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(Development of Infinite-Dimensional Noncommutative Analysis)

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A Class of Dirac-Type Operators on the Abstract Boson-Fermion Fock Space and Their Strong Anticommutativity

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1 Introduction

In a previous paper [4], we introduced a family \( \{ Q_S \mid S \in C(\mathcal{H}, \mathcal{K}) \} \) of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) over the pair \( \langle \mathcal{H}, \mathcal{K} \rangle \) of two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), where the index set \( C(\mathcal{H}, \mathcal{K}) \) of the family is the set of all densely defined closed linear operators from \( \mathcal{H} \) to \( \mathcal{K} \), and investigated fundamental properties of them. As is shown in [4], this class of Dirac-type operators has a connection with supersymmetric quantum field theory (SQFT) [19]. Namely \( Q_S \) gives an abstract form of free supercharges in some models of SQFT. Interacting models of SQFT can be constructed from perturbations of \( Q_S \) [4]. For related aspects and further developments, see, e.g., [1], [2], [3], [5], [6], [10], [14], [16], [17], [20], [21].

Generally speaking, Dirac-type operators have something to do with a notion of anticommutativity, because they are related to representations of Clifford algebras, and this aspect may be an essential feature of Dirac-type operators (cf. [7], [8], [9], [11], [12]). A proper notion of anticommutativity, i.e., \textit{strong anticommutativity}, of (unbounded) self-adjoint operators was given in [27] and developed by some authors (e.g., [25], [22], [7], [9], [11], [12]). In a recent paper [15], a theorem on the strong anticommutativity of two Dirac operators \( Q_S \) and \( Q_T \) was established with application to constructing representations on \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) of a supersymmetry algebra arising in a two-dimensional relativistic SQFT.

The aim of this note is to review fundamental aspects of the theory of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space and to present a summary of the results on their strong anticommutativity obtained in [15].
2 Dirac-type operators on the abstract Boson-Fermion Fock space—a brief review

Let $\mathcal{H}$ be a Hilbert space and $\otimes^n\mathcal{H}$ be the $n$-fold tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$). We denote by $S_n$ (resp. $A_n$) the symmetrizer (resp. the anti-symmetrizer) on $\otimes^n\mathcal{H}$ and by $S_n(\otimes^n\mathcal{H})$ (resp. $A_n(\otimes^n\mathcal{H})$) its range, which is called the $n$-fold symmetric (resp. anti-symmetric) tensor product of $\mathcal{H}$. The Boson Fock space $\mathcal{F}_b(\mathcal{H})$ and the Fermion Fock space $\mathcal{F}_f(\mathcal{H})$ over $\mathcal{H}$ are respectively defined by

$$\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} S_n(\otimes^n\mathcal{H}), \quad \mathcal{F}_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} A_n(\otimes^n\mathcal{H})$$

(e.g., [23, §II.4], [18, §5.2]). Let $\mathcal{K}$ be a Hilbert space. Then the Boson-Fermion Fock space $\mathcal{F}(\mathcal{H},\mathcal{K})$ associated with the pair $\langle \mathcal{H},\mathcal{K} \rangle$ is defined by

$$\mathcal{F}(\mathcal{H},\mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K})$$

the tensor product Hilbert space of the Boson Fock space over $\mathcal{H}$ and the Fermion Fock space over $\mathcal{K}$. We denote by $\mathcal{C}(\mathcal{H},\mathcal{K})$ the set of densely defined, closed linear operators from $\mathcal{H}$ to $\mathcal{K}$.

We first present the definitions of basics objects in the Boson Fock space and the Fermion Fock space. More detailed descriptions on Fock space objects can be found, e.g., in [23, §II.4, Example 2], [24, §X.7] and [18, §5.2].

For each vector $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H})$ ($\Psi^{(n)} \in S_n(\otimes^n\mathcal{H})$), we use the natural identification of $\Psi^{(n)}$ with $\{0,\cdots,0,\Psi^{(n)},0,\cdots\} \in \mathcal{F}_b(\mathcal{H})$. The same applies to vectors in other infinite direct sums of Hilbert spaces.

For a subset $V$ of a Hilbert space, we denote by $\mathcal{L}V$ the subspace algebraically spanned by all the vectors of $V$.

