Conditions for Choquet Integral Representation

(Choquet 積分表現のための条件)

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1 Introduction

The Choquet integral with respect to a fuzzy measure is a functional on the class $B$ of measurable functions, that is comonotonically additive and monotone (for short c.m.).

Sugeno et al. [15] proved that a c.m. functional $I$ can be represented by a Choquet integral with respect to a regular fuzzy measure when the domain of $I$ is the class $K^+$ of nonnegative continuous functions with compact support on a locally compact Hausdorff space. In [8, 9], it is proved that a c.m. functional is a rank- and sign-dependent functional, that is, the difference of two Choquet integrals. This functional is used in utility theory [5] and cumulative prospect theory [17, 18]. It is also proved that a rank- and sign-dependent functional is a c.m. functional if the universal set $X$ is not compact.

In this paper, we discuss the conditions for which a c.m. functional can be represented by one Choquet integral. We define the conjugate conditions and show their basic proper-
ties in Section 4. The conjugate conditions are stronger than the boundedness and a c.m. functional \( I \) is represented by one Choquet integral when \( I \) satisfies one of the conjugate conditions. Conversely if a c.m. functional \( I \) is represented by one Choquet integral, \( I \) satisfies the conjugate condition when the universal set \( X \) is separable.

The proof of the main theorem is shown in Section 5 and the other proofs are omitted.

2 Preliminaries

In this section, we define the fuzzy measure, the Choquet integral and the rank- and sign-dependent functional, and show their basic properties.

Throughout the paper we assume that \( X \) is a locally compact Hausdorff space, \( \mathcal{B} \) is the class of Borel subsets, \( \mathcal{O} \) is the class of open subsets and \( \mathcal{C} \) is the class of compact subsets.

**Definition 2.1.** [14] A fuzzy measure \( \mu \) is an extended real valued set function,

\[
\mu : \mathcal{B} \rightarrow \overline{\mathbb{R}^+}
\]

with the following properties,

1. \( \mu(\emptyset) = 0 \)
2. \( \mu(A) \leq \mu(B) \) whenever \( A \subset B, A, B \in \mathcal{B} \)

where \( \overline{\mathbb{R}^+} = [0, \infty] \) is the set of extended nonnegative real numbers.

When \( \mu(X) < \infty \), we define the conjugate \( \mu^c \) of \( \mu \) by

\[
\mu^c(A) = \mu(X) - \mu(A^c)
\]

for \( A \in \mathcal{B} \).
Definition 2.2. Let $\mu$ be a fuzzy measure on measurable space $(X, B)$.

$\mu$ is said to be outer regular if

$$\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$$

for all $B \in B$.

The outer regular fuzzy measure $\mu$ is said to be regular, if for all $O \in \mathcal{O}$

$$\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}.$$  

We denote by $K$ the class of continuous functions with compact support, by $K^+$ the class of nonnegative continuous functions with compact support and by $K_1^+$ the class of nonnegative continuous functions with compact support that satisfies $0 \leq f \leq 1$.

We denote $\text{supp}(f)$ the support of $f \in K$, that is,

$$\text{supp}(f) = \text{cl}\{x | f(x) \neq 0\}.$$  

Definition 2.3. [1, 6] Let $\mu$ be a fuzzy measure on $(X, B)$.

(1) The Choquet integral of $f \in K^+$ with respect to $\mu$ is defined by

$$\int fd\mu = \int_0^\infty \mu_f(r)dr,$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

(2) Suppose $\mu(X) < \infty$. The Choquet integral of $f \in K$ with respect to $\mu$ is defined by

$$\int fd\mu = (C) \int f^+d\mu - (C) \int f^-d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. When the right hand side is $\infty - \infty$, the Choquet integral is not defined.
Definition 2.4. [3] Let $f, g \in K$. We say that $f$ and $g$ are \textit{comonotonic} if
\[ f(x) < f(x') \Rightarrow g(x) \leq g(x') \]
for $x, x' \in X$. We denote $f \sim g$, when $f$ and $g$ are comonotonic.

Definition 2.5. Let $I$ be a real valued functional on $K$.

We say $I$ is \textit{comonotonically additive} iff
\[ f \sim g \Rightarrow I(f + g) = I(f) + I(g) \]
for $f, g \in K^+$, and $I$ is \textit{monotone} iff
\[ f \leq g \Rightarrow I(f) \leq I(g) \]
for $f, g \in K^+$.

