A Variant of Itô-Clark Type Formula  
in Historical Stochastic Analysis*

Isamu DÔKU (道工 勇)  
Department of Mathematics, Saitama University  
Urawa 338-8570, Japan

§1. Introduction

We consider a version of Itô-Clark type stochastic integration formula (e.g. [U95, p.92] )  
in the theory of historical superprocesses. The key idea of demonstration of the Itô-Clark  
formula is to derive a variant of Evans-Perkins type stochastic integration by parts with  
respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two  
reasons why this type of integration by parts formula is so important. For one thing, it can  
provides with a new formula of transformations of stochastic integrals closely connected  
with the so-called historical processes. In fact the establishment of the formula asserts  
that a product of historical functionals of a specific class and stochastic integral relative  
to the orthogonal martingale measure in the Walsh sense [W86] is, in its mathematical  
expectation form, equivalent to a certain expression of integration that is involved with  
stochastic integral with respect to a Dawson-Perkins historical process [DP91] associated  
with a reference Hunt process. In addition, it also allows us to interpret that the formula  
is nothing but a variant of stochastic integration by parts in an abstract level, that is very  
useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes.  
For another, it has an extremely remarkable meaning on an applicational basis. By making  
use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind  
of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any $L^2$ functional of superprocess may  
be represented as a constant $C_0$ plus a stochastic integral with respect to the associated  
orthogonal martingale measure $M$ (e.g. [EP94] ). Recently they have obtained the explicit  
representations involving multiple stochastic integrals for a quite general functional of the

*Research supported in part by JMESC Grant-in-Aids SR(C) 07640280 and CR(A) 09304022.
so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95]. Based upon the previous results (1994), they derived partial analogue of the Itô-Wiener chaos expansion in superprocess setting by taking advantage of the "stochastic integral formula" in question.

Lastly we shall give a rough idea of what the integration formula is like, but in the form as simple as possible. First of all, let us consider the functional $F(H)$ of a historical process $H$ with branching mechanism $\Phi$ for a real valued function $F$ on $C([0, \infty); M_F(D))$ with the space $D$ of $E$-valued cadlag paths. Actually, this $F$ should lie in a suitable admissible subspace $U(M(D))$ of $C(C([0, \infty); M_F(D)); \mathbb{R})$. Next consider a stochastic integral $J(\Xi; M) = \int \int \Xi(s, y) dM$ of a bounded predictable function $\Xi$ relative to the orthogonal martingale measure $M$ in the Walsh (1986) sense. Then we make a product $F(H) \cdot J(\Xi; M)$. On the other hand, consider the integral of another type $J(F, \Xi; H) = \int \int I[F] \Xi(s, y) dH ds$ for some predictable function $I[F]$ which is determined by the functional $F(H)$ given. Thus we attain the integration formula if we take the mathematical expectation of both terms, i.e., $\mathbb{E}[F(H) \cdot J(\Xi; M)] = \mathbb{E}[J(F, \Xi; H)]$.

§2. Notation and Preliminaries

Let $C = C^{d} = C([0, \infty), \mathbb{R}^{d})$ denote the space of $\mathbb{R}^{d}$-valued continuous paths on $\mathbb{R}_{+} = [0, \infty)$ with the compact-open topology. $C = B(C)$ is its Borel $\sigma$-field and

$$C_t = B_t(C) = \sigma(y(s), s \leq t)$$

denotes its canonical filtration. For $y, w \in C^{d}$ and $s \geq 0$, we define the stopped path by $y^s(t) = y(t \wedge s)$ and let

$$y/s/w = \begin{cases} y(t), & \text{for } t < s, \\ w(t - s), & \text{for } t \geq s. \end{cases} \tag{1}$$

$M_F(C)$ is the space of finite measures on $C$ with the topology of weak convergence and we define

$$M_F(C)^t := \{m \in M_F(C); y = y^t, \text{ m - a.s. } y \}, \quad t \geq 0.$$ 

If $P_x$ denotes Wiener measure on $(C, B(C))$ starting at $x$, $\tau \geq 0$, and $m \in M_F(C)^\tau$, define $P_{\tau, m} \in M_F(C)$ by

$$P_{\tau, m}(A) := \int_C P_y(\{w; y/\tau/w \in A\}) dm(y).$$

Let

$$\Omega_H[\tau, \infty) := \{H \in C([\tau, \infty), M_F(C)); H_t \in M_F(C)^t, \forall t \geq \tau\},$$

and put $\Omega_H := \Omega_H[0, \infty)$. We use $\mathcal{H}$ for the totality of Borel sets of $\Omega_H$. We use the notation $H_t(\omega) = \omega(t)$ for $\omega \in \Omega_H$ as for the canonical realization of historical process.
Fix $0 \leq t_1 < \cdots < t_n$ and $\psi \in C^2_0(\mathbb{R}^{nd})$. For $y \in C$ we set

$$
\bar{\psi}(y) \equiv \psi(t_1, \cdots, t_n)(y) = \psi(y(t_1), \cdots, y(t_n)),
$$

and $\bar{\psi}(t, y) = \bar{\psi}(y')$. $\psi_i$ (resp. $\psi_{ij}$) stands for the first (resp. second) order partials $\partial_i \psi$ (resp. $\partial_{ij} \psi$) of $\psi$. $\nabla \bar{\psi} : [0, \infty) \times C \to \mathbb{R}^d$ is the $(C_t)$-predictable process whose $j$-th component at $(t, y)$ is given by

$$
\sum_{i=0}^{n-1} I(t < t_{i+1}) \partial_{i} \partial_{j} \bar{\psi}(t, y)
$$

While, for $1 \leq i, j \leq d$, $\hat{\psi}_{ij} : [0, \infty) \times C \to \mathbb{R}$ is the $(C_t)$-predictable process defined by

$$
\hat{\psi}_{ij}(t, y) := \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} I(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i} \partial_{ld+j} \bar{\psi}(t, y).
$$

