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A Class of Dirac-Type Operators on the Abstract Boson-Fermion Fock Space and Their Strong Anticommutativity
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A Class of Dirac-Type Operators on the Abstract Boson-Fermion Fock Space and Their Strong Anticommutativity

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1 Introduction

In a previous paper [4], we introduced a family \( \{ Q_S | S \in C(\mathcal{H}, \mathcal{K}) \} \) of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) over the pair \( (\mathcal{H}, \mathcal{K}) \) of two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), where the index set \( C(\mathcal{H}, \mathcal{K}) \) of the family is the set of all densely defined closed linear operators from \( \mathcal{H} \) to \( \mathcal{K} \), and investigated fundamental properties of them. As is shown in [4], this class of Dirac-type operators has a connection with supersymmetric quantum field theory (SQFT) [19]. Namely \( Q_S \) gives an abstract form of free supercharges in some models of SQFT. Interacting models of SQFT can be constructed from perturbations of \( Q_S \) [4]: For related aspects and further developments, see, e.g., [1], [2], [3], [5], [6], [10], [14], [16], [17], [20], [21].

Generally speaking, Dirac-type operators have something to do with a notion of anticommutativity, because they are related to representations of Clifford algebras, and this aspect may be an essential feature of Dirac-type operators (cf. [7], [8], [9], [11], [12]). A proper notion of anticommutativity, i.e., strong anticommutativity, of (unbounded) self-adjoint operators was given in [27] and developed by some authors (e.g., [25], [22], [7], [9], [11], [12]). In a recent paper [15], a theorem on the strong anticommutativity of two Dirac operators \( Q_S \) and \( Q_T \) was established with application to constructing representations on \( \mathcal{F}(\mathcal{H}, \mathcal{K}) \) of a supersymmetry algebra arising in a two-dimensional relativistic SQFT.

The aim of this note is to review fundamental aspects of the theory of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space and to present a summary of the results on their strong anticommutativity obtained in [15].
2 Dirac-type operators on the abstract Boson-Fermion Fock space—a brief review

Let $\mathcal{H}$ be a Hilbert space and $\otimes^n\mathcal{H}$ be the $n$-fold tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$; $\otimes^0(\mathcal{H}) := \mathbb{C}$). We denote by $S_n$ (resp. $A_n$) the symmetrizer (resp. the anti-symmetrizer) on $\otimes^n\mathcal{H}$ and by $S_n(\otimes^n\mathcal{H})$ (resp. $A_n(\otimes^n\mathcal{H})$) its range, which is called the $n$-fold symmetric (resp. anti-symmetric) tensor product of $\mathcal{H}$. The Boson Fock space $\mathcal{F}_b(\mathcal{H})$ and the Fermion Fock space $\mathcal{F}_f(\mathcal{H})$ over $\mathcal{H}$ are respectively defined by

$$
\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} S_n(\otimes^n\mathcal{H}), \quad \mathcal{F}_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} A_n(\otimes^n\mathcal{H})
$$

(2.1)

(e.g., [23, §II.4], [18, §5.2]). Let $\mathcal{K}$ be a Hilbert space. Then the Boson-Fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with the pair $\langle \mathcal{H}, \mathcal{K} \rangle$ is defined by

$$
\mathcal{F}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K}),
$$

(2.2)

the tensor product Hilbert space of the Boson Fock space over $\mathcal{H}$ and the Fermion Fock space over $\mathcal{K}$. We denote by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$.

We first present the definitions of basics objects in the Boson Fock space and the Fermion Fock space. More detailed descriptions on Fock space objects can be found, e.g., in [23, §II.4, Example 2], [24, §X.7] and [18, §5.2].

For each vector $\Psi = \{\Psi(n)\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H})$ ($\Psi(n) \in S_n(\otimes^n\mathcal{H})$), we use the natural identification of $\Psi(n)$ with $\{0, \cdots, 0, \Psi(n), 0, \cdots\} \in \mathcal{F}_b(\mathcal{H})$. The same applies to vectors in other infinite direct sums of Hilbert spaces.

For a subset $V$ of a Hilbert space, we denote by $L V$ the subspace algebraically spanned by all the vectors of $V$.

