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A Class of Dirac-Type Operators on the Abstract Boson-Fermion Fock Space and Their Strong Anticommutativity

Asao Arai (新井朝雄)
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
e-mail: arai@math.sci.hokudai.ac.jp

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1 Introduction

In a previous paper [4], we introduced a family \( \{ Q_S | S \in C(\mathcal{H},\mathcal{K}) \} \) of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space \( \mathcal{F}(\mathcal{H},\mathcal{K}) \) over the pair \( \langle \mathcal{H},\mathcal{K} \rangle \) of two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), where the index set \( C(\mathcal{H},\mathcal{K}) \) of the family is the set of all densely defined closed linear operators from \( \mathcal{H} \) to \( \mathcal{K} \), and investigated fundamental properties of them. As is shown in [4], this class of Dirac-type operators has a connection with supersymmetric quantum field theory (SQFT) [19]. Namely \( Q_S \) gives an abstract form of free supercharges in some models of SQFT. Interacting models of SQFT can be constructed from perturbations of \( Q_S \) [4]. For related aspects and further developments, see, e.g., [1], [2], [3], [5], [6], [10], [14], [16], [17], [20], [21].

Generally speaking, Dirac-type operators have something to do with a notion of anticommutativity, because they are related to representations of Clifford algebras, and this aspect may be an essential feature of Dirac-type operators (cf. [7], [8], [9], [11], [12]). A proper notion of anticommutativity, i.e., strong anticommutativity, of (unbounded) self-adjoint operators was given in [27] and developed by some authors (e.g., [25], [22], [7], [9], [11], [12]). In a recent paper [15], a theorem on the strong anticommutativity of two Dirac operators \( Q_S \) and \( Q_T \) was established with application to constructing representations on \( \mathcal{F}(\mathcal{H},\mathcal{K}) \) of a supersymmetry algebra arising in a two-dimensional relativistic SQFT.

The aim of this note is to review fundamental aspects of the theory of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space and to present a summary of the results on their strong anticommutativity obtained in [15].
2 Dirac-type operators on the abstract Boson-Fermion Fock space—a brief review

Let $\mathcal{H}$ be a Hilbert space and $\otimes^n\mathcal{H}$ be the $n$-fold tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots; \otimes^0(\mathcal{H}) := \mathbb{C}$). We denote by $S_n$ (resp. $A_n$) the symmetrizer (resp. the anti-symmetrizer) on $\otimes^n\mathcal{H}$ and by $S_n(\otimes^n\mathcal{H})$ (resp. $A_n(\otimes^n\mathcal{H})$) its range, which is called the $n$-fold symmetric (resp. anti-symmetric) tensor product of $\mathcal{H}$. The Boson Fock space $\mathcal{F}_b(\mathcal{H})$ and the Fermion Fock space $\mathcal{F}_f(\mathcal{H})$ over $\mathcal{H}$ are respectively defined by

$$
\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} S_n(\otimes^n\mathcal{H}), \quad \mathcal{F}_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} A_n(\otimes^n\mathcal{H})
$$

(2.1)

(e.g., [23, §II.4], [18, §5.2]). Let $\mathcal{K}$ be a Hilbert space. Then the Boson-Fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with the pair $\langle \mathcal{H}, \mathcal{K} \rangle$ is defined by

$$
\mathcal{F}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K}),
$$

(2.2)

the tensor product Hilbert space of the Boson Fock space over $\mathcal{H}$ and the Fermion Fock space over $\mathcal{K}$. We denote by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$.

We first present the definitions of basics objects in the Boson Fock space and the Fermion Fock space. More detailed descriptions on Fock space objects can be found, e.g., in [23, §II.4, Example 2], [24, §X.7] and [18, §5.2].

For each vector $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H})$ ($\Psi^{(n)} \in S_n(\otimes^n\mathcal{H})$), we use the natural identification of $\Psi^{(n)}$ with $\{0, \cdots, 0, \Psi^{(n)}, 0, \cdots\} \in \mathcal{F}_b(\mathcal{H})$. The same applies to vectors in other infinite direct sums of Hilbert spaces.