Let us define the domains

$$
D_0 := \bigcup_{n=1}^{\infty} \{\psi(t_1, \cdots, t_n); 0 \leq t_1 < \cdots < t_n, \psi \in C^\infty_0(\mathbb{R}^{nd})\} \cup \{1\},
$$

$$
\hat{D}_0 := \{\bar{\psi}; \bar{\psi}(t, y) = \bar{\psi}(y') \text{ for some } \bar{\psi} \in D_0\}.
$$

Let $\hat{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq \tau}, \mathbb{P})$ be a filtered probability space and let $(\omega, y) = (\omega, y_1, \cdots, y_d)$ denote sample points in $\hat{\Omega} = \Omega \times C^d$. Here $\tau \geq 0$ is fixed. When $f$ is a function on $[\tau, \infty) \times \hat{\Omega}$ taking values in a normed linear space $(E, \| \|)$, then a bounded $(\mathcal{F}_t)$-stopping time $T$ is a reducing time for $f$ if and only if

$$
I(\tau < t \leq T) \| f(t, \omega, y) \|
$$

is uniformly bounded. In addition we say that a sequence $\{T_n\}$ reduces $f$ if and only if each $T_n$ reduces $f$ and $T_n \nearrow \infty$ holds $\mathbb{P}$-a.s. We say that $f$ is locally bounded if such a sequence $\{T_n\}$ exists. We assume that

(1) $\gamma \in [0, \infty), a \in S^d, b \in \mathbb{R}^d$ and $g \in \mathbb{R}$ are $(\mathcal{F}_t^\tau)$-predictable processes on $[\tau, \infty) \times \hat{\Omega}$ such that $\Lambda = (\gamma, a, b, g \gamma^{-1} I(\hat{\gamma} \neq \bar{\gamma}))$ is locally bounded.

Notice that the above assumption implies that $g$ is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For $\tau \geq 0$ and $m \in M_F(C)^\tau$, we define

$$
A_{\tau, m}(\bar{\psi})(t, y) = A(\bar{\psi})(t, y) := \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(t, y) \psi_{ij}(t, y) + b(t, \omega, y) \cdot \nabla \bar{\psi}(t, y) + g(t, \omega, y) \bar{\psi}(y')
$$

for $\bar{\psi} \in D_0$. We write $\langle \mu, f \rangle$ or sometimes $\mu(f)$ for the integral $\int f \, d\mu$ when $\mu$ is a measure and $f$ is a suitable $\mu$-integrable function. Suggested by [DkTn98], we may define
**Definition.** (cf. [P95], §2) A predictable process \( K = \{K_t, t \geq \tau\} \) on \( \Omega \) with sample paths a.s. in \( \Omega_H[\tau, \infty) \) is a generalized \( \{\gamma, a, b, g\} \)-historical process (GHP) (or \( (A, -\gamma \lambda^2/2) \)-historical process) if and only if \( K_t \in M_F(C)^t \) for all \( t \geq \tau, \) a.s. and \( \mathbb{P}[K_\tau(1)] < \infty, \) and if there exists a probability measure \( \mathcal{P} \) on \( \Omega_H[\tau, \infty) \) such that it satisfies the martingale problem (MP) with initial data \( \{\tau, m\} \) and \( \{\gamma, a, b, g\} \): for \( \forall \overline{\psi} \in \mathcal{D}_0, \)

\[
Z_t(\overline{\psi}) = \langle K_t, \overline{\psi} \rangle - \langle m, \overline{\psi} \rangle - \int_{\tau}^{t} \langle K_s, A(\overline{\psi})(_{S}) \rangle ds, \quad t \geq \tau,
\]

is a continuous \((\mathcal{F}_t)\)-local martingale satisfying \( Z_\tau(\overline{\psi}) = 0 \) and

\[
\langle Z(\overline{\psi}) \rangle_t = \int_{\tau}^{t} \int \gamma(s, \omega, y)\overline{\psi}(y)^2 K_s(dy) ds, \quad \forall t \geq \tau, \ a.s.
\]

**Remark.** The existence and uniqueness of the law of \( K \) is essentially due to [F88] (cf. [DIP89]).

Set \( T_s = [s, \infty), \) and in particular \( T_0 = [\tau, \infty). \) Define \( C(M_F(C)) := C(T_0; M_F(C)) \), and we write \( C(t) = (\tau, t] \times C \) for the integral domain. When \( \mathcal{F} \) is the \( \sigma \)-field or the usual filtration, then \( f \in \mathcal{F} \) indicates that the function \( f \) is \( \mathcal{F} \)-measurable and \( \mathcal{P}(\mathcal{F}) \) is the totality of \((\mathcal{F})\)-predictable functions, and \( bP(\mathcal{F}) \) denotes the whole space of functions that are all bounded elements of \( \mathcal{P}(\mathcal{F}). \) We use the symbol \( U(M_F(C)) \) for an admissible subset of the space \( C(C(M_F(C)); \mathbb{R}); \) more precisely \( U(M_F(C)) \) is the totality of real valued continuous functions \( F \) on \( C(M_F(C)) \) such that for some compactly supported finite measure \( L(dt) \) on \( T_0, \) the estimate

\[
|\Delta F(h, g)| \leq \int_{T_0} g(t, C) L(dt)
\]

holds for all \( h, g \in C(M_F(C)), \) where we define \( \Delta F(x, y) := F(x+y) - F(x). \)