Let $\Omega_b := \{1, 0, 0, \cdots\} \in \mathcal{F}_b(\mathcal{H})$, the boson Fock vacuum in $\mathcal{F}_b(\mathcal{H})$. For a subspace $D$ of $\mathcal{H}$, we define

$$
\mathcal{F}_{b, \text{fin}}(D) := L \{\Omega_b, S_n(f_1 \otimes \cdots \otimes f_n)|n \in \mathbb{N}, f_j \in D, j = 1, \cdots, n\}. \tag{2.3}
$$

If $D$ is dense, then $\mathcal{F}_{b, \text{fin}}(D)$ is dense in $\mathcal{F}_b(\mathcal{H})$.

For each $f \in \mathcal{H}$, there exists a unique densely defined closed (unbounded) linear operator $a(f)$ on $\mathcal{F}_b(\mathcal{H})$, called boson annihilation operators (its adjoint $a(f)^*$ is called a boson creation operator), such that (i) for all $f \in \mathcal{H}$, $a(f)\Omega_b = 0$, (ii) for all $n \in \mathbb{N}$, $f_j \in \mathcal{H}$, $j = 1, \cdots, n$,

$$
a(f) S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f, f_j)_{\mathcal{H}} S_{n-1}(f_1 \otimes \cdots \hat{f}_j \otimes \cdots f_n),
$$

where $\hat{f}_j$ indicates the omission of $f_j$, and (iii) $\mathcal{F}_{b, \text{fin}}(\mathcal{H})$ is a core of $a(f)$. We have

$$
S_n(\otimes^n\mathcal{H}) = \overline{\{a(f_1)^* \cdots a(f_n)^*\Omega_b|f_j \in \mathcal{H}, j = 1, \cdots, n\}}, \tag{2.4}
$$

where $\overline{\{ \cdot \}}$ denotes the closure of the set $\{ \cdot \}$. The set $\{a(f), a(f)^*|f \in \mathcal{H}\}$ satisfies the canonical commutation relations

$$
[a(f), a(g)^*] = (f, g)_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0
$$
for all $f, g \in \mathcal{H}$ on $\mathcal{F}_{b, \text{fin}}(\mathcal{H})$.

A similar consideration can be done in the Fermion Fock space $\mathcal{F}_f(\mathcal{K})$. The fermion Fock vacuum $\Omega_f$ in $\mathcal{F}_f(\mathcal{K})$ is defined by $\Omega_f := \{1, 0, \cdots\} \in \mathcal{F}_b(\mathcal{K})$. For a subspace $D$ of $\mathcal{K}$, we define
\[
\mathcal{F}_{f, \text{fin}}(D) := \mathcal{L}\{\Omega_f, A_n(u_1 \otimes \cdots \otimes u_n) | n \geq 1, u_j \in D, j = 1, \cdots, n\}. \tag{2.5}
\]
If $D$ is dense, then $\mathcal{F}_{f, \text{fin}}(D)$ is dense in $\mathcal{F}_f(\mathcal{K})$.

For each $u \in \mathcal{K}$, there exists a unique bounded linear operator $b(u)$ on $\mathcal{F}_f(\mathcal{K})$, called fermion annihilation operators on $\mathcal{F}_f(\mathcal{K})$ ($b(u)^*$ is called a fermion creation operator), such that (i) for all $u \in \mathcal{K}$, $b(u)\Omega_b = 0$, (ii) for all $n \in \mathbb{N}$, $u_j \in \mathcal{K}$, $j = 1, \cdots, n$
\[
b(u)A_n(u_1 \otimes \cdots \otimes u_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1} (u, u_j)_{\mathcal{H}} S_{n-1}(u_1 \otimes \cdots \otimes \hat{u}_j \otimes \cdots \otimes u_n). \tag{2.6}
\]
We have
\[
A_n(\otimes^n \mathcal{K}) = \mathcal{L}\{b(u_1)^* \cdots b(u_n)^* \Omega_f | u_j \in \mathcal{K}, j = 1, \cdots, n\}. \tag{2.7}
\]
The set $\{b(u), b(u)^* | u \in \mathcal{K}\}$ satisfies the canonical anti-commutation relations
\[
\{b(u), b(v)^*\} = (u, v)_\mathcal{K}, \quad \{b(u), b(v)\} = 0, \quad \{b(u)^*, b(v)^*\} = 0
\]
for all $u, v \in \mathcal{K}$, where $\{A, B\} := AB + BA$.

