(C)\, R(s,t,y;d\xi) = 1$$

holds (cf. Lemma 3, [Dk99a]) with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:

$$\mathbf{P}\int\int_{C(t)} \{Q(s,y^{s-})L^*[g]\}K_s(dy)ds = \lim_{l\to\infty} \mathbf{P}\int\int_{C(t)} \{Q(s,y^{s-})(L^* \circ W[l])[g]\}\, K_s(dy)ds$$

holds with $g \in C(M_F(C))$ for all $t > \tau$, and

$$\begin{aligned}
&\mathbf{P} \int\int_{C(t)} Pr[F](s,y)Z(s,y)K_s(dy)ds \\
&= \lim_{l\to\infty} \mathbf{P} \int\int_{C(t)} Pr[F\circ W[l]](s,y)Z(s,y)\,K_s(dy)ds. \qquad (17)
\end{aligned}$$

holds for all $t > \tau$ if $Z \in \mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$. Since, for each $n \geq 1$, $\mathbf{P}\{K_t(C)^n\}$ is uniformly bounded on compact intervals, we can readily deduce that $\mathbf{P}\{(L^* \circ W[l])[K]^n\}$ is bounded in l for each $n \geq 1$. Moreover,

$$\mathbf{P}\Big\{F(K) \int\int_{C(t)} \Psi(s,y)\,dM(s,y)\Big\} = \lim_{l\to\infty} \mathbf{P}\Big\{\big(F\circ W[l]\big)(K) \int\int_{C(t)} \Psi(s,y)\,dM(s,y)\Big\}.$$

To complete the extension discussion in this section we have only to observe that $F\circ W[l]$ satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 2 in §7 is valid. Hence, combining the above results, we obtain

$$\begin{aligned}
\mathbf{P}\Big\{F(K) \int\int_{C(t)} \Psi(s,y)\,dM\Big\} &= \lim_{l\to\infty} \mathbf{P}\Big\{\big(F\circ W[l]\big)(K) \int\int_{C(t)} \Psi(s,y)\,dM\Big\} \\
&= \lim_{l\to\infty} \mathbf{P} \int\int_{C(t)} Pr[F\circ W[l]]\ \gamma(s,y)\Psi(s,y)\,K_s(dy)ds \\
&= \mathbf{P} \int\int_{C(t)} Pr[F](s,y)\ \gamma(s,y)\Psi(s,y)\,K_s(dy)ds,
\end{aligned}$$

which concludes Theorem 3.

§8.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk99a].

Thanks to (A.1), it suffices to verify the integral formula for a special $\{\gamma_N, a_N, b_n, 0\}$-historical process $K_{\cdot\wedge T_N}$ under $\mathbf{Q}_N$ instead of the generalized K (GHP) with $\mathbf{P}$. Indeed, since $d\mathbf{P} = \mathcal{E}(t\wedge T_N)^{-1}d\mathbf{Q}_N$, what we have to show is as follows:
(The Modified Stochastic Integration By Parts Formula)

$$\begin{aligned}
&\mathbf{Q}_N \ \Big\{\mathcal{E}(t\wedge T_N)^{-1}\cdot\Phi(K_{\cdot\wedge T_N}) \int\int_{C(t)} \Psi(s,y)dM(s,y)\Big\} \\
&= \ \mathbf{Q}_N \Big\{\mathcal{E}(t\wedge T_N)^{-1} \int\int_{C(t)} Pr[\Phi](s,y)\,\gamma(s,y)\,\Psi(s,y)K_{s\wedge T_N}(dy)ds\Big\}.
\end{aligned}$$

Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for $\varphi = \Psi\cdot\mathcal{E}^{-1}$. Hence, by the auguments on exponential martingale formalism for the historical process, $\Lambda[\Psi\cdot\mathcal{E}^{-1}](t)$ is a $\mathcal{H}_t$-martingale and the measure $\mathbf{Q}_N[\Psi\cdot\mathcal{E}^{-1}]$ is given by $\mathbf{Q}_N[\{\cdot\}\Lambda[\Psi\cdot\mathcal{E}^{-1}]]$. Then it follows from Dawson's Girsanov theorem (Proposition 2 in §7) that, for any positive ε,

$$\mathbf{Q}_N\{\Phi(K_{\cdot\wedge T_N})\} = \mathbf{Q}_N[\varepsilon\Psi\mathcal{E}^{-1}]\Big\{\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}])\Big\}.$$