**§3. Predictable Representation Property**

Let \( \{T_N\} \) be a reducing sequence. Take a sequence \( \{\overline{\psi}_n\}, \overline{\psi}_n \in \mathcal{D}_0 \) such that \( \overline{\psi}_n \) converges bounded pointwise (bp for short) to \( \psi, \) namely,

\[
\overline{\psi}_n \to \psi, \quad \text{bp (} n \to \infty\).
\]

An application of dominated convergence theorem together with the local boundedness of \( \gamma \) implies that

\[
\langle Z(\overline{\psi}_n - \overline{\psi}_m) \rangle_t \to 0 \quad \text{as } n, m \to \infty
\]

for \( \forall t \geq \tau, \) a.s. Therefore we obtain

**Proposition 1.** There is an a.s. continuous adapted process \( \{Z_t(\psi); t \geq \tau\} \) such that

\[
\sup_{\tau \leq t \leq N} \left| Z_t(\overline{\psi}_n) - Z_t(\psi) \right| \to 0
\]
holds in probability (w.r.t. $P$) as $n \to \infty$ for $\forall N > \tau$.

To proceed our discussion, we need the following lemmas.

**Lemma 1.** (cf. Corollary 2.2, p.11, [P95]) Let $T$ be a reducing time for $(\gamma, g)$. Then we have

(a) $0 < P[K_T(1)] \leq P[\sup_{s \leq t \leq T} |K_t(1)| + (Z(1))_T] < \infty.$

(b) If $P[K_T(1)^p] < \infty$ for $p \in \mathbb{N}$, then

$$P \left\{ \left( \sup_{s \leq t \leq T} |K_t(1)| \right)^p + (Z(1))_T^p \right\} < \infty.$$

**Lemma 2.** (cf. [EP94, p.123]) $D_0$ is dense in $\mathcal{B}(C)$ relative to the bounded pointwise convergence topology.

We may use Lemma 1 to obtain

$$\sup_{T \leq t \leq T_N} |Z_t(\bar{\psi}_n) - Z_t(\bar{\psi})| \to 0 \text{ in } L^2$$

as $n \to \infty$, for $\forall N \in \mathbb{N}$. Clearly $Z_t(\bar{\psi})$ is a continuous $(\mathcal{F}_t)$-local martingale whose quadratic variation process is given by

$$(Z(\bar{\psi}))_t = \int_{\tau}^{t} \int_{C} \gamma(s, \omega, y)\psi(y)^2K_s(dy)ds.$$  

By virtue of Lemma 2, it is a routine work to show that this $Z_t$ extends to an orthogonal martingale measure

$$\{Z_t(\bar{\psi}); t \geq \tau, \psi \in \mathcal{B}(C) \}.$$  

Consequently, the mapping $t \mapsto Z_t(\bar{\psi})$ is a continuous local martingale satisfying Eq.(3) for each $\psi \in \mathcal{B}(C)$, and $\psi \mapsto Z_{t\wedge T_N}(\psi)$ is an $L^2$-valued measure on $\mathcal{B}(C)$ for each $t \geq \tau$, $N \in \mathbb{N}$. By a trivial localization argument, we may define the stochastic integral

$$Z_t(\psi) = \int_{\tau}^{t} \int_{C} \psi(s, \omega, y) dM(s, y)$$  

( $\exists$ an orthogonal martingale measure $M = M^K$ in the sense of Walsh [W86, Chapter 2] ) such that

$$(Z(\psi))_t = \int_{\tau}^{t} \langle K_s, \gamma(s, \omega)\psi(s, \omega)^2 \rangle ds,$$  

$\forall t \geq \tau$, a.s., as long as $\psi$ belongs to $L^2_{loc}(K, P)$. Here $L^2_{loc}(K, P)$ denotes the $L^2$ space of $(\mathcal{F}_t \times C)_{t \geq \tau}$-predictable functions $f$ and

$$\int_{\tau}^{t} \gamma(s, y)f(s, y)^2K_s(dy)ds < \infty$$

for $\forall t \geq \tau$, $P$-a.s.
We write $f \in L^2(K, \mathbb{P})$ (resp. $L^\infty_\infty(K, \mathbb{P})$) if, in addition,
\[
\mathbb{P}\left\{ \int_{\tau}^{t} \int \gamma(s,\omega,y)f(s,\omega,y)^2K_s(dy)ds \right\} < \infty, \quad \forall t > 0,
\]
respectively,
\[
\mathbb{P}\left\{ \int_{\tau}^{\infty} \int \gamma(s,\omega,y)f(s,\omega,y)^2K_s(dy)ds \right\} < \infty.
\]

**Theorem 1.** (Predictable Representation Property) If $V \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then there is an $f$ in $L^\infty_\infty(K, \mathbb{P})$ such that
\[
V = \mathbb{P}[V] + \int_{\tau}^{\infty} \int f(s,\omega,y)dM^K(s, y), \quad \mathbb{P}-a.s. \quad (6)
\]

**Proof.** It is sufficient to verify (6) for the particular case where $V$ is a square integrable martingale $M_t$. Then Jacod's general theory (cf. Theorem 2 and Proposition 2 of [J77]) provides with a stochastic integral representation of $M_t$. For the rest, it goes almost similarly as in the proof of Theorem 1.1 [EP94, p.124].