The Fock vacuum in the Boson-Fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by
\[
\Omega := \Omega_b \otimes \Omega_f. \tag{2.8}
\]
The annihilation operators $a(f)$ and $b(u)$ are extended to operators on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ as
\[
A(f) := a(f) \otimes I, \quad B(u) := I \otimes b(u), \tag{2.9}
\]
where $I$ denotes identity operator.

For a linear operator $A$, we denote by $D(A)$ its domain. Let $S \in C(\mathcal{H}, \mathcal{K})$. Then we define
\[
D_S := \mathcal{L}\left\{A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega | n, p \geq 0, f_j \in D(S), j = 1, \cdots, n, u_k \in D(S^*), k = 1, \cdots, p\right\}
\[
= \mathcal{F}_{b, \text{fin}}(D(S)) \otimes_{\text{alg}} \mathcal{F}_{f, \text{fin}}(D(S^*)), \tag{2.10}
\]
where $\otimes_{\text{alg}}$ denotes algebraic tensor product. It follows that $D_S$ is dense in $\mathcal{F}$. The following proposition is proved in [4].

**Proposition 2.1** There exists a unique densely defined closed linear operator $d_S$ on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with the following properties: (i) $D_S$ is a core of $d_S$; (ii) for each vector $\Psi \in D_S$ of the form
\[
\Psi = A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega, \tag{2.11}
\]
$d_S$ acts as

\[ d_S \Psi = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{j=1}^{n} A(f_1)^* \cdots A(f_j)^* \cdots A(f_n)^* B(S f_j)^* B(u_1)^* \cdots B(u_p)^* \Omega & \text{for } n \geq 1, \end{cases} \]

where $A(f_j)^*$ indicates the omission of $A(f_j)^*$. Moreover the following (a)-(d) hold:

(a) $d_S^2 = 0$.

(b) For each complete orthonormal system (CONS) $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ with $e_n \in D(S^*)$,

\[ d_S \Psi = \sum_{n=1}^{\infty} A(S^* e_n) B(e_n)^* \Psi, \quad \Psi \in D_S, \]

where the convergence is taken in the strong topology of $\mathcal{F}(\mathcal{H}, \mathcal{K})$.

(c) For each CONS $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ with $\phi_n \in D(S)$, we have

\[ (\Phi, d_S \Psi)_{\mathcal{F}(\mathcal{H}, \mathcal{K})} = \lim_{N \to \infty} \left( \Phi, \sum_{n=1}^{N} A(\phi_n) B(S \phi_n)^* \Psi \right)_{\mathcal{F}(\mathcal{H}, \mathcal{K})}, \quad \Phi, \Psi \in D_S. \]

(d) $D_S \subset D(d_S^*)$ and

\[ d_S^* \Psi = \sum_{k=1}^{p} (-1)^{k-1} A(S^* u_k)^* A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_k)^* \cdots B(u_p)^* \Omega \]

for vectors $\Psi$ of the form (2.11) with $p \geq 1$. In the case $p = 0$, we have $d_S^2 \Psi = 0$.

A Dirac-type operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by

\[ Q_S = d_S + d_S^* \] (2.12)

with $D(Q_S) = D(d_S) \cap D(d_S^*)$.