Let $\Omega_b := \{1,0,0,\cdots\} \in \mathcal{F}_b(\mathcal{H})$, the boson Fock vacuum in $\mathcal{F}_b(\mathcal{H})$. For a subspace $\mathcal{D}$ of $\mathcal{H}$, we define

$$\mathcal{F}_{b,\text{fin}}(\mathcal{D}) := \mathcal{L}\{\Omega_b, S_n(f_1 \otimes \cdots \otimes f_n)|n \in \mathbb{N}, f_j \in \mathcal{D}, j=1,\cdots,n\}.$$  

(2.3)

If $\mathcal{D}$ is dense, then $\mathcal{F}_{b,\text{fin}}(\mathcal{D})$ is dense in $\mathcal{F}_b(\mathcal{H})$.

For each $f \in \mathcal{H}$, there exists a unique densely defined closed (unbounded) linear operator $a(f)$ on $\mathcal{F}_b(\mathcal{H})$, called boson annihilation operators (its adjoint $a(f)^* \equiv$ called a boson creation operator), such that (i) for all $f \in \mathcal{H}$, $a(f)\Omega_b = 0$, (ii) for all $n \in \mathbb{N}, f_j \in \mathcal{H}$, $j = 1,\cdots,n$,

$$a(f)S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f,f_j)_{\mathcal{H}} S_{n-1}(f_1 \otimes \cdots \hat{f}_j \otimes \cdots \otimes f_n),$$

where $\hat{f}_j$ indicates the omission of $f_j$, and (iii) $\mathcal{F}_{b,\text{fin}}(\mathcal{H})$ is a core of $a(f)$. We have

$$S_n(\otimes^n\mathcal{H}) = \overline{\{a(f_1)^* \cdots a(f_n)^*\Omega_b|f_j \in \mathcal{H}, j = 1,\cdots,n\}},$$

(2.4)

where $\overline{\{\cdot\}}$ denotes the closure of the set $\{\cdot\}$. The set $\{a(f), a(f)^*|f \in \mathcal{H}\}$ satisfies the canonical commutation relations

$$[a(f),a(g)^*] = (f,g)_\mathcal{H}, \quad [a(f),a(g)] = 0, \quad [a(f)^*,a(g)^*] = 0$$
for all \( f, g \in \mathcal{H} \) on \( \mathcal{F}_{\text{b,fin}}(\mathcal{H}) \).

A similar consideration can be done in the Fermion Fock space \( \mathcal{F}_f(\mathcal{K}) \). The fermion Fock vacuum \( \Omega_f \) in \( \mathcal{F}_f(\mathcal{K}) \) is defined by \( \Omega_f := \{1,0,0,\cdots\} \in \mathcal{F}_b(\mathcal{K}) \). For a subspace \( D \) of \( \mathcal{K} \), we define
\[
\mathcal{F}_{\text{f,fin}}(D) := \mathcal{L}\{\Omega_f, A_n(u_1 \otimes \cdots \otimes u_n) | n \geq 1, u_j \in D, j = 1, \cdots, n\}.
\]

(2.5)

If \( D \) is dense, then \( \mathcal{F}_{\text{f,fin}}(D) \) is dense in \( \mathcal{F}_f(\mathcal{K}) \).

For each \( u \in \mathcal{K} \), there exists a unique bounded linear operator \( b(u) \) on \( \mathcal{F}_f(\mathcal{K}) \), called fermion annihilation operators on \( \mathcal{F}_f(\mathcal{K}) \), such that (i) for all \( u \in \mathcal{K}, b(u)\Omega_f = 0, \) (ii) for all \( n \in \mathbb{N}, u_j \in \mathcal{K}, j = 1, \cdots, n \)
\[
b(u)A_n(u_1 \otimes \cdots \otimes u_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1} (u, u_j)_\mathcal{H} S_{n-1}(u_1 \otimes \cdots \otimes \hat{u}_j \otimes \cdots \otimes u_n).
\]

(2.6)

We have
\[
A_n(\otimes^n \mathcal{K}) = \mathcal{L}\{b(u_1)^* \cdots b(u_n)^* \Omega_f | u_j \in \mathcal{K}, j = 1, \cdots, n\}.
\]

The set \( \{b(u), b(u)^* | u \in \mathcal{K}\} \) satisfies the canonical anti-commutation relations
\[
\{b(u), b(v)^*\} = (u, v)_\mathcal{K}, \quad \{b(u), b(v)\} = 0, \quad \{b(u)^*, b(v)^*\} = 0
\]
for all \( u, v \in \mathcal{K} \), where \( \{A, B\} := AB + BA \).