§4. Canonical Measure and Campbell Measure

For $y \in D = D(\mathbb{R}_+; \mathbb{R}^d)$, we define $y^r(s)$ as $y(s)$ if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s, y)$ is a $\sigma$-finite measure on $C(M_F(D))$ such that
\[
Q(s, y^r; \{ h \in C(M_F(D)); \tau \leq \exists t \leq s, h(t) \neq 0 \}) = 0,
\]
which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ [D93] associated with the law of $K_t = K(t)$ and the path restriction mapping $\pi$ (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here $R$ is characterized by
\[
\log \mathcal{P}_s[\exp(K_t, -\varphi)] = \int_{\mathcal{M}_F(D)} \left( e^{-(\zeta, \varphi)} - 1 \right) R(s, t, y; d\zeta)
\]
(cf. Lemma 1 in [Dk99c]; see also [DP91, Proposition 3.3, pp.38-39]). Let $F$ be a real valued Borel function on $C(M_F(C))$. Assume that
\[
I^Q_{s,y}[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h, g)Q(s, y^r; dg)
\]
(7)
is well-defined and bounded below for all $s > \tau$, $y \in C$, and $h \in C(M_F(C))$. For a bounded ($\mathcal{F}_t$)-stopping time $T$, we define the Campbell measure $P_T$ associated with $K(t)$ by
\[
P_T(A \times B) := \mathbb{P}(K(T,A) \cdot I_B(K(T))) / m(C)
\]
(8)
for any $A \times B \in (C \times \Omega, C \times F)$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_r = m$. Since the mapping $(s, y, \omega) \mapsto I_{s,y}^Q(\Delta F)(K(\omega))$ is bounded below and measurable with respect to the product of the predictable $\sigma$-field associated with the filtration $(C_i)$ and the $\sigma$-field $F$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(C \times F_t)_{t \geq r}$-predictable function $Pr[F](s, y, \omega) : (r, \infty) \times C \times \Omega \rightarrow \mathbb{R}$ such that

$$P_{T}\{I_{s}^{Q}(\Delta F)/(C \times F_{T})\} = Pr[F](T, \omega, y)$$

holds $P_{T}$-a.s. for all bounded $(F_t)$-predictable stopping times $T > s$. It is quite interesting to note that in particular

$$P \int_{C} I_{s}^{Q}(\Delta F)(T, y)K(T, dy) = P \int_{C} Pr[F](T, y)K(T, dy).$$

We shall introduce an approximation map. For each $l \in \mathbb{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j) ; 1 \leq j \leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \cdots < t^{(l)}(k[l]) < \infty$,

$$\lim_{l \rightarrow \infty} \{\sup_{k} \Delta t[l; k]\} = 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map $W[l]$ from $C(M_{F}(C))$ into $C(M_{F}(C))$ is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i) + 1) - Sb(t^{(l)}(i)) \cdot g(t^{(l)}(i))\} \Delta t[l; i]^{-1}$$

if $t \in [t^{(l)}(i), t^{(l)}(i + 1)]$, and := $g(t^{(l)}(k[l]))$ if $t \geq t^{(l)}(k[l])$, for any element $g$ of $C(M_{F}(C))$ with $Sb(k) = k - t$. Immediately we get

**Lemma 3.** (cf. Lemma 4 [DK98a]) Let $F$ be an element of $C(C(M_{F}(C)) \times \mathbb{R})$. Then for all $g \in C(M_{F}(C))$

$$\lim_{l \rightarrow \infty} (F \circ W[l])(g) = F(g).$$

§5. Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section. $C^4$ denotes the image of $C$ under the map: $y \mapsto y^t$. We define a measure $K^*[s, t]$ on $C^*$ by $K^*[s, t](F) := K_t(\{y : y^s \in F\})$. Then the measure $K^*[s, t]$ is atomic with a finite set of atoms, and we write $L[s, t] \subset C^*$ for the locations of these atoms. For $s \in (a, b]$, let $\lambda_{s}[\varphi]$ be the random measure on $C$ that places mass $\varphi(s, y)$ at each point $y$ in $(L[b, c])^* = L[s, c]$. With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure $Q_N$ on $(\Omega, F)$ such that

$$\frac{dQ_{N}}{dP} \bigg|_{F_{i}} = \exp \left\{ \int_{\tau}^{t^{\Delta^{TN}}} \int g\gamma^{-1}(s)I(g(s) \neq 0) dM^{K}(s, y) - \frac{1}{2} \int_{\tau}^{t^{\Delta^{TN}}} g^{2}\gamma^{-1}(s)I(g(s) \neq 0) K_{s}(dy) ds \right\}.$$
For brevity’s sake we rather write $\mathcal{E}(t \wedge T_N)$ than the above. On this account, $K_{\wedge T_N}$ satisfies the martingale problem $(\text{MP})[\gamma_N, a_N, b_N, 0]$ instead of $(\text{MP})[\gamma, a, b, g]$, where we set $f_N := f \cdot \mathbf{1}(\tau < t \leq T_N)$. Moreover, for $s \in (a, b], y \in C^s$, the symbol $\mathcal{M}[s, y]$ denotes the mapping of the set of functions $\{m : (\tau, \infty) \rightarrow M_F(C)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s, y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \geq s$.

Let us now introduce assumptions for our principal results.

(A.1) $g : [\tau, \infty) \times \Omega \times C \rightarrow \mathbf{R}$ is a $(\mathcal{F}_t \times C_t)^*$-predictable process such that $g_{\gamma^{-1}} \cdot \mathbf{1}(g \neq 0)$ is locally bounded.