Immediately,

$$\begin{aligned} &\mathbf{Q}_N \ \Big\{\Phi(K_{\cdot\wedge T_N})\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\Big\} \\ +\ &\mathbf{Q}_N\Big\{\Big(\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}])-\Phi(K_{\cdot\wedge T_N})\Big)\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\Big\} \\ =\ &\mathbf{Q}_N\Big\{\Phi(K_{\cdot\wedge T_N})-\Phi(K_{\cdot\wedge T_N}[\varepsilon\Phi\mathcal{E}^{-1}])\Big\}. \end{aligned}$$

For simplicity we denote by I_1 (resp. I_2) the first (resp. second) term at the left hand side of the above equality, and put

$$I_3 = \text{the right hand side with the minus sign.}$$

Then we find that the convergence

$$\varepsilon^{-1}\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\to\int\int_{C(t)}\Psi(s,y)\ \mathcal{E}(t\wedge T_N)^{-1}\ dM(s,y),\qquad \mathbf{Q}_N-a.s.\quad(\varepsilon\to 0)$$

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\ I_1 = \mathbf{Q}_N\left\{\Phi(K_{\cdot\wedge T_N})\cdot\int\int_{C(t)}\Psi(s,y)\ \mathcal{E}(t\wedge T_N)^{-1}\ dM(s,y)\right\}.$$

Paying attention to the fact that

$$\lim_{\varepsilon\downarrow 0}K^*[\varepsilon\Psi\mathcal{E}^{-1};C](t)=0,\qquad \mathbf{Q}_N-a.s.,$$

we can show that $\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\ I_2=0$, as well.

It remains to treat the third term I_3. In order to discuss the convergence of I_3 divided by ε, we need the following:

Key Lemma (cf. Lemma 12, [Dk99a])

$$\begin{aligned} \mathbf{Q}_N\int\int\Big\{\Phi(\mathcal{M}[s,y]K_{\cdot\wedge T_N})\ &-\ \Phi(K_{\cdot\wedge T_N})\Big\}\Lambda_{\Psi\cdot\mathcal{E}^{-1}}(ds\otimes dy) \\ &= -\ \mathbf{Q}_N\int\int Pr[\Phi]\gamma(s,y)\Psi(s,y)\mathcal{E}^{-1}(t\wedge T_N)dK_{s\wedge T_N}(y)ds. \end{aligned}$$

On the other hand, for $\varepsilon > 0$ we have

$$\begin{aligned} & \mathbf{Q}_N[\Phi(K[\varepsilon\varphi]) - \Phi(K) \,/\, \mathcal{F}] \\ = \;& \varepsilon \cdot \mathrm{e}^{-\varepsilon\Lambda_\varphi((\tau,\infty)\times C)} \int\int_{C(\infty)} \{\Phi(\mathcal{M}[s,y]K) - \Phi(K)\}\Lambda_\varphi(ds\otimes dy) + R(\varepsilon,\Phi,\varphi) \quad (18) \end{aligned}$$

where the residue function R satisfies $|R(\varepsilon,\Phi,\varphi)| \le o(\varepsilon)$. From (18) we get the convergence

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} I_3 = -\mathbf{Q}_N \int\int_{C(t)} Pr[\Phi]\ \gamma(s,y)\cdot\Psi\ \mathcal{E}^{-1}\ dK_{s\wedge T_N} ds. \quad (19)$$

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

§8.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any $y \in C^s$, $R(s,t,y)$ denotes the canonical measure (cf §4) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually, R is a σ-finite measure such that

$$R(s,t,y;M_F(C)) = r_{s,t}.$$

Here the crucial point is that the total mass $r_{s,t}$ does not depend on y. So $r_{s,t}^{-1}dR(s,t,y)$ becomes a probability measure. It is interesting to note that K_t is a sum of independent nonzero clusters with laws $r_{s,t}^{-1}R(s,t,y;dh)$, conditional on $L[s,t]$ (see §5). Furthermore, conditional on $\mathcal{F}_s$, $L[s,t]$ can be regarded as a Poisson point process with intensity $r_{s,t}\gamma(s)K_s$. This is one of the most important points for the computation in terms of clusters growing from the points of $L[s,t_{l+1}]$ in what follows. We define a measure S by the following equation: for $\forall g \in b\mathcal{B}([M_F(C)]^{k-l} \to \mathbf{R})$,

$$\begin{aligned} & \int g(\eta_{l+1},\cdots,\eta_k) S_{s,y}(d\eta_{l+1}\otimes\cdots\otimes d\eta_k) \\ & \qquad = \int g(h(t_{l+1}),\cdots,h(t_k))\cdot \mathbf{I}\{h(t_{l+1}) \ne 0\} Q(s,y;dh) \end{aligned}$$