The Fock vacuum in the Boson-Fermion Fock space \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) is defined by
\[
\Omega := \Omega_b \otimes \Omega_f.
\]

(2.7)

The annihilation operators \( a(f) \) and \( b(u) \) are extended to operators on \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) as
\[
A(f) := a(f)^* \otimes I, \quad B(u) := I \otimes b(u),
\]
where \( I \) denotes identity operator.

For a linear operator \( A \), we denote by \( D(A) \) its domain. Let \( S \in C(\mathcal{H}, \mathcal{K}) \). Then we define
\[
\mathcal{D}_S := \mathcal{L}\{A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega | n, p \geq 0, f_j \in D(S),
\]
\[
\quad \quad \quad \quad  j = 1, \cdots, n, u_k \in D(S^*), k = 1, \cdots, p\} = \mathcal{F}_{b,\text{fin}}(D(S)) \otimes_{\text{alg}} \mathcal{F}_{\text{f,fin}}(D(S^*)),
\]

(2.10)

where \( \otimes_{\text{alg}} \) denotes algebraic tensor product. It follows that \( \mathcal{D}_S \) is dense in \( \mathcal{F} \). The following proposition is proved in [4].

**Proposition 2.1** There exists a unique densely defined closed linear operator \( d_S \) on \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) with the following properties: (i) \( \mathcal{D}_S \) is a core of \( d_S \); (ii) for each vector \( \Psi \in \mathcal{D}_S \) of the form
\[
\Psi = A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega,
\]

(2.11)
$d_S$ acts as
\[
    d_S \Psi = \hat{0} \quad \text{for } n = 0,
\]
\[
    d_S \Psi = \sum_{j=1}^{n} A(f_1)^* \cdots A(f_j)^* A(f_n)^* B(S f_j)^* B(u_1)^* \cdots B(u_p)^* \Omega \quad \text{for } n \geq 1,
\]
where $A(f_j)^*$ indicates the omission of $A(f_j)^*$. Moreover the following (a)-(d) hold:

(a) $d_S^2 = 0$.

(b) For each complete orthonormal system (CONS) \(\{e_n\}_{n=1}^\infty\) of \(\mathcal{K}\) with \(e_n \in D(S^*)\),
\[
    d_S \Psi = \sum_{n=1}^\infty A(S^* e_n) B(e_n)^* \Psi, \quad \Psi \in D_S,
\]
where the convergence is taken in the strong topology of \(\mathcal{F}(\mathcal{H}, \mathcal{K})\).

(c) For each CONS \(\{\phi_n\}_{n=1}^\infty\) of \(\mathcal{H}\) with \(\phi_n \in D(S)\), we have
\[
    (\Phi, d_S \Psi)_{\mathcal{F}(\mathcal{H}, \mathcal{K})} = \lim_{N \to \infty} \left( \Phi, \sum_{n=1}^{N} A(\phi_n) B(S \phi_n)^* \Psi \right)_{\mathcal{F}(\mathcal{H}, \mathcal{K})}, \quad \Phi, \Psi \in D_S.
\]

(d) \(D_S \subset D(d_S^*)\) and
\[
    d_S^* \Psi = \sum_{k=1}^{p} (-1)^{k-1} A(S^* u_k)^* A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_k)^* \cdots B(u_p)^* \Omega
\]
for vectors \(\Psi\) of the form (2.11) with \(p \geq 1\). In the case \(p = 0\), we have \(d_S^2 \Psi = 0\).

A Dirac-type operator on \(\mathcal{F}(\mathcal{H}, \mathcal{K})\) is defined by
\[
    Q_S = d_S + d_S^* \quad \text{(2.12)}
\]
with \(D(Q_S) = D(d_S) \cap D(d_S^*)\).