(A.2) For any predictable function $f$ on $[\tau, \infty) \times I \times C^* \times \Omega$, the counting measure $n^*$ satisfies
\[
P \int_C n^*((s, t] \times I) G_t(dx) = m(C^*)(t-s)
\]
where $G_t$ is a marked historical process corresponding to $K$ and $N_t$ is the martingale measure associated with $G_t$ (cf. §7 for details).

(A.3) There exists a random measure $\Lambda_{\varphi}$ on $(\tau, \infty) \times C$ such that
\[
\int \int_{C(\infty)} f(s, y) \Lambda_{\varphi}(ds \otimes dy) = \int_a^b \int_C f(s, y) \lambda_s[\varphi](dy)ds
\]
holds for any suitable predictable function $f$.

(A.4) $\Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1}$ is uniformly bounded in $s$, $K_s$-a.e. $y$, $Q_N$-a.s.

(A.5) There exists some constant $C_0 (>0)$ such that
\[
\int \int_{C(t)} \Psi(s, y)^2 \mathcal{E}(t \wedge T_N)^{-2} \gamma(s, y) K_s(dy)ds \leq C_0
\]
holds $Q_N$-a.s., for all $t \geq \tau$.

Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

§6. Main Results: Stochastic Integration Formulae

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let $K$ be a predictable measure-valued process whose law is specified by a general martingale problem $(\text{MP})[\tau, K, \gamma, a, b, g]$.

**Theorem 2.** (cf. [Dk98b]) Assume that $\Phi : C(M_F(C)) \rightarrow \mathbf{R}$ is a cylinder function with bounded representing function $\varphi : [M(C)]^k \rightarrow \mathbf{R}$ and base $\tau < t(1) < \cdots < t(k)$, such that
\[
|\Delta \varphi(\alpha, \beta)| \leq c_0 \sum_j \beta_j(C)
\]
for some positive constant $c_0$, for all $\alpha, \beta = (\beta_j) \in [M(C)]^k$. Then for $t > \tau$
\[
P \left\{ \Phi(K) \int \int_{C(t)} \Psi(s, y)dM^K(s, y) \right\} = P \int \int_{C(t)} Pr[\Phi](s, y)\Psi(s, y)\gamma(s, y)K_s(dy)ds
\]
holds where $\Psi$ is a bounded $(C_t \times \mathcal{F}_t)_{t \geq \tau}$-predictable function, $K_t$ is a GHP, and $Pr[\Phi]$ is a predictable function determined by (9) in accordance with the given $\Phi$.

**Remark 1.** The assertion of the above theorem is quite similar to Theorem 2.4 (p.1785, §2, [EP95]).

**Theorem 3.** (Stochastic Integration By Parts) Let $F \in U(M_F(C))$. If $\Psi$ is an element of $bP(C_t \times \mathcal{F}_t)$, then for all $t > s$,

$$\mathbb{P}\left\{ F(K) \int_{C(t)} \Psi(s,y) dM^K(s,y) \right\} = \mathbb{P} \int_{C(t)} Pr[F](s,y) \alpha(s,y) \Psi(s,y) K_s(dy) ds.$$

(10)

**Remark 2.** Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional $F(K)$. As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it (cf. §3, [Dk98a]). We shall refer to this generalization in §8.

**Theorem 4.** (Itô-Clark Type Formula) Let $F \in U(M_F(C))$.

$$F(K) = \mathbb{P}[F(K)] + \int_{\tau}^{\infty} \int Pr[F](s,y) dM^K(s,y)$$

where $Pr[F](s,y)$ is a $\mathcal{P}(C_t \times \mathcal{F}_t)$-measurable version (relative to $P_T$) of

$$P_T \left[ \int_{C(M_F(C))} \Delta F(K,h) Q(s, y^{s-}; dh) / (D \times \mathcal{F})_T \right].$$

§7. Marked Historical Processes and the Girsanov-Dawson-Perkins Theorem

Set $I = [0, 1]$, $E^* = C \times I$ and $C^* = C(\mathbb{R}_+, E^*)$, and let $C^*$ (resp. $C^*_t$) be the Borel $\sigma$-field (resp. the canonical filtration) of $C^*$. Put $x = (y, n) \in E^*$. Let $G$ be the corresponding counterpart historical process of $K$ starting at $(\tau, \mu)$, defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbb{P})$. Suppose that $\varphi : (\tau, \infty) \times C \times \Omega \to I$ be an element of $\mathcal{P}(C_t \times \mathcal{F}_t)$. Given any cadlag function $n : \mathbb{R}_+ \to I$, we can construct a $\sigma$-finite counting measure $n^*$ on $\mathbb{R}_+ \times I$ by assigning an atom of mass one to each point $(s, z)$ such that $n(s) - n(s-) = z \neq 0$. Put

$$A(t, x, \omega) := n^*(\{(s, z) \in [\tau, t) \times I; \ \varphi(s, y, \omega) > z\})$$

(12)
and $B(t,x,\omega) = I\{A(t,x,\omega) = 0\}$. Then we can define an $M_F(C)$-valued process $K[\varphi](t)$ by
\[ K[\varphi;J](t) := \int_{C^*} I\{J\}(y) B(t,x) G_t\,d\mathcal{I} \] (13)

Put
\[ I_1(\varphi, N) = \int_{C^*(t)} \varphi(s,y) dN(s,x), \quad I_2(\varphi, G) = \int_{C^*(t)} \gamma(s,y) \varphi(s,y)^2 G_s\,ds \]
with $C^*(t) = (\tau, t] \times C^*$. Then we define
\[ \Lambda[\varphi](t) := \exp\{I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G)\}. \] (14)