For a subset $V$ of a Hilbert space, we denote by $LV$ the subspace algebraically spanned by all the vectors of $V$.

Let $\Omega_b := \{1, 0, 0, \cdots\} \in \mathcal{F}_b(\mathcal{H})$, the boson Fock vacuum in $\mathcal{F}_b(\mathcal{H})$. For a subspace $\mathcal{D}$ of $\mathcal{H}$, we define

$$
\mathcal{F}_{b, \text{fin}}(\mathcal{D}) := \mathcal{L}\{\Omega_b, S_n(f_1 \otimes \cdots \otimes f_n)\} \text{if } f_j \in \mathcal{D}, j = 1, \cdots, n.
$$

(2.3)

If $\mathcal{D}$ is dense, then $\mathcal{F}_{b, \text{fin}}(\mathcal{D})$ is dense in $\mathcal{F}_b(\mathcal{H})$.

For each $f \in \mathcal{H}$, there exists a unique densely defined closed (unbounded) linear operator $a(f)$ on $\mathcal{F}_b(\mathcal{H})$, called boson annihilation operators (its adjoint $a(f)^*$ is called a boson creation operator), such that (i) for all $f \in \mathcal{H}$, $a(f)\Omega_b = 0$, (ii) for all $n \in \mathbb{N}$, $f_j \in \mathcal{H}$, $j = 1, \cdots, n$,

$$
a(f)S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f, f_j)_{\mathcal{H}} S_{n-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_n),
$$

where $\hat{f}_j$ indicates the omission of $f_j$, and (iii) $\mathcal{F}_{b, \text{fin}}(\mathcal{H})$ is a core of $a(f)$. We have

$$
S_n(\otimes^n\mathcal{H}) = \overline{\{(a(f_1)^* \cdots a(f_n)^*\Omega_b | f_j \in \mathcal{H}, j = 1, \cdots, n\}},
$$

(2.4)

where $\overline{\{\}}$ denotes the closure of the set $\{\}$. The set $\{a(f), a(f)^* | f \in \mathcal{H}\}$ satisfies the canonical commutation relations

$$
[a(f), a(g)^*] = (f, g)_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0
$$
for all $f, g \in \mathcal{H}$ on $\mathcal{F}_{b, \mathrm{fin}}(\mathcal{H})$.

A similar consideration can be done in the Fermion Fock space $\mathcal{F}_{f}(\mathcal{K})$. The fermion Fock vacuum $\Omega_{f}$ in $\mathcal{F}_{f}(\mathcal{K})$ is defined by $\Omega_{f} := \{1, 0, 0, \cdots\} \in \mathcal{F}_{b}(\mathcal{K})$. For a subspace $D$ of $\mathcal{K}$, we define

$$\mathcal{F}_{f, \mathrm{fin}}(D) := \mathcal{L}\{\Omega_{f}, A_{n}(u_{1} \otimes \cdots \otimes u_{n})|n \geq 1, u_{j} \in D, j = 1, \cdots, n\}. \quad (2.5)$$

If $D$ is dense, then $\mathcal{F}_{f, \mathrm{fin}}(D)$ is dense in $\mathcal{F}_{f}(\mathcal{K})$.

For each $u \in \mathcal{K}$, there exists a unique bounded linear operator $b(u)$ on $\mathcal{F}_{f}(\mathcal{K})$, called fermion annihilation operators on $\mathcal{F}_{f}(\mathcal{K})$ ($b(u)^{*}$ is called a fermion creation operator), such that (i) for all $u \in \mathcal{K}$, $b(u)\Omega_{b} = 0$, (ii) for all $n \in \mathbb{N}$, $u_{j} \in \mathcal{K}$, $j = 1, \cdots, n$

$$b(u)A_{n}(u_{1} \otimes \cdots \otimes u_{n}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1}(u, u_{j})_{\mathcal{H}} S_{n-1}(u_{1} \otimes \cdots \otimes \hat{u}_{j} \otimes \cdots \otimes u_{n}). \quad (2.6)$$

We have

$$A_{n}(\otimes^{n} \mathcal{K}) = \mathcal{L}\{b(u_{1})^{*} \cdots b(u_{n})^{*}\Omega_{f}|u_{j} \in \mathcal{K}, j = 1, \cdots, n\}. \quad (2.7)$$

The annihilation operators $a(f)$ and $b(u)$ are extended to operators on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ as

$$A(f) := a(f) \otimes I, \quad B(u) := I \otimes b(u), \quad (2.8)$$

where $I$ denotes identity operator.