where $Q(s,y;dh)$ is a σ-finite measure on $C(M_F(C))$ (cf. Eq.(7) in §4). $S_{s,y}^*$ is the normalization of $S_{s,y}$, given by $dS_{s,y}^* := r_{s,t_{l+1}}^{-1} dS_{s,y}$. Moreover, we define

$$\begin{aligned} \Xi(s;E) \;:=\; & \int\int\cdots(k-l)\cdots\int \varphi(K(t_1),\cdots,K(t_l),\sum_{i=1}^m \eta_{l+1}^i,\cdots,\sum_{i=1}^m \eta_k^i) \\ & \times \bigotimes_{i=1}^m S_{s,y}^*(d\eta_{l+1}^i\otimes\cdots\otimes d\eta_k^i), \end{aligned}$$

where $E = \{y_1,\cdots,y_m\} (\ne \emptyset)$.

Take the mass φ as $(\Psi\mathcal{E}^{-1})(s,y)$ at each point y (cf. §5). For simplicity we set

$$\Delta[\Phi](\mathcal{M};s,y,K) := \Phi(\mathcal{M}[s,y]K_{\cdot\wedge T_N}) - \Phi(K_{\cdot\wedge T_N}).$$

Recall the assumption (A.3). Immediately we can get

$$\begin{aligned}
&\mathbf{Q}_N \int\int_{C(\infty)} \Delta[\Phi](\mathcal{M};s,y,K)\Lambda_{\Psi\mathcal{E}^{-1}}(ds\otimes dy)\\
=\;& \mathbf{Q}_N \int_{a+}^{b}\int_C \Delta[\Phi](\mathcal{M};s,y,K)\lambda_s[\Psi\mathcal{E}^{-1}](dy)ds\\
=\;& \int_{a+}^{b} ds\mathbf{Q}_N\left\{\sum_{y\in L[s,u]} \Delta[\Phi](\mathcal{M};s,y,K)\cdot(\Psi\mathcal{E}^{-1})(s,y)\right\}.
\end{aligned}$$

In the following calculation, we may take much advantage of those concepts such as i) the Markov property of K_t; ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location $L[s,t_{l+1}]$. Hence we can proceed with the computation. In fact,

$$\begin{aligned}
&\mathbf{Q}_N \left\{\sum_{y\in L[s,u]} \Delta[\Phi](\mathcal{M};s,y,K)\cdot(\Psi\mathcal{E}^{-1})(s,y)\right\}\\
=\;& \mathbf{Q}_N\left\{\mathbf{P}\left[\sum_{y\in L[s,u]} \mathbf{P}\{\Delta[\Phi]\cdot\Psi\mathcal{E}^{-1}|\mathcal{F}_s\bigvee\sigma(L[s,u])\}\middle|\mathcal{F}_s\right]\right\}\\
=\;& \mathbf{Q}_N\left\{\mathbf{P}\left[\sum_{y\in L[s,u]} \{\Xi(s;L[s,u]\setminus\{y\}) - \Xi(s;L[s,u])\}\cdot\Psi\mathcal{E}^{-1}\middle|\mathcal{F}_s\right]\right\} \qquad (20)
\end{aligned}$$

It is easy to see the following lemma.

Lemma 4. *The last expression of* (20) *is equivalent to*

$$\begin{aligned}
&\mathbf{Q}_N \int_C (\Psi\mathcal{E}^{-1})(s,y)\cdot r_{s,t_{l+1}}\gamma(s,y)K_{s\wedge T_N}(dy)\Big[\exp\left(-r_{s,t_{l+1}}K_s(C)\right)\cdot\\
\times\;& \sum_{m=0}^{\infty}\frac{1}{m!}\int\int\cdots(m)\cdots\int_{[C]^m}\{\Xi(s;\{y_1,\cdots,y_m\}) - \Xi(s;\{y_1,\cdots,y_m,y\})\}\cdot\\
\times\;& (r_{s,t_{l+1}})^m K_s^{\otimes m}(dy_1,\cdots,dy_m)\Big].
\end{aligned}$$

