

NONSTANDARD METHODS ON REPRESENTATIONS OF THE
CANONICAL COMMUTATION RELATIONS

名古屋大学人間情報学研究科 山下秀康 (Hideyasu Yamashita)
名古屋大学人間情報学研究科 小澤正直 (Masanao Ozawa)

1 NONSTANDARD ANALYSIS

There are several different formulations of nonstandard analysis. This paper adopts the set-theoretical approach based on superstructures instituted by Robinson and Zakon [2] and follows the up-to-date description by Chang and Keisler [3].

For any set X , let $S(X)$ denote the set of all subsets of X . The *superstructure* over X , denoted by $V(X)$, is defined by the following recursion:

$$V_0(X) = X, \quad V_{n+1}(X) = V_n(X) \cup S(V_n(X)), \quad V(X) = \bigcup_{n \in \mathbf{N}} V_n(X),$$

where \mathbf{N} is the set of natural numbers. The set X is called a *base set* if $\emptyset \notin X$ and for all $x \in X$ we have $x \cap V(X) = \emptyset$.

The language \mathcal{L} which describes $V(X)$ consists of logical connectives $\neg, \wedge, \vee, \Rightarrow$, quantifiers \forall, \exists , individual variables x', x'', \dots , individual constants C_u for all $u \in V(X)$, and two binary predicate constants $=, \in$. A *formula* of \mathcal{L} is constructed from the above constituents in the usual way. We will use the following abbreviations, called *bounded quantifiers*: $(\forall x \in y)\phi$ means $(\forall x)[x \in y \Rightarrow \phi]$, $(\exists x \in y)\phi$ means $(\exists x)[x \in y \wedge \phi]$. A *bounded formula* is a formula in which every quantifier occurs as a bounded quantifier. We will write $\phi[u_1, \dots, u_n]$ for $\phi(C_{u_1}, \dots, C_{u_n})$.

For any formula ϕ in \mathcal{L} , the relation $V(X) \models \phi$ is defined by the following rules:

- (i) $V(X) \models C_u = C_v$ if and only if u and v are identical.
- (ii) $V(X) \models C_u \in C_v$ if and only if u is an element of v .
- (iii) $V(X) \models \neg\phi$ if and only if $V(X) \not\models \phi$ does not hold.

- (iv) $V(X) \models \phi_1 \wedge \phi_2$ if and only if $V(X) \models \phi_1$ and $V(X) \models \phi_2$.
- (v) $V(X) \models \phi_1 \vee \phi_2$ if and only if $V(X) \models \phi_1$ or $V(X) \models \phi_2$.
- (vi) $V(X) \models \phi_1 \Rightarrow \phi_2$ if and only if $V(X) \models \phi_1$ then $V(X) \models \phi_2$.
- (vii) $V(X) \models (\forall x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for all u in $V(X)$.
- (viii) $V(X) \models (\exists x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for some u in $V(X)$.

A *nonstandard universe* is a triple $\langle V(X), V(Y), \star \rangle$ consisting of superstructures $V(X)$, $V(Y)$, and a map $\star : V(X) \rightarrow V(Y)$ satisfying the following conditions (i)–(iii):

- (i) X and Y are infinite base sets.
- (ii) (**Transfer Principle**) The map $\star : a \mapsto \star a$ is an injective mapping from $V(X)$ into $V(Y)$, and for any bounded formula $\phi(x_1, \dots, x_n)$ in \mathcal{L} ,

$$V(X) \models \phi[u_1, \dots, u_n] \quad \text{if and only if} \quad V(Y) \models \phi[\star u_1, \dots, \star u_n]$$

for any u_1, \dots, u_n in $V(X)$.

- (iii) $\star X = Y$.

An element $u \in V(Y) \setminus Y$ is called an *internal set* if there is $x \in V(X)$ such that $u \in \star x$. Let α be a cardinal. A nonstandard universe $\langle V(X), V(Y), \star \rangle$ is said to be α -*saturated* if it satisfies the following condition:

- (iv) (**Saturation Principle**) Every family of less than α internal sets with the finite intersection property has nonempty intersection.