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$-martingale. The new probability space $(\Omega, \mathcal{H}, \mathbb{P}[\varphi])$ is defined by
\[ \mathbb{P}[\varphi]\{F\} := \mathbb{P}\{F \cdot \Lambda[\varphi](t)\} \] (cf. $[\text{Dk98a}]$) for any $F \in b\mathcal{H}_t$ with
\[ \mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \] (15)
(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

**Proposition 2.** (cf. Theorem 5.1, p.1798, [EP95]) *The law of $K[\varphi]$ under $\mathbb{P}[\varphi]$ is equivalent to the law of $K$ under $\mathbb{P}$.*

### §8. Sketch of Proofs of Main Theorems

#### §8.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §6, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function $L^*$ on $C(M_F(C))$ by
\[ L^*[g] := \int_{T_0} g(t,C) L(dt) = \langle L, g(\cdot, C) \rangle. \] (16)

In connection with the measure $L$ (see §2), we introduce the finite measure $L(l) \equiv L(l, dt)$ which concentrates its mass on $\{t^{(l)}(j); 0 \leq j \leq k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, C) \rangle$ for $g \in C(M_F(C))$. Recall that
\[ \int g(t,C) Q(s,y; dg) = \int \xi(C) R(s,t,y; d\xi) = 1 \]
holds (cf. Lemma 3, [Dk99a]) with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:
\[ \mathbb{P} \int \int_{C(t)} \{Q(s,y^s) L^*[g]\} K_s(dy)\,ds = \lim_{l \to \infty} \mathbb{P} \int \int_{C(t)} \{Q(s,y^s) (L^* \circ W[l])[g]\} K_s(dy)\,ds \]
holds with $g \in C(M_F(C))$ for all $t > \tau$, and
\[
\mathbf{P} \int \int_{C(t)} Pr[F](s,y)Z(s,y)K_{a}(dy)ds
= \lim_{l \to \infty} \mathbf{P} \int \int_{C(t)} Pr[F \circ W[l]](s,y)Z(s,y)K_{a}(dy)ds.
\] (17)
holds for all $t > \tau$ if $Z \in \mathcal{P}(C_t \times \mathcal{F}_t)$. Since, for each $n \geq 1$, \[\mathbf{P}\{K_t(C)^n\}\] is uniformly bounded on compact intervals, we can readily deduce that \[\mathbf{P}\{(L^* \circ W[l])^{K^n}\}\] is bounded in $l$ for each $n \geq 1$. Moreover,
\[
\mathbf{P}\{F(K) \int \int_{C(t)} \Psi(s,y) dM(S,y)\} = \lim_{l \to \infty} \mathbf{P}\{(F \circ W[l])(K) \int \int_{C(t)} \Psi(s,y) dM(S,y)\}.
\]
To complete the extension discussion in this section we have only to observe that $F \circ W[l]$ satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk9a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 2 in §7 is valid. Hence, combining the above results, we obtain
\[
\mathbf{P}\{F(K) \int \int_{C(t)} \Psi(s,y) dM(S,y)\} = \lim_{l \to \infty} \mathbf{P}\{(F \circ W[l])(K) \int \int_{C(t)} \Psi(s,y) dM(S,y)\}.
\]
which concludes Theorem 3.

§8.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk9a].

Thanks to (A.1), it suffices to verify the integral formula for a special \{\gamma_N, a_N, b_n, 0\}-historical process $K_{\wedge T_N}$ under $\mathbf{Q}_N$ instead of the generalized $K$ (GHP) with $\mathbf{P}$. Indeed, since $d\mathbf{P} = \mathcal{E}(t \wedge T_N)^{-1} d\mathbf{Q}_N$, what we have to show is as follows:

(The Modified Stochastic Integration By Parts Formula)
\[
\mathbf{Q}_N \left\{ (t \wedge T_N)^{-1} \cdot \Phi(K_{\wedge T_N}) \int \int_{C(t)} \Psi(s,y) dM(S,y) \right\} = \mathbf{Q}_N \left\{ (t \wedge T_N)^{-1} \int \int_{C(t)} Pr[\Phi](s,y) \gamma(s,y) \Psi(s,y) K_{\wedge T_N}(dy)ds \right\}.
\]
Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for $\varphi = \Psi \cdot \mathcal{E}^{-1}$. Hence, by the arguments on exponential martingale
formalism for the historical process, $\Lambda[\Psi \cdot \mathcal{E}^{-1}](t)$ is a $\mathcal{H}_t$-martingale and the measure $Q_N[\Psi \cdot \mathcal{E}^{-1}]$ is given by $Q_N[\{\} \Lambda[\Psi \cdot \mathcal{E}^{-1}]]$. Then it follows from Dawson's Girsanov theorem (Proposition 2 in §7) that, for any positive $\epsilon$,

$$Q_N\{\Phi(K_{\wedge T_N})\} = Q_N[\epsilon \Psi \mathcal{E}^{-1}][\Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}])].$$

Immediately,

$$Q_N\{\Phi(K_{\wedge T_N}) \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1)\} + Q_N\{\Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}]) \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1)\} = Q_N\{\Phi(K_{\wedge T_N}) - \Phi(K_{\wedge T_N}[\epsilon \Psi \mathcal{E}^{-1}])\}.$$

For simplicity we denote by $I_1$ (resp. $I_2$) the first (resp. second) term at the left hand side of the above equality, and put

$$I_3 = \text{the right hand side with the minus sign.}$$

Then we find that the convergence

$$\epsilon^{-1} \cdot (\Lambda[\epsilon \Psi \mathcal{E}^{-1}](t) - 1) \to \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y), \quad Q_N - a.s. \quad (\epsilon \to 0)$$

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1} I_1 = Q_N\{\Phi(K_{\wedge T_N}) \cdot \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y)\}.$$

Paying attention to the fact that

$$\lim_{\epsilon \downarrow 0} K^*[\epsilon \Psi \mathcal{E}^{-1}, C](t) = 0, \quad Q_N - a.s.,$$

we can show that $\lim_{\epsilon \downarrow 0} \epsilon^{-1} I_2 = 0$, as well.