For a linear operator $A$, we denote by $D(A)$ its domain. Let $S \in C(\mathcal{H}, \mathcal{K})$. Then we define

$$D_{S} := \mathcal{L}\{A(f_{1})^{*} \cdots A(f_{n})^{*}B(u_{1})^{*} \cdots B(u_{p})^{*}\Omega|n, p \geq 0, f_{j} \in D(S),$$

$$j = 1, \cdots, n, \quad u_{k} \in D(S^{*}), k = 1, \cdots, p\},$$

$$= \mathcal{F}_{b,\mathrm{fin}}(D(S)) \otimes_{\text{alg}} \mathcal{F}_{f,\mathrm{fin}}(D(S^{*})), \quad (2.10)$$

where $\otimes_{\text{alg}}$ denotes algebraic tensor product. It follows that $D_{S}$ is dense in $\mathcal{F}$. The following proposition is proved in [4].

**Proposition 2.1** There exists a unique densely defined closed linear operator $d_{S}$ on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with the following properties: (i) $D_{S}$ is a core of $d_{S}$; (ii) for each vector $\Psi \in D_{S}$ of the form

$$\Psi = A(f_{1})^{*} \cdots A(f_{n})^{*}B(u_{1})^{*} \cdots B(u_{p})^{*}\Omega, \quad (2.11)$$
$d_S$ acts as
\[
\begin{align*}
    d_S \Psi &= 0 & \text{for } n = 0, \\
    d_S \Psi &= \sum_{j=1}^{n} A(f_1)^* \cdots A(f_j)^* \cdots A(f_n)^* B(S f_j)^* B(u_1)^* \cdots B(u_p)^* \Omega & \text{for } n \geq 1,
\end{align*}
\]

where $A(f_j)^*$ indicates the omission of $A(f_j)^*$. Moreover the following (a)-(d) hold:
(a) $d_S^2 = 0$.
(b) For each complete orthonormal system (CONS) $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ with $e_n \in D(S^*)$,
\[
d_S \Psi = \sum_{n=1}^{\infty} A(S^* e_n) B(e_n)^* \Psi, \quad \Psi \in D_S,
\]
where the convergence is taken in the strong topology of $\mathcal{F}(\mathcal{H}, \mathcal{K})$.
(c) For each CONS $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ with $\phi_n \in D(S)$, we have
\[
(\Phi, d_S \Psi)_{\mathcal{F}(\mathcal{H}, \mathcal{K})} = \lim_{N \to \infty} \left( \Phi, \sum_{n=1}^{N} A(\phi_n) B(S \phi_n)^* \Psi \right)_{\mathcal{F}(\mathcal{H}, \mathcal{K})}, \quad \Phi, \Psi \in D_S.
\]
(d) $D_S \subset D(d_S^*)$ and
\[
d_S^* \Psi = \sum_{k=1}^{p} (-1)^{k-1} A(S^* u_k)^* A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_k)^* \cdots B(u_p)^* \Omega
\]
for vectors $\Psi$ of the form (2.11) with $p \geq 1$. In the case $p = 0$, we have $d_S^* \Psi = 0$.

A Dirac-type operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by
\[
Q_S = d_S + d_S^*
\]
with $D(Q_S) = D(d_S) \cap D(d_S^*)$.