In this paper, we always work with a nonstandard universe $\langle V(X), V(Y), \star \rangle$ which is α -saturated with $\text{card}(V(X)) < \alpha$; such a nonstandard universe is said to be *polysaturated*. We also assume that the base X includes the complex numbers \mathbf{C} and any other structures under consideration such as given groups and Hilbert spaces.

For a set S , let ${}^\circ S = \{\star s \mid s \in S\}$. We identify $\star z$ with z for all $z \in \mathbf{C}$. Hence, ${}^\circ S = S$ if S is a subset of \mathbf{C} , e.g., ${}^\circ \mathbf{C} = \mathbf{C}$, ${}^\circ \mathbf{R} = \mathbf{R}$ (the real numbers), ${}^\circ \mathbf{Z} = \mathbf{Z}$ (the integers), and ${}^\circ \mathbf{N} = \mathbf{N}$. Let \mathbf{R}^+ , $\star \mathbf{R}_0$, $\star \mathbf{R}_0^+$, $\star \mathbf{R}_\infty^+$, and $\star \mathbf{N}_\infty$ denote the sets of positive real numbers, infinitesimal hyperreal numbers, positive infinitesimal hyperreal numbers, positive infinite hyperreal numbers

and infinite hypernatural numbers, respectively. It is shown that ${}^*\mathbf{N}_\infty = {}^*\mathbf{N} \setminus \mathbf{N}$. We write $x \sim \infty$ if $x \in \mathbf{R}_\infty^+$, and $0 < x < \infty$ if $x \in \text{fin}{}^*\mathbf{R}^+ = {}^*\mathbf{R}^+ \setminus \mathbf{R}_\infty^+$. If $r \in {}^*\mathbf{R}$ and $|r| < \infty$, the standard part of r is denoted by ${}^{\circ}r$. If $r \sim \infty$, we write ${}^{\circ}r = \infty$. Let $x, y \in {}^*\mathbf{R}^+$. We say that x is of the *order* of y , in symbols $x \asymp y$, iff $0 < x/y < \infty$ and $0 < y/x < \infty$. We write $x \ll y$ if $x/y \approx 0$. For a hyperfinite (\star -finite) set F , let $|F|$ denote the internal cardinal number of F .

Let (X, \mathcal{O}) be a topological space. Let \mathcal{O}_x denote the system of open neighborhoods of $x \in X$. The *monad* of $x \in X$ is the subset of *X defined by $\text{mon}_{\mathcal{O}}(x) = \bigcap \{ {}^*O \mid O \in \mathcal{O}_x \}$. The set of *near standard* points is the subset of *X defined by $\text{ns}({}^*X) = \bigcup \{ \text{mon}_{\mathcal{O}}(x) \mid x \in X \}$. It is shown that (X, \mathcal{O}) is Hausdorff if and only if $x \neq y$ implies $\text{mon}_{\mathcal{O}}(x) \cap \text{mon}_{\mathcal{O}}(y) = \emptyset$. Thus for any Hausdorff space (X, \mathcal{O}) , we can define the equivalence relation $\approx_{\mathcal{O}}$ on $\text{ns}{}^*X$ so that $a \approx_{\mathcal{O}} b$ iff $a \in \text{mon}_{\mathcal{O}}(x)$ and $b \in \text{mon}_{\mathcal{O}}(x)$ for some $x \in X$. Let $(X, \|\cdot\|)$ be an internal normed linear space. Define the relation \approx on X so that $x \approx y$ iff $\|x - y\| \approx 0$. The *principal galaxy* of X is the subset of *X defined by $\text{fin}(X) = \{x \in X \mid \|x\| < \infty\}$. For $x \in \text{fin}(X)$, let \hat{x} denote the equivalence class $\hat{x} = \{y \in X \mid x \approx y\}$. Let $\hat{X} = \{\hat{x} \mid x \in \text{fin}(X)\}$. Define the (standard) norm $\|\cdot\|$ on \hat{X} by $\|\hat{x}\| = {}^{\circ}\|x\|$ for all $x \in \text{fin}X$. Then $(\hat{X}, \|\cdot\|)$ turns out to be a Banach space, called the *standardization* of $(X, \|\cdot\|)$. In a similar way, the standardization is defined for any internal pre-Hilbert space $(X, \langle \cdot, \cdot \rangle)$, and it turns to be a Hilbert space.