A simple computation implies that the integral expression in Lemma 4 is also equal to

$$\begin{aligned}
&\mathbf{Q}_N \int_C (\Psi\mathcal{E}^{-1})(s,y)\gamma(s,y)K_{s\wedge T_N}(dy)\cdot\Big[\int\int\cdots(k-l)\cdots\int_{[M_F(C)]^{k-l}}\\
\times\;& \mathbf{P}\{\varphi(K(t_1),\cdots,K(t_k)) - \varphi(K(t_1),K(t_l),K(t_{l+1})+\eta_{l+1},\cdots,K(t_k)+\eta_k)|\mathcal{F}_s\}\\
\times\;& r_{s,t_{l+1}}\cdot S^*_{s,y^{s-}}(d\eta_{l+1}\otimes\cdots\otimes d\eta_k)\Big]. \qquad (21)
\end{aligned}$$

While, taking (7), (8) in §4, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

Lemma 5. *The followinf equality holds for all* s, y:

$$\begin{aligned} Pr \quad [\Phi](s,y) &= \int\int \cdots (k-l) \cdots \int r_{s,t_{l+1}} \cdot S^*_{s,y^{s-}}(d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \cdot \\ \times \quad \mathbf{P}\{\varphi(K(t_1), &\cdots, K(t_l), K(t_{l+1}) + \eta_{l+1}, \cdots, K(t_k) + \eta_k) - \varphi(K(t_1), \cdots, K(t_k)) | \mathcal{F}_s\}. \end{aligned}$$

Therefore, an application of the above proposition with Lemma 4 implies

$$\begin{aligned} &- \mathbf{Q}_N \int\int_{C(t)} Pr[\Phi](\gamma \cdot \Psi\mathcal{E}^{-1})(s,y) dK_{s\wedge T_N} ds \\ &= \int_{\tau+}^t ds \left\{ \mathbf{Q}_N \int_C (-Pr[\Phi])\gamma \cdot \Psi\mathcal{E}^{-1} dK_{s\wedge T_N} ds \right\} = \int_{\tau+}^t Eq.(21) ds = \int_{\tau+}^t Eq.(20) ds \\ &= \mathbf{Q}_N \int\int_{C(t)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda_{\Psi\mathcal{E}^{-1}}(ds \otimes dy), \end{aligned}$$

which completes the proof.

§9. Itô-Clark Formula: Proof of Theorem 4

Since $\mathbf{P}[K_t(C)^2]$ is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity $\mathbf{P}[F(K)^2]$. Therefore we can apply Theorem 1 (§3) for $F(K)$ to obtain that

$$F(K) = \mathbf{P}[F(K)] + \int_\tau^\infty \int_C f(s,y) dM^K(s,y), \mathbf{P} - a.s. \tag{22}$$

holds for some f in $L^2_\infty(K, \mathbf{P})$. While, it follows from the covariance formula in the theroy of stochastic integration that

$$\begin{aligned} \mathbf{P} &\left[\left(\int\int_{C(\infty)} f(s,y) dM^K(s,y) \right) \left(\int\int_{C(t)} \Psi(s,y) dM^K(s,y) \right) \right] \quad (23) \\ = \mathbf{P} &\left[\int_\tau^t \int_C f(s,y)\Psi(s,y)\gamma(s,y) K_s(dy) ds \right] \end{aligned}$$

for all $t > \tau$ and Ψ in $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$. Rewriting the left hand side of Eq.(23) we get

$$\mathbf{P}\left[F(K) \int_\tau^t \int_C \Psi(s,y) dM^K(s,y) \right] \tag{24}$$

by employing the predictable representation property (22). Hence we may apply Theorem 3 (§6) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded $(\mathcal{C}_t \times \mathcal{F}_t)$-predictable functions. So that, from (23)

$$\mathbf{P} \int\int_{C(t)} f(s,y)\Psi(s,y)\gamma(s,y) dK_s ds = \mathbf{P} \int\int_{C(t)} Pr[F](s,y)\Psi(s,y)\gamma(s,y) dK_s ds.$$

On this account, the general theory of Hilbert spaces shows that

$$\mathbf{P}\int_{\tau}^{t}\int_{C}\{f(s,y)-Pr[F](s,y)\}^2\gamma(s,y)K_s(dy)ds=0.$$

Therefore the uniqueness argument allows us to conclude that $\int\int_{C(t)} fdM$ is equivalent to $\int\int_{C(t)} Pr[F]dM$, $\mathbf{P}$-a.s. Note that $Pr[F](s,y)$ become null for K_s-a.s. y, for any $s>t$, by its construction, as long as we choose t largely enough for the support of m to be contained in $[\tau,t]$. Consequently, the above integral $\int\int Pr[F]dM$ can be replaced by $\int\int_{C(\infty)} Pr[F]dM$, which completes the proof. This goes quite similarly as in the proof of Theroem 2.5 in [EP95].

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