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Then there is a unique self-adjoint operator $A_n$ on $\otimes^n \mathcal{X}$ such that $D_{\text{alg}}(A)$ is a core of $D(A_n)$ and, for all $f_j \in D(A)$, $j = 1, \cdots, n$, $A_n(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^{n} f_1 \otimes \cdots \otimes f_{j-1} \otimes A f_j \otimes f_{j+1} \otimes \cdots \otimes f_n$ ([23, §VIII.10, Corollary]). Putting $A_0 = 0$, one can define a self-adjoint operator

\[ d\Gamma(A) := \bigoplus_{n=0}^{\infty} A_n \] (2.13)

on $\bigoplus_{n=0}^{\infty} \otimes^n \mathcal{X}$, called the second quantization of $A$ ([23, §VIII.10, Example 2], [18, §5.2]). It is easy to show that $d\Gamma(A)$ is reduced by $\mathcal{F}_#(\mathcal{X})$ ($# = b, f$). We denote the reduced part of $d\Gamma(A)$ to $\mathcal{F}_#(\mathcal{X})$ by $d\Gamma_#(A)$. We put

\[ N_# := d\Gamma_#(I), \] (2.14)
called the number operator on $\mathcal{F}_{\#}(\mathcal{X})$.
Let
\[ \Gamma_{\#} = (-1)^{I \otimes N_{\#}}. \]  
(2.15)
We introduce an operator
\[ \Delta_{S} := d\Gamma(B(S^*S)) \otimes I + I \otimes d\Gamma(S^*S) \]  
(2.16)
acting in $\mathcal{F}(\mathcal{H},\mathcal{K})$, which is nonegative and self-adjoint (cf. [23, §VIII.10, Corollary]). For a linear operator $A$ on a Hilbert space, we set
\[ C^\infty(A) := \cap_{n=1}^\infty D(A^n). \]
Let
\[ D_S^\infty = \mathcal{L}\{A(f_1)^* \cdots A(f_n)^*B(u_1)^* \cdots B(u_p)^*\Omega | n, p \geq 0, f_j \in C^\infty(S^*S), j = 1, \cdots, n, u_k \in C^\infty(SS^*), k = 1, \cdots, p\}. \]  
(2.17)

**Theorem 2.2** [4]

(i) The operator $Q_{S}$ is self-adjoint, and essentially self-adjoint on every core of $\Delta_{S}$. In particular, $Q_{S}$ is essentially self-adjoint on $D_S^\infty$.

(ii) The operator $\Gamma_{\#}$ leaves $D(Q_{S})$ invariant and
\[ \Gamma_{\#}Q_{S} + Q_{S}\Gamma_{\#} = 0 \]
on $D(Q_{S})$.

(iii) The following operator equations hold:
\[ \Delta_{S} = Q_{S}^2 = d_{S}^*d_{S} + d_{S}d_{S}^*. \]

**Remark 2.1** The operators $d_{S}$ and $d_{S}^*$ leave $D_S^\infty$ invariant and so does $Q_{S}$.

Because of part (iii) of Theorem 2.2, we call the operator $\Delta_{S}$ the Laplacian associated with the Dirac-type operator $Q_{S}$. 
3 Strong anticommutativity of the Dirac-type operators

Let $A$ and $B$ be self-adjoint operators on a Hilbert space. We say that $A$ and $B$ strongly commute if their spectral measures commute. On the other hand, $A$ and $B$ are said to strongly anticommute if $e^{itB}A \subset Ae^{-itB}$ for all $t \in \mathbb{R}$ ([27], [22]). It turns out that this definition is symmetric in $A$ and $B$ [22].

For various Dirac-type operators, the notion of strong anticommutativity plays an important role ([7], [8], [10], [11]).

For each $S \in C(\mathcal{H}, \mathcal{K})$, the operator

$$L_S := \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

acting in $\mathcal{H} \oplus \mathcal{K}$ is self-adjoint. This operator is an abstract Dirac operator on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ [26, Chapter 5].

The strong anticommutativity of $Q_S$ and $Q_T$ ($S, T \in C(\mathcal{H}, \mathcal{K})$) is characterized as follows.

**Theorem 3.1** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Then $Q_S$ and $Q_T$ strongly anticommute if and only if $L_S$ and $L_T$ strongly anticommute. In that case, $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $Q_{S \pm T} = Q_S \pm Q_T$.

This theorem is one of the main results of the paper [15], which establishes a beautiful correspondence between the strong anticommutativity of $L_S$ and $L_T$ and that of $Q_S$ and $Q_T$.