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Then there is a unique self-adjoint operator $A_n$ on $\otimes^n \mathcal{X}$ such that $\otimes_n^{|A|} D(A)$ is a core of $D(A_n)$ and, for all $f_j \in D(A)$, $j = 1, \cdots, n$, $A_n(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^{n} f_1 \otimes \cdots \otimes f_{j-1} \otimes A f_j \otimes f_{j+1} \otimes \cdots \otimes f_n$ ([23, §VIII.10, Corollary]). Putting $A_0 = 0$, one can define a self-adjoint operator
\[
d\Gamma(A) := \bigoplus_{n=0}^{\infty} A_n
\]
on $\bigoplus_{n=0}^{\infty} \otimes^n \mathcal{X}$, called the second quantization of $A$ ([23, §VIII.10, Example 2], [18, §5.2]). It is easy to show that $d\Gamma(A)$ is reduced by $\mathcal{F}_\#(\mathcal{X})$ ($\# = b, f$). We denote the reduced part of $d\Gamma(A)$ to $\mathcal{F}_\#(\mathcal{X})$ by $d\Gamma_\#(A)$. We put
\[
N_\# := d\Gamma_\#(I),
\]
called the number operator on $\mathcal{F}_\#(\mathcal{X})$.

Let
\[ \Gamma_\# = (-1)^{I \otimes N_\#}. \]  
(2.15)

We introduce an operator
\[ \Delta_S := d\Gamma_h(S^*S) \otimes I + I \otimes d\Gamma_f(SS^*) \]  
(2.16)
acting in $\mathcal{F}(\mathcal{H},\mathcal{K})$, which is nonegative and self-adjoint (cf. [23, §VIII.10, Corollary]). For a linear operator $A$ on a Hilbert space, we set
\[ C^\infty(A) := \cap_{n=1}^\infty D(A^n). \]

Let
\[ D_S^\infty \]  
(2.17)
\[ = \mathcal{L}\left\{ A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega \left| n, p \geq 0, f_j \in C^\infty(S^*S), \right. \right. \\
j = 1, \cdots, n, u_k \in C^\infty(SS^*), k = 1, \cdots, p \right\}. \]

**Theorem 2.2** [4]

(i) The operator $Q_S$ is self-adjoint, and essentially self-adjoint on every core of $\Delta_S$. In particular, $Q_S$ is essentially self-adjoint on $D_S^\infty$.

(ii) The operator $\Gamma_\#$ leaves $D(Q_S)$ invariant and
\[ \Gamma_\# Q_S + Q_S \Gamma_\# = 0 \]
on $D(Q_S)$.

(iii) The following operator equations hold:
\[ \Delta_S = Q_S^2 = d_S^* d_S + d_S d_S^*. \]

**Remark 2.1** The operators $d_S$ and $d_S^*$ leave $D_S^\infty$ invariant and so does $Q_S$.

Because of part (iii) of Theorem 2.2, we call the operator $\Delta_S$ the Laplacian associated with the Dirac-type operator $Q_S$. 


3 Strong anticommutativity of the Dirac-type operators

Let $A$ and $B$ be self-adjoint operators on a Hilbert space. We say that $A$ and $B$ *strongly commute* if their spectral measures commute. On the other hand, $A$ and $B$ are said to *strongly anticommute* if $e^{itB}A \subseteq Ae^{-itB}$ for all $t \in \mathbb{R}$ ([27], [22]). It turns out that this definition is symmetric in $A$ and $B$ [22].

For various Dirac-type operators, the notion of strong anticommutativity plays an important role ([7], [8], [10], [11]).

For each $S \in C(\mathcal{H}, \mathcal{K})$, the operator

$$L_S := \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

acting in $\mathcal{H} \oplus \mathcal{K}$ is self-adjoint. This operator is an abstract Dirac operator on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ [26, Chapter 5].

The strong anticommutativity of $Q_S$ and $Q_T$ ($S, T \in C(\mathcal{H}, \mathcal{K})$) is characterized as follows.

**Theorem 3.1** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Then $Q_S$ and $Q_T$ strongly anticommute if and only if $L_S$ and $L_T$ strongly anticommute. In that case, $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $Q_{S \pm T} = Q_S \pm Q_T$.

This theorem is one of the main results of the paper [15], which establishes a beautiful correspondence between the strong anticommutativity of $L_S$ and $L_T$ and that of $Q_S$ and $Q_T$.