For a (standard) normed linear space $(X, \|\cdot\|)$, we abbreviate $\widehat{{}^*X}$ to \hat{X} . In this case, the Banach space $(\hat{X}, \|\cdot\|)$ is called the *nonstandard hull* of $(X, \|\cdot\|)$.

Let \mathcal{H} be an internal Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ an internal bounded operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$, called the *standardization* of T , is defined by the relation $\hat{T}\hat{x} = \widehat{Tx}$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [5] and Hurd and Loeb [4].

2 CANONICAL COMMUTATION RELATIONS

Let \mathcal{H} be a Hilbert space, and $\{a_i \mid i \in I\}$ a family of linear operators on \mathcal{H} , with dense domains $\text{dom}(a_i)$, when I is finite or infinite. Let D be a dense subspace of \mathcal{H} . The pair $(\{a_i\}, D)$ is called a *representation of the CCR* if the

following conditions are satisfied for all i :

- (i) $D \subset \text{dom}(a_i)$, $D \subset \text{dom}(a_i^*)$.
- (ii) D is invariant under a_i and a_i^* .
- (iii) $[a_i, a_j^*] = \delta_{ij}$, $[a_i, a_j] = 0$ on D .

The number $n = |I|$ is called the *degree of freedom* of the CCR.

Let $U(\cdot)$ and $V(\cdot)$ be strongly continuous one-parameter unitary groups on \mathcal{H} . The pair (U, V) is a *representation of the Weyl CCR* (of one degree of freedom) if

$$U(p)V(q) = e^{ipq}V(q)U(p) \quad (1)$$

for all $p, q \in \mathbf{R}$.

Let \mathbf{H}_1 be the group $(\mathbf{R} \times \mathbf{R} \times \mathbf{R}, \cdot)$ with group law

$$(p, q, t) \cdot (p', q', t') = (p + p', q + q', t + t' + pq').$$

Then, we get a strongly-continuous representation of \mathbf{H}_1 by

$$(p, q, t) \mapsto W(p, q, t) := e^{it}V(q)U(p),$$

where (U, V) is a representation of the Weyl CCR. The representation W is called a *Weyl representation* of \mathbf{H}_1 .

Let $U(p)$ and $V(q)$ be the unitary operators on $L^2(\mathbf{R})$ defined by

$$(U(p)f)(x) = e^{ip}f(x),$$

$$(V(q)f)(x) = f(x - t).$$

Then, $\rho(p, q, t) := e^{it}V(q)U(p)$ is a Weyl representation of \mathbf{H}_1 . Let us call this the *Schrödinger representation* of \mathbf{H}_1 .

3 NONSTANDARD REPRESENTATIONS OF THE CCR

Let $\nu \in {}^*\mathbf{N}$ be infinite, and $\{a_i | i \in \mathbf{N}\}$ be a sequence of internal operators on ${}^*\mathbf{C}^\nu$, or equivalently, be $\nu \times \nu$ internal matrices. Let $D \subset \text{fin}{}^*\mathbf{C}^\nu$ be an external subspace, invariant under a_i and a_i^* for all i . The pair $(\{a_i\}, D)$ is called a *hyperfinite representation of the CCR* if

$$\| [a_i, a_j^*]\xi - \delta_{ij}\xi \| \approx 0,$$

for all $\xi \in D$.

We easily see the following:

Lemma 3.1 *If $\xi, \eta \in D$ and $\xi \approx \eta$, then $a_i \xi \approx a_i \eta$ and $a_i^* \xi \approx a_i^* \eta$.*

This allows us to define the operators \hat{a}_i and \hat{a}_i^* on the Hilbert space $\hat{D}^{\perp\perp}$ by

$$\hat{a}_i \hat{\xi} = \widehat{a_i \xi}, \quad \hat{a}_i^* \hat{\xi} = \widehat{a_i^* \xi}.$$

We call \hat{a}_i and \hat{a}_i^* the *standard part* of a_i and a_i^* , respectively.