To prove Theorem 3.1, we need some fundamental facts in the theory of strongly anticommuting self-adjoint operators [27, 22] as well as its applications, together with the following lemma. For the details, see [15].

**Lemma 3.2** Let $S, T \in C(\mathcal{H}, \mathcal{K})$. Suppose that $L_S$ and $L_T$ strongly anticommute. Then the following (i)-(v) hold:

(i) $S \pm T \in C(\mathcal{H}, \mathcal{K})$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) $|S|$ and $|T|$ strongly commute.

(iii) $|S^*|$ and $|T^*|$ strongly commute.

(iv) $D(S^*S) \cap D(T^*T) \subset D(T^*S) \cap D(S^*T)$ and, for all $f \in D(S^*S) \cap D(T^*T)$,

$$(T^*S + S^*T)f = 0.$$ 

(v) $D(SS^*) \cap D(TT^*) \subset D(TS^*) \cap D(ST^*)$ and, for all $u \in D(SS^*) \cap D(TT^*)$,

$$(TS^* + ST^*)u = 0.$$ 

\(^1\)The authors of [27] and [22] call this notion simply anticommutativity, but, to be definite, we call it strong anticommutativity.
In terms of $S$ and $T$, a necessary and sufficient condition for $L_S$ and $L_T$ to strongly anticommute is given as follows.

**Proposition 3.3** Let $S, T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then $L_S$ and $L_T$ strongly anticommute if and only if the following (i) and (ii) hold:

(i) $S \pm T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and $(S \pm T)^* = S^* \pm T^*$.

(ii) For all $f, g \in D(S) \cap D(T)$ and $u, v \in D(S^*) \cap D(T^*)$,

$$(Sf, Tg) + (Tf, Sg) = 0, \quad (S^*u, T^*v) + (T^*u, S^*v) = 0.$$

4 Application to constructing representations of a supersymmetry algebra

We consider Fock space representations of the algebra $\mathcal{A}_{\text{SUSY}}$ generated by four elements $Q_1, Q_2, H, P$ with defining relations

$$Q_1^2 = H + P, \quad Q_2^2 = H - P, \quad Q_1Q_2 + Q_2Q_1 = 0. \quad (4.1)$$

This algebra is called a supersymmetry algebra, which arises in a relativistic SQFT in the two-dimensional space-time ([19], [13]). The elements $H, P$ and $Q_j$ ($j = 1, 2$) are called the Hamiltonian, the momentum operator and the supercharge, respectively.

We recall a definition from [13]. Let $\mathcal{F}$ be a Hilbert space, $D$ a dense subspace of $\mathcal{F}$, and $H, P, Q_1, Q_2$ be linear operators on $\mathcal{F}$. We say that $\{\mathcal{F}, D, H, P, Q_1, Q_2\}$ is a symmetric representation of $\mathcal{A}_{\text{SUSY}}$ if $H, P, Q_1$ and $Q_2$ are symmetric and leave $D$ invariant satisfying $(4.1)$ on $D$. A symmetric representation $\{\mathcal{F}, D, H, P, Q_1, Q_2\}$ of $\mathcal{A}_{\text{SUSY}}$ is said to be integrable if (i) $H, P, Q_1$ and $Q_2$ are essentially self-adjoint (denote their closures by $\bar{H}, \bar{P}, \bar{Q}_1$ and $\bar{Q}_2$, respectively); (ii) $\{\bar{H}, \bar{P}, \bar{Q}_1\}$ and $\{\bar{H}, \bar{P}, \bar{Q}_2\}$ are families of strongly commuting self-adjoint operators, respectively; (iii) $\bar{H}$ and $\bar{P}$ satisfy the relativistic spectral condition

$$\pm \bar{P} \leq \bar{H}. \quad (4.2)$$

Suppose that $L_S$ and $L_T$ strongly anticommute. Then, by Lemma 3.3(ii) and (iii), $S^*S$ and $T^*T$ strongly commute, and $SS^*$ and $TT^*$ strongly commute. Hence $S^*S + T^*T$ and $SS^* + TT^*$ are nonnegative, self-adjoint, and $S^*S - T^*T$ and $SS^* - TT^*$ are essentially self-adjoint. Therefore we can define self-adjoint operators