Let \(A\) be a self-adjoint operator on a Hilbert space \(\mathcal{X}\). Then there is a unique self-adjoint operator \(A_n\) on \(\otimes^n \mathcal{X}\) such that \(\otimes_n^\text{alg} D(A)\) is a core of \(D(A_n)\) and, for all \(f_j \in D(A)\), \(j = 1, \ldots, n\), \(A_n(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^{n} f_1 \otimes \cdots \otimes f_{j-1} \otimes A f_j \otimes f_{j+1} \otimes \cdots \otimes f_n\) ([23, §VIII.10, Corollary]). Putting \(A_0 = 0\), one can define a self-adjoint operator
\[
    d\Gamma(A) := \bigoplus_{n=0}^{\infty} A_n \quad \text{(2.13)}
\]
on \(\bigoplus_{n=0}^{\infty} \otimes^n \mathcal{X}\), called the second quantization of \(A\) ([23, §VIII.10, Example 2], [18, §5.2]). It is easy to show that \(d\Gamma(A)\) is reduced by \(\mathcal{F}_\#(\mathcal{X})\) (\(\# = b, f\)). We denote the reduced part of \(d\Gamma(A)\) to \(\mathcal{F}_\#(\mathcal{X})\) by \(d\Gamma_\#(A)\). We put
\[
    N_\# := d\Gamma_\#(I), \quad \text{(2.14)}
\]
called the number operator on $\mathcal{F}_\#(\mathcal{X})$.

Let
\[ \Gamma_\# = (-1)^{I \otimes N_\#}. \]  
\[ \text{(2.15)} \]

We introduce an operator
\[ \Delta_S := d \Gamma_h(S^*S) \otimes I + I \otimes d \Gamma_f(SS^*) \]  
acting in $\mathcal{F}(\mathcal{H}, \mathcal{K})$, which is nonegative and self-adjoint (cf. [23, §VIII.10, Corollary]). For a linear operator $A$ on a Hilbert space, we set
\[ C^\infty(A) := \cap_{n=1}^\infty D(A^n). \]

Let
\[ \mathcal{D}_S^\infty = \mathcal{L} \left\{ A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega \bigg| n, p \geq 0, \ f_j \in C^\infty(S^*S), \ j = 1, \cdots, n, \ u_k \in C^\infty(SS^*), \ k = 1, \cdots, p \right\}. \]  
\[ \text{(2.17)} \]

**Theorem 2.2** [4]

(i) The operator $Q_S$ is self-adjoint, and essentially self-adjoint on every core of $\Delta_S$. In particular, $Q_S$ is essentially self-adjoint on $\mathcal{D}_S^\infty$.

(ii) The operator $\Gamma_\#$ leaves $D(Q_S)$ invariant and
\[ \Gamma_\# Q_S + Q_S \Gamma_\# = 0 \]
on $D(Q_S)$.

(iii) The following operator equations hold:
\[ \Delta_S = Q_S^2 = d_S^* d_S + d_S d_S^*. \]

**Remark 2.1** The operators $d_S$ and $d_S^*$ leave $\mathcal{D}_S^\infty$ invariant and so does $Q_S$.

Because of part (iii) of Theorem 2.2, we call the operator $\Delta_S$ the Laplacian associated with the Dirac-type operator $Q_S$. 


3 Strong anticommutativity of the Dirac-type operators

Let $A$ and $B$ be self-adjoint operators on a Hilbert space. We say that $A$ and $B$ strongly commute if their spectral measures commute. On the other hand, $A$ and $B$ are said to strongly anticommute if $e^{itB}A \subset Ae^{-itB}$ for all $t \in \mathbb{R}$ ([27], [22]). It turns out that this definition is symmetric in $A$ and $B$ [22].

For various Dirac-type operators, the notion of strong anticommutativity plays an important role ([7], [8], [10], [11]).

For each $S \in C(\mathcal{H}, \mathcal{K})$, the operator

$$L_S := \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

acting in $\mathcal{H} \oplus \mathcal{K}$ is self-adjoint. This operator is an abstract Dirac operator on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ [26, Chapter 5].

The strong anticommutativity of $Q_S$ and $Q_T$ ($S, T \in C(\mathcal{H}, \mathcal{K})$) is characterized as follows.

**Theorem 3.1** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Then $Q_S$ and $Q_T$ strongly anticommute if and only if $L_S$ and $L_T$ strongly anticommute. In that case, $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $Q_{S \pm T} = Q_S \pm Q_T$.