4 HYPERFINITE HEISENBERG GROUP

This section reviews the results given by Ojima and Ozawa [7].

Let $K \in {}^*\mathbf{N}$ be infinite, and $\mathbf{K} = \langle {}^*\mathbf{Z}/K{}^*\mathbf{Z}, \otimes, \oplus \rangle$ be a ring of residue classes modulo K . Define an inner product on ${}^*\mathbf{C}^{\mathbf{K}}$ by

$$\langle f, g \rangle := \sum_{k \in \mathbf{K}} \overline{f(k)} g(k) \Delta x,$$

for $f, g \in {}^*\mathbf{C}^{\mathbf{K}}$, $\Delta x \in {}^*\mathbf{R}$, $\Delta x > 0$. Define \mathbf{H} to be $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ equipped with the group law

$$(k, l, m)(k', l', m') = (k \oplus k', l \oplus l', m \oplus m' \oplus (k \otimes l')).$$

Let us call \mathbf{H} the *hyperfinite Heisenberg group*. Let $W(k, l, m)$ be the internal operator on ${}^*\mathbf{C}^{\mathbf{K}}$ defined by $W(k, l, m)f(k') = e^{2\pi i(m+lk')/K} f(k' \oplus k)$.

Proposition 4.1 *The map $W: (k, l, m) \mapsto W(k, l, m)$ is an internal irreducible unitary representation of \mathbf{H} .*

We call it the *hyperfinite Schrödinger representation* of \mathbf{H} .

Proposition 4.2 *The map \hat{W} is a unitary representation of \mathbf{H} on ${}^*\widehat{\mathbf{C}}^{\mathbf{K}}$.*

Let $\Delta x = K^{-1/2}$, and $\Delta(p, q, t) = (\lfloor p/\Delta x \rfloor, \lfloor q/\Delta x \rfloor, \lfloor t/\Delta x \rfloor)$.

Theorem 4.3 Let $\text{fin}(\mathbf{H})$ be the subgroup of \mathbf{H} defined by

$$\text{fin}(\mathbf{H}) = \{(k, l, m) \mid |k\Delta x| < \infty, |l\Delta x| < \infty, |m\Delta x^2| < \infty\}.$$

Then, there is $f \in \text{fin}({}^*\mathbf{C}^{\mathbf{K}})$ satisfying the following. Let \mathcal{H} be the closed subspace of $\widehat{{}^*\mathbf{C}^{\mathbf{K}}}$ such that

$$\mathcal{H} = \{\widehat{W}(k, l, m)\hat{f} \mid (k, l, m) \in \text{fin}(\mathbf{H})\}^{\perp\perp}.$$

For any $(k, l, m) \in \text{fin}(\mathbf{H})$, let $\widetilde{W}(k, l, m)$ be the restriction of $\widehat{W}(k, l, m)$ to \mathcal{H} . Then the map $(p, q, t) \mapsto \widetilde{W}(\Delta(p, q, t))$ is a strongly continuous unitary representation of \mathbf{H}_1 , unitarily equivalent to the Schrödinger representation ρ .

5 HYPERFINITE PARAFERMI OPERATORS

This section reviews the results given by Yamashita [7].

Let $\nu \in \mathbf{N}$ and $d \in \mathbf{N}$. Suppose that $b_1, \dots, b_\nu \in M(d, \mathbf{C})$ (i.e., b_1, \dots, b_ν are finite-dimensional matrices). The matrices b_1, \dots, b_ν are called the *annihilation operators* of parafermi oscillators of order $p \in \mathbf{N}$ if they satisfy

$$\begin{aligned} [b_k, [b_l^*, b_m]] &= 2\delta_{kl}b_m, \\ [b_k, [b_l^*, b_m^*]] &= 2\delta_{kl}b_m^* - 2\delta_{km}b_l^*, \\ [b_k, [b_l, b_m]] &= 0, \end{aligned}$$

and the uniqueness of vacuum $|0\rangle$, and,

$$b_k b_l^* |0\rangle = \delta_{kl} p |0\rangle.$$

The matrices b_1^*, \dots, b_ν^* are called the *creation operators* of parafermi oscillators of order p . The *hyperfinite annihilation operators* of parafermi oscillators are the internal matrices defined by substituting ${}^*\mathbf{N}$ and ${}^*\mathbf{C}$ for \mathbf{N} and \mathbf{C} in the above definition, respectively.