It remains to treat the third term $I_3$. In order to discuss the convergence of $I_3$ divided by $\epsilon$, we need the following:

**Key Lemma** (cf. Lemma 12, [Dk99a])

$$Q_N \int \int \{\Phi(\mathcal{M}[s, y]K_{\wedge T_N}) - \Phi(K_{\wedge T_N})\} \Lambda[\psi \mathcal{E}^{-1}(ds \otimes dy) = - Q_N \int \int Pr[\Phi] \gamma(s, y) \Psi(s, y) \mathcal{E}^{-1}(t \wedge T_N)dK_{\wedge T_N}(y)ds.$$
On the other hand, for $\varepsilon > 0$ we have

$$Q_N[\Phi(K[\varepsilon] \varphi]) - \Phi(K) / F] = \varepsilon \cdot e^{-\varepsilon A \varphi((\mathcal{T}\infty) \times \mathcal{C} - \mathcal{F})} \Lambda_{\varphi}(ds \otimes dy) + R(\varepsilon, \Phi, \varphi)$$

(18)

where the residue function $R$ satisfies $|R(\varepsilon, \Phi, \varphi)| \leq o(\varepsilon)$. From (18) we get the convergence

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_3 = -Q_N \int \int \mathcal{C}(t) Pr[\Phi] \gamma(s, y) \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge \tau_N} ds. \quad (19)$$

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

§8.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any $y \in C^s$, $R(s, t, y)$ denotes the canonical measure (cf §4) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually, $R$ is a $\sigma$-finite measure such that

$$R(s, t, y; M_F(C)) = r_{s,t}.$$

Here the crucial point is that the total mass $r_{s,t}$ does not depend on $y$. So $r_{s,t}^{-1} dR(s, t, y)$ becomes a probability measure. It is interesting to note that $K_t$ is a sum of independent nonzero clusters with laws $r_{s,t}^{-1} R(s, t, y; dh)$, conditional on $L[s, t]$ (see §5). Furthermore, conditional on $\mathcal{F}_s$, $L[s, t]$ can be regarded as a Poisson point process with intensity $r_{s,t} \gamma(s) K_s$. This is one of the most important points for the computation in terms of clusters growing from the points of $L[s, t_{l+1}]$ in what follows. We define a measure $S$ by the following equation: for $\forall g \in bB([M_F(C)]^{k-l} \to \mathbb{R})$,

$$\int g(\eta_{l+1}, \cdots, \eta_k) S_{s,y} (d\eta_{l+1} \otimes \cdots \otimes d\eta_k) = \int g(h(t_{l+1}), \cdots, h(t_k)) \cdot I{h(t_{l+1}) \neq 0} Q(s, y; dh)$$

where $Q(s, y; dh)$ is a $\sigma$-finite measure on $C(M_F(C))$ (cf. Eq.(7) in §4). $S_{s,y}^*$ is the normalization of $S_{s,y}$, given by $dS_{s,y}^* := r_{s,t}^{-1} dS_{s,y}$. Moreover, we define

$$\Xi(s; E) := \int \cdots \int \varphi(K(t_1), \cdots, K(t_l), \sum_{i=1}^{m} \eta_i(t_{l+1}), \cdots, \sum_{i=1}^{m} \eta_k(t_k)) \times \bigotimes_{i=1}^{m} S_{s,y}^* (d\eta_{l+1}^i \otimes \cdots \otimes d\eta_k^i),$$

where $E = \{y_1, \cdots, y_m\}(\neq \emptyset)$. 
Take the mass \( \varphi \) as \( (\Psi \mathcal{E}^{-1})(s, y) \) at each point \( y \) (cf. §5). For simplicity we set
\[
\Delta[\Phi](M; s, y, K) := \Phi(M[s, y]K_{\wedge \tau_N}) - \Phi(K_{\wedge \tau_N}).
\]
Recall the assumption (A.3). Immediately we can get
\[
Q_N \int \int_{C(\infty)} \Delta[\Phi](M; s, y, K) \lambda_{\Psi \mathcal{E}^{-1}}(ds \otimes dy)
= Q_N \int_{a+}^{b} \int_{C} \Delta[\Phi](M; s, y, K) \lambda_{s}[\Psi \mathcal{E}^{-1}](dy)ds.
\]
\[
= \int_{a+}^{b} ds Q_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](M; s, y, K) \cdot (\Psi \mathcal{E}^{-1})(s, y) \right\}.
\]
In the following calculation, we may take much advantage of those concepts such as i) the Markov property of \( K_t \); ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location \( L[s, t_{l+1}] \). Hence we can proceed with the computation. In fact,
\[
Q_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](M; s, y, K) \cdot (\Psi \mathcal{E}^{-1})(s, y) \right\}
= Q_N \left\{ \mathbb{P} \left[ \sum_{y \in L[s, u]} \mathbb{P} \{ \Delta[\Phi] \cdot \Psi \mathcal{E}^{-1} | \mathcal{F}_s \vee \sigma(L[s, u]) \} | \mathcal{F}_s \right] \right\}
= Q_N \left\{ \mathbb{P} \left[ \sum_{y \in L[s, u]} \{ \Xi(s; L[s, u] \setminus \{y\}) - \Xi(s; L[s, u]) \} \cdot \Psi \mathcal{E}^{-1} | \mathcal{F}_s \right] \right\}.
\]
It is easy to see the following lemma.