This theorem is one of the main results of the paper [15], which establishes a beautiful correspondence between the strong anticommutativity of $L_S$ and $L_T$ and that of $Q_S$ and $Q_T$.

To prove Theorem 3.1, we need some fundamental facts in the theory of strongly anticommuting self-adjoint operators [27, 22] as well as its applications, together with the following lemma. For the details, see [15].

**Lemma 3.2** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Suppose that $L_S$ and $L_T$ strongly anticommute. Then the following (i)-(v) hold:

(i) $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) $|S|$ and $|T|$ strongly commute.

(iii) $|S^*|$ and $|T^*|$ strongly commute.

(iv) $D(S^*S) \cap D(T^*T) \subset D(T^*S) \cap D(S^*T)$ and, for all $f \in D(S^*S) \cap D(T^*T)$,

$$(T^*S + S^*T)f = 0.$$ 

(v) $D(SS^*) \cap D(TT^*) \subset D(TS^*) \cap D(ST^*)$ and, for all $u \in D(SS^*) \cap D(TT^*)$,

$$(TS^* + ST^*)u = 0.$$ 

1The authors of [27] and [22] call this notion simply anticommutativity, but, to be definite, we call it strong anticommutativity.
In terms of $S$ and $T$, a necessary and sufficient condition for $L_S$ and $L_T$ to strongly anticommute is given as follows.

**Proposition 3.3** Let $S, T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then $L_S$ and $L_T$ strongly anticommute if and only if the following (i) and (ii) hold:

(i) $S \pm T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) For all $f, g \in D(S) \cap D(T)$ and $u, v \in D(S^*) \cap D(T^*)$,

$$(Sf, Tg) + (Tf, Sg) = 0, \quad (S^*u, T^*v) + (T^*u, S^*v) = 0.$$

4 Application to constructing representations of a supersymmetry algebra

We consider Fock space representations of the algebra $A_{\text{SUSSY}}$ generated by four elements $Q_1, Q_2, H, P$ with defining relations

$$Q_1^2 = H + P, \quad Q_2^2 = H - P, \quad Q_1 Q_2 + Q_2 Q_1 = 0. \quad (4.1)$$

This algebra is called a supersymmetry algebra, which arises in a relativistic SQFT in the two-dimensional space-time ([19], [13]). The elements $H, P$ and $Q_j$ ($j = 1, 2$) are called the Hamiltonian, the momentum operator and the supercharge, respectively.

We recall a definition from [13]. Let $\mathcal{F}$ be a Hilbert space, $\mathcal{D}$ a dense subspace of $\mathcal{F}$, and $H, P, Q_1, Q_2$ be linear operators on $\mathcal{F}$. We say that $\{\mathcal{F}, \mathcal{D}, H, P, Q_1, Q_2\}$ is a symmetric representation of $A_{\text{SUSSY}}$ if $H, P, Q_1$ and $Q_2$ are symmetric and leave $\mathcal{D}$ invariant satisfying (4.1) on $\mathcal{D}$. A symmetric representation $\{\mathcal{F}, \mathcal{D}, H, P, Q_1, Q_2\}$ of $A_{\text{SUSSY}}$ is said to be integrable if (i) $H, P, Q_1$ and $Q_2$ are essentially self-adjoint (denote their closures by $\overline{H}, \overline{P}, \overline{Q}_1$ and $\overline{Q}_2$, respectively); (ii) $\{\overline{H}, \overline{P}, \overline{Q}_1\}$ and $\{\overline{H}, \overline{P}, \overline{Q}_2\}$ are families of strongly commuting self-adjoint operators, respectively; (iii) $\overline{H}$ and $\overline{P}$ satisfy the relativistic spectral condition

$$\pm \overline{P} \leq \overline{H}. \quad (4.2)$$

Suppose that $L_S$ and $L_T$ strongly anticommute. Then, by Lemma 3.3(ii) and (iii), $S^*S$ and $T^*T$ strongly commute, and $SS^*$ and $TT^*$ strongly commute. Hence $S^*S + T^*T$ and $SS^* + TT^*$ are nonnegative, self-adjoint, and $S^*S - T^*T$ and $SS^* - TT^*$ are essentially self-adjoint. Therefore we can define self-adjoint operators