Green [8] has given a class of representations of the above commutation relations of the parafermi creation and annihilation operators. In the so-called the *Green representation* for the cases of order p , the parafermi operators b_k are expressed by the form

$$b_k = \sum_{\alpha=1}^p b_k^{(\alpha)},$$

where the *Green-component* operators $b_k^{(\alpha)}$ satisfy the commutation relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)*}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

$$[b_k^{(\alpha)}, b_l^{(\beta)*}] = [b_k^{(\alpha)}, b_l^{(\beta)}] = 0 \quad (\alpha \neq \beta);$$

where $\{A, B\} = AB + BA$, and the uniqueness of vacuum $|0\rangle$ such that

$$b_k^{(\alpha)}|0\rangle = 0 \quad \text{for all } k, \alpha.$$

The Green representation is essentially equivalent to the tensor product representation of the Clifford algebra representation of $so(2\nu)$. In fact, we easily verify that $e_1, \dots, e_{2\nu}$ defined by $e_{2k-1} = i(b_k^* + b_k)$ and $e_{2k} = b_k^* - b_k$ form the generators of a Clifford algebra, i.e., $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ ($i \neq j$). Thus, we can construct a $2^{2\nu}$ -dimensional representation of Green components by using a spin representation of the Clifford algebra as follows. Let $V_k^{(\alpha)} \simeq \mathbb{C}^2$ ($k = 1, \dots, \nu$, $\alpha = 1, \dots, p$). The Pauli matrices $\sigma_{1,k}^{(\alpha)}$, $\sigma_{2,k}^{(\alpha)}$ and $\sigma_{3,k}^{(\alpha)}$ that act on $V_k^{(\alpha)}$ are represented as

$$\sigma_{1,k}^{(\alpha)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2,k}^{(\alpha)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3,k}^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $V^{(\alpha)}$ by

$$V^{(\alpha)} = V_1^{(\alpha)} \otimes \dots \otimes V_\nu^{(\alpha)},$$

and define $\hat{\sigma}_{c,k}^{(\alpha)}$ and $\gamma_i^{(\alpha)}$ ($k = 1, \dots, \nu$, $i = 1, \dots, 2\nu$) that acts on $V^{(\alpha)}$ by

$$\hat{\sigma}_{c,k}^{(\alpha)} = \overbrace{1 \otimes \dots \otimes 1}^{k-1} \otimes \sigma_{c,k}^{(\alpha)} \otimes 1 \otimes \dots \otimes 1, \quad c = 1, 2, 3,$$

$$\gamma_{2k-1}^{(\alpha)} = \hat{\sigma}_{2,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \dots \hat{\sigma}_{3,\nu}^{(\alpha)},$$

$$\gamma_{2k}^{(\alpha)} = -\hat{\sigma}_{1,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \dots \hat{\sigma}_{3,\nu}^{(\alpha)}.$$

The operators $b_k^{(\alpha)}$ ($k = 1, \dots, \nu$) defined by

$$b_k^{(\alpha)} = \frac{1}{2}(\gamma_{2k-1}^{(\alpha)} - i\gamma_{2k}^{(\alpha)}),$$

satisfy the relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)*}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

for all $k, l = 1, \dots, \nu$.