**Lemma 4.** The last expression of (20) is equivalent to
\[
Q_N \int_{C} (\Psi \mathcal{E}^{-1})(s, y) \cdot r_{s, t_{l+1}} \gamma(s, y) K_{s \wedge \tau_N}(dy) \left[ \exp \left( -r_{s, t_{l+1}} K_s(C) \right) \cdot \sum_{m=0}^{\infty} \frac{1}{m!} \int \cdots (m) \cdots \int_{[C]^m} \Xi(s; \{y_1, \cdots, y_m\}) - \Xi(s; \{y_1, \cdots, y_m, y\}) \right] \cdot (r_{s, t_{l+1}})^m K_s \otimes \cdots \otimes dy_m.
\]

A simple computation implies that the integral expression in Lemma 4 is also equal to
\[
Q_N \int_{C} (\Psi \mathcal{E}^{-1})(s, y) \gamma(s, y) K_{s \wedge \tau_N}(dy) \cdot \left[ \int \cdots (k - l) \cdots \int_{[M_F(C)]^{k-l}} \mathbb{P} \{ \varphi(K(t_1), \cdots, K(t_k)) - \varphi(K(t_1), K(t_i), K(t_{l+1}) + \eta_{l+1}, \cdots, K(t_k) + \eta_k) \mid \mathcal{F}_s \} \right] \times r_{s, t_{l+1}} \cdot S^*(s, y) - (d\eta_{l+1} \otimes \cdots \otimes d\eta_k).
\]

(21)
While, taking (7), (8) in §4, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

**Lemma 5.** *The following equality holds for all s, y:*

\[
Pr\left[\Phi(s, y) = \int \cdots (k-l) \cdots \int r_{s,t_{1+}} \cdot s^{*}1s,y \epsilon -(d\eta_{l+1} \otimes \cdots \otimes d\eta_{k}) \right].
\]

\[\times \mathbb{P}\{\varphi(K(t_{1}), \cdots, K(t_{l}), K(t_{l+1}) + \eta_{l+1}, \cdots, K(t_{k}) + \eta_{k}) - \varphi(K(t_{1}), \cdots, K(t_{k})) | \mathcal{F}_{s}\}.
\]

Therefore, an application of the above proposition with Lemma 4 implies

\[
-Q_{N} \int_{C(t)} Pr[\Phi](\gamma \cdot \Psi^{-1})(s, y) dK_{s \wedge T_{N}}.ds
\]

\[\int_{\tau}^{t} ds \left\{ Q_{N} \int_{C} (-Pr[\Phi]) \gamma \cdot \Psi^{-1} dK_{s \wedge T_{N}}.ds \right\} = \int_{\tau}^{t} Eq.(21) ds = \int_{\tau}^{t} Eq.(20) ds
\]

\[= Q_{N} \int_{C(t)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda \Psi^{-1}(ds \otimes dy),
\]

which completes the proof.

**§9. Itô-Clark Formula: Proof of Theorem 4**

Since \(P[K_{t}(C)^{2}]\) is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity \(P[F(K)^{2}]\). Therefore we can apply Theorem 1 (§3) for \(F(K)\) to obtain that

\[F(K) = P[F(K)] + \int_{\tau}^{\infty} \int_{C} f(s, y) dM^{K}(s, y), P \text{ - a.s.} (22)\]

holds for some \(f\) in \(L_{C}^{2}(K, P)\). While, it follows from the covariance formula in the theory of stochastic integration that

\[P \left[ \left( \int_{C(\infty)} f(s, y) dM^{K}(s, y) \right) \left( \int_{C(t)} \Psi(s, y) dM^{K}(s, y) \right) \right] \]

\[\Rightarrow P \left[ \int_{\tau}^{t} \int_{C} f(s, y) \Psi(s, y) \gamma(s, y) K_{s}(dy)ds \right] \]

for all \(t > \tau \) and \(\Psi\) in \(bP(C \times F_{t})\). Rewriting the left hand side of Eq.(23) we get

\[P \left[ F(K) \int_{\tau}^{t} \int_{C} \Psi(s, y) dM^{K}(s, y) \right] \]

by employing the predictable representation property (22). Hence we may apply Theorem 3 (§6) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded \((C_{t} \times F_{t})\)-predictable functions. So that, from (23)

\[
P \int_{C(t)} f(s, y) \Psi(s, y) \gamma(s, y) dK_{s}ds = P \int_{C(t)} Pr[F](s, y) \Psi(s, y) \gamma(s, y) dK_{s}ds.
\]
On this account, the general theory of Hilbert spaces shows that
\[
P \int_{\tau}^{t} \int_{C} \{f(s, y) - Pr[F](s, y)\}^2 \gamma(s, y) K_s(dy) ds = 0.
\]
Therefore the uniqueness argument allows us to conclude that \(\int \int_{C(0)} f dM\) is equivalent to \(\int \int_{C(0)} Pr[F] dM\), \(\mathcal{P}\)-a.s. Note that \(Pr[F](s, y)\) become null for \(K_s\)-a.s. \(y\), for any \(s > t\), by its construction, as long as we choose \(t\) largely enough for the support of \(m\) to be contained in \([\tau, t]\). Consequently, the above integral \(\int \int Pr[F] dM\) can be replaced by \(\int \int_{C(\infty)} Pr[F] dM\), which completes the proof. This goes quite similarly as in the proof of Theorem 2.5 in [EP95].

References