$$H_{S,T} := \frac{1}{2} \{d\Gamma_b((S^*S + T^*T) \otimes I + I \otimes d\Gamma_t(SS^* + TT^*))\}, \quad (4.3)$$

$$P_{S,T} := \frac{1}{2} \{d\Gamma_b((S^*S - T^*T) \otimes I + I \otimes d\Gamma_t(SS^* - TT^*))\}^- \quad (4.4)$$

where for a closable linear operator $A$, $A^-$ (or $A^*$) denotes its closure. Note that $H_{S,T}$ is nonnegative, but $P_{S,T}$ may be neither bounded below nor bounded above.

For a self-adjoint operator $A$, we denote by $E_A$ its spectral measure. Let

$$\mathcal{D}_{S,T} := \mathcal{L}(E_{[a,b]}([c,d])) \Psi | \Psi \in \mathcal{F}(\mathcal{H}, \mathcal{K}), 0 \leq a < b < \infty, 0 \leq c < d < \infty \}. \quad (4.5)$$

We can prove the following theorem (for the proof, see [15]).
Theorem 4.1 Let $S, T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and suppose that $L_S$ and $L_T$ strongly anticommute. Then $\{\mathcal{F}(\mathcal{H}, \mathcal{K}), D_{S,T}, H_{S,T}, P_{S,T}, Q_S, Q_T\}$ is an integrable representation of $A_{\text{SUSY}}$.

We give only one basic example from SQFT (for other examples, see [19], [4]).

Example Let $\mathcal{H} = \mathcal{K} = L^2(\mathbb{R})$ and $\mathbb{R} \ni p \to \omega(p)$ be a nonnegative function on $\mathbb{R}$ which is Borel measurable, almost everywhere (a.e.) finite with respect to the Lebesgue measure on $\mathbb{R}$, and satisfies $|p| \leq \omega(p)$, a.e. $p \in \mathbb{R}$.

Let $\nu(p) = \sqrt{\lambda p + \omega(p)}$ with $\lambda \in [0, 1]$ (a constant parameter) and $\theta(p)$ be an a.e. finite real-valued Borel measurable function on $\mathbb{R}$. Define the operators $S$ and $T$ on $L^2(\mathbb{R})$ to be the multiplication operators by the functions

$$S(p) := i\nu(p)e^{i\theta(p)}, \quad T(p) := \nu(-p)e^{i\theta(p)},$$

respectively. Then it is easy to see that $S$ and $T$ satisfy the conditions (i) and (ii) in Proposition 3.3 with $D(T) = D(S) = D(S^*) = D(T^*)$ and

$$S^*S = SS^* = \lambda p + \omega, \quad T^*T = TT^* = -\lambda p + \omega,$$

$$S^*T = TS^* = -i\sqrt{\omega^2 - \lambda^2 p^2}, \quad T^*S = ST^* = i\sqrt{\omega^2 - \lambda^2 p^2}.$$ 

Hence, by Proposition 3.3, $L_S$ and $L_T$ strongly anticommute. Therefore, by Theorem 4.1, $\{\mathcal{F}(L^2(\mathbb{R}), L^2(\mathbb{R})), D_{S,T}, H_{S,T}, P_{S,T}, Q_S, Q_T\}$ with these $S$ and $T$ is an integrable representation of $A_{\text{SUSY}}$. We have

$$H_{S,T} = d\Gamma_b(\omega) \otimes I + I \otimes d\Gamma_f(\omega),$$

$$P_{S,T} = \lambda\{d\Gamma_b(p) \otimes I + I \otimes d\Gamma_f(p)\}.$$ 

Note that $H_{S,T}$ and $P_{S,T}$ are independent of $\theta$.

If $\omega(p) = \sqrt{p^2 + m^2}$ with a constant $m \geq 0$, $\lambda = 1$ and $\theta = 0$, then $H_{S,T}$ and $P_{S,T}$ are respectively the Hamiltonian and the momentum operator of a free relativistic SQFT in the two-dimensional space-time, called the $N = 1$ Wess-Zumino model (cf. [19]).

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