Define V and $\tilde{b}_k^{(\alpha)}$ ($k = 1, \dots, \nu$) acting on V by

$$V = V^{(1)} \otimes \dots \otimes V^{(p)},$$

$$\tilde{b}_k^{(\alpha)} = \overbrace{1 \otimes \dots \otimes 1}^{\alpha-1} \otimes b_k^{(\alpha)} \otimes 1 \otimes \dots \otimes 1.$$

We see that for all $k, l = 1, \dots, \nu$,

$$\{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)*}\} = \delta_{kl}, \quad \{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)}\} = 0,$$

$$[\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)*}] = [\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)}] = 0, \quad (\alpha \neq \beta).$$

Let $|0\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ denote the unit vector satisfying $b_k^{(\alpha)}|0\rangle_k^{(\alpha)} = 0$. Define $|0\rangle^{(\alpha)}$ and $|0\rangle$ by

$$|0\rangle^{(\alpha)} = |0\rangle_1^{(\alpha)} \otimes \dots \otimes |0\rangle_\nu^{(\alpha)},$$

$$|0\rangle = |0\rangle^{(\alpha)} \otimes \dots \otimes |0\rangle^{(p)}.$$

Now, we find that $\tilde{b}_1^{(\alpha)}, \dots, \tilde{b}_\nu^{(\alpha)}$ are $2^{p\nu}$ -dimensional representations of the Green components and $|0\rangle$ is the vacuum. Thus, $b_k = \sum_{\alpha=1}^p \tilde{b}_k^{(\alpha)}$, ($k = 1, \dots, \nu$) are $2^{p\nu}$ -dimensional representations of annihilation operators of ν parafermi oscillators of order p . Let us call the above representation of the algebra of the parafermi oscillators the *spin representation*.

Define $\sigma_{\pm, k}^{(\alpha)}$ by

$$\sigma_{\pm, k}^{(\alpha)} = (\sigma_{1, k}^{(\alpha)} \pm i\sigma_{2, k}^{(\alpha)})/2,$$

and $|1\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ by

$$|1\rangle_k^{(\alpha)} = \sigma_{+, k}^{(\alpha)}|0\rangle_k^{(\alpha)}.$$

The set of vectors

$$\{(|e_1^{(1)}\rangle_1^{(1)} \dots |e_\nu^{(1)}\rangle_\nu^{(1)}) \dots (|e_1^{(p)}\rangle_1^{(p)} \dots |e_\nu^{(p)}\rangle_\nu^{(p)}) : e_k^{(\alpha)} = 0, 1\}$$

(\otimes 's are omitted) is a complete orthonormal system of V . We write the vectors simply as $\{|e_k^{(\alpha)}\rangle\}$.

The number operator N on V and the related operators $N_k, N^{(\alpha)}$ are defined as follows:

$$N_k^{(\alpha)} = \frac{1}{2}(1 + \sigma_{3, k}^{(\alpha)}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned}\hat{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes N_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{\nu-k}, \\ \tilde{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{\alpha-1} \otimes \hat{N}_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{p-\alpha}, \\ N_k &= \sum_{\alpha=1}^p \tilde{N}_k^{(\alpha)}, \quad N^{(\alpha)} = \sum_{k=1}^{\nu} \tilde{N}_k^{(\alpha)}, \quad N = \sum_{\alpha=1}^p N^{(\alpha)},\end{aligned}$$

We see that

$$N|\{e_k^{(\alpha)}\}\rangle = n|\{e_k^{(\alpha)}\}\rangle,$$

where n is the number of $e_k^{(\alpha)}$'s that is equal to 1. It is easily shown that

$$\tilde{b}_k^{(\alpha)*} \tilde{b}_k^{(\alpha)} = \tilde{N}_k^{(\alpha)}, \quad \tilde{b}_k^{(\alpha)} \tilde{b}_k^{(\alpha)*} = 1 - \tilde{N}_k^{(\alpha)}, \quad N_k = \frac{1}{2}([b_k^*, b_k] + p),$$

$$[N_k, N_l] = 0, \quad N_k b_k = b_k(N_k - 1), \quad N_k b_k^* = b_k^*(N_k + 1), \quad \text{etc.}$$

Lemma 5.1 *Suppose that the hyperfinite parafermi annihilation operators b_1, \dots, b_ν are represented by the spin representation, and that their order p is an infinite hypernatural number (b_1, \dots, b_ν are $2^{p\nu} \times 2^{p\nu}$ internal matrices acting on ${}^*C^{2^{p\nu}}$). If $|\xi\rangle \in {}^*C^{2^{p\nu}}$ satisfies $\langle \xi | \xi \rangle, \langle \xi | N^2 | \xi \rangle < \infty$, and $k \neq l$ ($k, l = 1, 2, \dots, \nu$), then*