To prove Theorem 3.1, we need some fundamental facts in the theory of strongly anticommuting self-adjoint operators [27, 22] as well as its applications, together with the following lemma. For the details, see [15].

**Lemma 3.2** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Suppose that $L_S$ and $L_T$ strongly anticommute. Then the following (i)-(v) hold:

(i) $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) $|S|$ and $|T|$ strongly commute.

(iii) $|S^*|$ and $|T^*|$ strongly commute.

(iv) $D(S^*S) \cap D(T^*T) \subset D(T^*S) \cap D(S^*T)$ and, for all $f \in D(S^*S) \cap D(T^*T)$,

$$ (T^*S + S^*T)f = 0. $$

(v) $D(SS^*) \cap D(TT^*) \subset D(TS^*) \cap D(ST^*)$ and, for all $u \in D(SS^*) \cap D(TT^*)$,

$$ (TS^* + ST^*)u = 0. $$

The authors of [27] and [22] call this notion simply anticommutativity, but, to be definite, we call it *strong anticommutativity*.\(^1\)
In terms of $S$ and $T$, a necessary and sufficient condition for $L_S$ and $L_T$ to strongly anticommute is given as follows.

**Proposition 3.3** Let $S, T \in C(H, K)$. Then $L_S$ and $L_T$ strongly anticommute if and only if the following (i) and (ii) hold:

(i) $S \pm T \in C(H, K)$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) For all $f, g \in D(S) \cap D(T)$ and $u, v \in D(S^*) \cap D(T^*)$,

$$(Sf, Tg) + (Tf, Sg) = 0, \quad (S^*u, T^*v) + (T^*u, S^*v) = 0.$$ 

4 Application to constructing representations of a supersymmetry algebra

We consider Fock space representations of the algebra $A_{\text{SUSY}}$ generated by four elements $Q_1, Q_2, H, P$ with defining relations

$$Q_1^2 = H + P, \quad Q_2^2 = H - P, \quad Q_1 Q_2 + Q_2 Q_1 = 0. \quad (4.1)$$

This algebra is called a supersymmetry algebra, which arises in a relativistic SQFT in the two-dimensional space-time ([19], [13]). The elements $H, P$ and $Q_j$ ($j = 1, 2$) are called the Hamiltonian, the momentum operator and the supercharge, respectively.

We recall a definition from [13]. Let $\mathcal{F}$ be a Hilbert space, $D$ a dense subspace of $\mathcal{F}$, and $H, P, Q_1, Q_2$ be linear operators on $\mathcal{F}$. We say that $\{\mathcal{F}, D, H, P, Q_1, Q_2\}$ is a symmetric representation of $A_{\text{SUSY}}$ if $H, P, Q_1$ and $Q_2$ are symmetric and leave $D$ invariant satisfying (4.1) on $D$. A symmetric representation $\{\mathcal{F}, D, H, P, Q_1, Q_2\}$ of $A_{\text{SUSY}}$ is said to be integrable if (i) $H, P, Q_1$ and $Q_2$ are essentially self-adjoint (denote their closures by $\bar{H}, \bar{P}, \bar{Q}_1$ and $\bar{Q}_2$, respectively); (ii) $\{\bar{H}, \bar{P}, \bar{Q}_1\}$ and $\{\bar{H}, \bar{P}, \bar{Q}_2\}$ are families of strongly commuting self-adjoint operators, respectively; (iii) $\bar{H}$ and $\bar{P}$ satisfy the relativistic spectral condition

$$\pm \bar{P} \leq \bar{H}. \quad (4.2)$$

Suppose that $L_S$ and $L_T$ strongly anticommute. Then, by Lemma 3.3(ii) and (iii), $S^*S$ and $T^*T$ strongly commute, and $SS^*$ and $TT^*$ strongly commute. Hence $S^*S + T^*T$ and $SS^* + TT^*$ are nonnegative, self-adjoint, and $S^*S - T^*T$ and $SS^* - TT^*$ are essentially self-adjoint. Therefore we can define self-adjoint operators