$$H_{S,T} := \frac{1}{2} \{d\Gamma_b(S^*S + T^*T) \otimes I + I \otimes d\Gamma_T SS^* + TT^*)\}, \quad (4.3)$$

$$P_{S,T} := \frac{1}{2} \{d\Gamma_b(S^*S - T^*T) \otimes I + I \otimes d\Gamma_T (SS^* - TT^*)\} \quad (4.4)$$

where for a closable linear operator $A$, $\overline{A}$ (or $A^-$) denotes its closure. Note that $H_{S,T}$ is nonnegative, but $P_{S,T}$ may be neither bounded below nor bounded above.

For a self-adjoint operator $A$, we denote by $E_A$ its spectral measure. Let

$$D_{S,T} := \mathcal{L}(E_{[a,b]}(\overline{[a,b]}))E_{[c,d]}(\overline{[c,d]})\Psi | \Psi \in \mathcal{F}(\mathcal{H}, \mathcal{K}), 0 \leq a < b < \infty, 0 \leq c < d < \infty \quad (4.5)$$

We can prove the following theorem (for the proof, see [15]).
Theorem 4.1 Let $S, T \in C(\mathcal{H}, \mathcal{K})$ and suppose that $L_S$ and $L_T$ strongly anticommute. Then $\{F(\mathcal{H}, \mathcal{K}), D_{S,T}, H_{S,T}, P_{S,T}, Q_S, QT\}$ is an integrable representation of $A_{\text{SUSSY}}$.

We give only one basic example from SQFT (for other examples, see [19], [4]).

Example Let $\mathcal{H} = \mathcal{K} = L^2(\mathbb{R})$ and $\mathbb{R} \ni p \mapsto \omega(p)$ be a nonnegative function on $\mathbb{R}$ which is Borel measurable, almost everywhere (a.e.) finite with respect to the Lebesgue measure on $\mathbb{R}$, and satisfies

$$|p| \leq \omega(p), \quad \text{a.e.} p \in \mathbb{R}.$$ 

Let

$$\nu(p) = \sqrt{\lambda p + \omega(p)}$$

with $\lambda \in [0, 1]$ (a constant parameter) and $\theta(p)$ be an a.e. finite real-valued Borel measurable function on $\mathbb{R}$. Define the operators $S$ and $T$ on $L^2(\mathbb{R})$ to be the multiplication operators by the functions

$$S(p) := i\nu(p)e^{i\theta(p)}, \quad T(p) := \nu(-p)e^{i\theta(p)},$$

respectively. Then it is easy to see that $S$ and $T$ satisfy the conditions (i) and (ii) in Proposition 3.3 with $D(T) = D(S) = D(S^*) = D(T^*)$ and

$$S^*S = SS^* = \lambda p + \omega, \quad T^*T = TT^* = -\lambda p + \omega,$$

$$S^*T = TS^* = -i\sqrt{\omega^2 - \lambda^2p^2}, \quad T^*S = ST^* = i\sqrt{\omega^2 - \lambda^2p^2}.$$ 

Hence, by Proposition 3.3, $L_S$ and $L_T$ strongly anticommute. Therefore, by Theorem 4.1, $\{F(L^2(\mathbb{R}), L^2(\mathbb{R})), D_{S,T}, H_{S,T}, P_{S,T}, Q_S, QT\}$ with these $S$ and $T$ is an integrable representation of $A_{\text{SUSSY}}$. We have

$$H_{S,T} = d\Gamma_b(\omega) \otimes I + I \otimes d\Gamma_f(\omega),$$

$$P_{S,T} = \lambda\{d\Gamma_b(p) \otimes I + I \otimes d\Gamma_f(p)\}.$$ 

Note that $H_{S,T}$ and $P_{S,T}$ are independent of $\theta$.

If $\omega(p) = \sqrt{p^2 + m^2}$ with a constant $m \geq 0$, $\lambda = 1$ and $\theta = 0$, then $H_{S,T}$ and $P_{S,T}$ are respectively the Hamiltonian and the momentum operator of a free relativistic SQFT in the two-dimensional space-time, called the $N = 1$ Wess-Zumino model (cf. [19]).

References