- (i) $[\beta_k, \beta_l]|\xi\rangle \approx [\beta_k, \beta_l^*]|\xi\rangle \approx 0$,
- (ii) $[\beta_k, \beta_k^*]|\xi\rangle \approx |\xi\rangle$,
- (iii) $\beta_k \beta_k^{*n} |\xi\rangle \approx (\beta_k^{*n} \beta_k + n \beta_k^{*n-1})|\xi\rangle$,

where $\beta_k = p^{-1/2} b_k$ (the normalization of b_k) and $n < \infty$.

Suppose that the number of the parafermi oscillators ν and their order p are infinite hypernatural numbers. When n_i is a nonnegative integer for any $i = 1, 2, \dots < \infty$, and the number of n_i 's such that $n_i \neq 0$ is finite, we will define $|n_1, n_2, \dots\rangle$ by

$$|n_1, n_2, \dots\rangle = \frac{b_1^{*n_1} b_2^{*n_2} \cdots |0\rangle}{\|b_1^{*n_1} b_2^{*n_2} \cdots |0\rangle\|}.$$

Since $b_1^{*n_1} b_2^{*n_2} \cdots$ is the product of a finite number of operators, it is well-defined. $N_k |n_1, n_2, \dots\rangle = n_k |n_1, n_2, \dots\rangle$ is easily shown, and hence, since N_k is hermitian, the set of the vectors of the form $|n_1, n_2, \dots\rangle$ is an orthonormal system.

Lemma 5.2 *The following relations hold:*

- (i) $\beta_k^* \beta_k |n_1, n_2, \dots\rangle \approx n_k |n_1, n_2, \dots\rangle,$
- (ii) $\beta_k \beta_k^* |n_1, n_2, \dots\rangle \approx (n_k + 1) |n_1, n_2, \dots\rangle,$
- (iii) $\|\beta_1^{*n_1} \beta_2^{*n_2} \dots |0\rangle\| \approx \sqrt{n_1! n_2! \dots},$
- (iv) $\beta_k^* |n_1, n_2, \dots\rangle \approx \sqrt{n_k + 1} |n_1, n_2, \dots, n_k + 1, \dots\rangle,$
- (v) $\beta_k |n_1, n_2, \dots\rangle \approx \sqrt{n_k} |n_1, n_2, \dots, n_k - 1, \dots\rangle.$

Define a set $D \subset {}^*C^{2p\nu}$ by

$$D = \left\{ \frac{\beta_{k_1}^* \dots \beta_{k_n}^* |0\rangle}{\|\beta_{k_1}^* \dots \beta_{k_n}^* |0\rangle\|} \mid n, k_1, \dots, k_n \in \mathbf{N} \right\} \cup \{|0\rangle\}.$$

Clearly, every vector in D is a normalized eigenvector of the number operator N with a finite eigenvalue. Let S denote the external subspace of ${}^*C^{2p\nu}$ spanned by D , i.e.,

$$S = \left\{ \sum_{i=1}^n c_i |\xi\rangle \mid c_i \in {}^*C, |c_i| < \infty, n \in \mathbf{N}, |\xi\rangle \in D \right\}.$$

The following theorem follows from Lemma 5.1 and 5.2.

Theorem 5.3 *The pair (β_k, S) ($k \in \mathbf{N}$) is a hyperfinite representation of CCR of countably-infinite degree of freedom, i.e. S is invariant with respect to β_k and β_k^* for every $k \in \mathbf{N}$, and*

$$[\beta_k, \beta_l] |\xi\rangle \approx 0,$$

$$[\beta_k, \beta_l^*] |\xi\rangle \approx \delta_{kl} |\xi\rangle,$$

for any $|\xi\rangle \in S$. Moreover, the uniqueness of vacuum is satisfied in the following sense: if $|\xi\rangle \in S$, $\langle \xi | \xi \rangle = 1$ and $\beta_k |\xi\rangle \approx 0$ for all $k \in \mathbf{N}$, then $|\xi\rangle \approx |0\rangle$.

References

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