$$H_{S,T} := \frac{1}{2} \{d\Gamma_b((S^*S + T^*T) \otimes I + I \otimes d\Gamma_t(S^*S^* + TT^*))\}, \quad (4.3)$$

$$P_{S,T} := \frac{1}{2} \{d\Gamma_b((S^*S - T^*T) \otimes I + I \otimes d\Gamma_t(SS^* - TT^*))\}^{-}. \quad (4.4)$$

where for a closable linear operator $A, \bar{A}$ (or $A^-$) denotes its closure. Note that $H_{S,T}$ is nonnegative, but, $P_{S,T}$ may be neither bounded below nor bounded above.

For a self-adjoint operator $A$, we denote by $E_A$ its spectral measure. Let

$$D_{S,T} := L\{E_{|Q_2|}(a,b)|E_{|Q_2|}([c,d])\Psi | \Psi \in \mathcal{F}(H, K), 0 \leq a < b < \infty, 0 \leq c < d < \infty\}. \quad (4.5)$$

We can prove the following theorem (for the proof, see [15]).
Theorem 4.1 Let $S, T \in C(H, K)$ and suppose that $L_S$ and $L_T$ strongly anticommute. Then $\{\mathcal{F}(H, K), D_{ST}, H_{ST}, P_{ST}, Q_S, Q_T\}$ is an integrable representation of $A_{\text{SUSSY}}$.

We give only one basic example from SQFT (for other examples, see [19], [4]).

Example Let $H = K = L^2(\mathbb{R})$ and $\mathbb{R} \ni p \rightarrow \omega(p)$ be a nonnegative function on $\mathbb{R}$ which is Borel measurable, almost everywhere (a.e.) finite with respect to the Lebesgue measure on $\mathbb{R}$, and satisfies

$$|p| \leq \omega(p), \quad \text{a.e. } p \in \mathbb{R}. $$

Let

$$ \nu(p) = \sqrt{\lambda p + \omega(p)} $$

with $\lambda \in [0, 1]$ (a constant parameter) and $\theta(p)$ be an a.e. finite real-valued Borel measurable function on $\mathbb{R}$. Define the operators $S$ and $T$ on $L^2(\mathbb{R})$ to be the multiplication operators by the functions

$$ S(p) := i\nu(p)e^{i\theta(p)}, \quad T(p) := \nu(-p)e^{i\theta(p)}, $$

respectively. Then it is easy to see that $S$ and $T$ satisfy the conditions (i) and (ii) in Proposition 3.3 with $D(T) = D(S) = D(S^*) = D(T^*)$ and

$$ S^*S = SS^* = \lambda p + \omega, \quad T^*T = TT^* = -\lambda p + \omega, $$

$$ S^*T = TS^* = -i\sqrt{\omega^2 - \lambda^2 p^2}, \quad T^*S = ST^* = i\sqrt{\omega^2 - \lambda^2 p^2}. $$

Hence, by Proposition 3.3, $L_S$ and $L_T$ strongly anticommute. Therefore, by Theorem 4.1, $\{\mathcal{F}(L^2(\mathbb{R}), L^2(\mathbb{R})), D_{ST}, H_{ST}, P_{ST}, Q_S, Q_T\}$ with these $S$ and $T$ is an integrable representation of $A_{\text{SUSSY}}$. We have

$$ H_{ST} = d\Gamma_b(\omega) \otimes I + I \otimes d\Gamma_f(\omega), $$

$$ P_{ST} = \lambda\{d\Gamma_b(p) \otimes I + I \otimes d\Gamma_f(p)\}. $$

Note that $H_{ST}$ and $P_{ST}$ are independent of $\theta$.

If $\omega(p) = \sqrt{p^2 + m^2}$ with a constant $m \geq 0$, $\lambda = 1$ and $\theta = 0$, then $H_{ST}$ and $P_{ST}$ are respectively the Hamiltonian and the momentum operator of a free relativistic SQFT in the two-dimensional space-time, called the $N = 1$ Wess-Zumino model (cf. [19]).

References