If a functional $I$ is comonotonically additive and monotone, we say that $I$ is a \textit{c.m. functional}.

Suppose that $I$ is a c.m. functional, then we have $I(af) = aI(f)$ for $a \geq 0$ and $f \in K^+$, that is, $I$ is positive homogeneous.

3 Representation and Boundedness

Definition 3.1. Let $I$ be a real valued functional on $K$. $I$ is said to be a \textit{rank- and sign-dependent functional} (for short a \textit{r.s.d. functional}) on $K$, if there exist two fuzzy measures $\mu^+, \mu^-$ such that for every $f \in K$
\[ I(f) = (C) \int f^+ d\mu^+ - (C) \int f^- d\mu^- \]
where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.
When $\mu^+ = \mu^-$, we say that the r.s.d. functional is the Šipoš functional [13]. If the r.s.d. functional is the Šipoš functional, we have $I(-f) = -I(f)$.

If $\mu^+(X) < \infty$ and $\mu^- = (\mu^+)^c$, we say that the r.s.d. functional is the Choquet functional.

**Theorem 3.2.** [8, 9] Let $I$ be a c.m. functional on $K$.

1. We put

\[
\mu_1^+(O) = \sup\{I(f) | f \in K_1^+, support(f) \subset O\},
\]

and

\[
\mu_1^+(B) = \inf\{\mu_1^+(O) | O \in \mathcal{O}, O \supset B\}
\]

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$.

Then $\mu_1^+$ is a regular fuzzy measure.

2. We put

\[
\mu_1^-(O) = \sup\{-I(-f) | f \in K_1^+, support(f) \subset O\},
\]

and

\[
\mu_1^-(B) = \inf\{\mu_1^-(O) | O \in \mathcal{O}, O \supset B\}
\]

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$.

Then $\mu_1^-$ is a regular fuzzy measure.

3. A c.m. functional is a r.s.d functional, that is,

\[
I(f) = (C) \int (f \vee 0) d\mu_1^+ - (C) \int -(f \wedge 0) d\mu_1^-
\]

for every $f \in K$. 


(4) If $X$ is compact, then a c.m. functional can be represented by one Choquet integral.

(5) If $X$ is locally compact but not compact, then a r.s.d functional is a c.m. functional.

**Definition 3.3.** Let $I$ be a c.m. functional on $K$. We say that $\mu_I^+$ defined in Theorem 3.2 is the regular fuzzy measure induced by the positive part of $I$, and $\mu_I^-$ the regular fuzzy measure induced by the negative part of $I$.

**Definition 3.4.** Let $I$ be a real valued functional on $K$.

1. $I$ is said to be **bounded above** if there exists $M > 0$ such that

   \[ I(f) \leq M\|f\| \]

   for all $f \in K$.

2. $I$ is said to be **bounded below** if there exists $M > 0$ such that

   \[ -M\|f\| \leq I(f) \]

   for all $f \in K$.

3. $I$ is said to be **bounded** if $I$ is bounded above and below.

**Proposition 3.5.** [8, 11] Let $I$ be a c.m. functional on $K$ and $\mu_I^+$ and $\mu_I^-$ the regular fuzzy measure induced by $I$.

1. $I$ is bounded above iff $\mu_I^+(X) < \infty$.

2. $I$ is bounded below iff $\mu_I^-(X) < \infty$.

**Proposition 3.6.** [10] Let $X$ be separable and $I$ be a c.m. functional on $K$ that is bounded, and $\mu_I^+$ and $\mu_I^-$ the regular fuzzy measure induced by $I$. 
(1) If \((C) \int fd\mu_{I}^{+} = (C) \int fd(\mu_{I}^{-})^{c}\) for all \(f \in K^{+}\), then \(\mu_{I}^{+}(C) = (\mu_{I}^{-})^{c}(C)\) for all \(C \in \mathcal{C}\).

(2) If \((C) \int fd\mu_{I}^{-} = (C) \int fd(\mu_{I}^{+})^{c}\) for all \(f \in K_{f}\), then \(\mu_{I}^{-}(C) = (\mu_{I}^{+})^{c}(C)\) for all \(C \in \mathcal{C}\).

Proposition 3.6 says that if a c.m. functional \(I\) is Choquet integral with respect to \(\mu_{I}^{+}\) then we have \(\mu_{I}^{-}(C) = (\mu_{I}^{+})^{c}(C)\) for every \(C \in \mathcal{C}\). Since \((\mu_{I}^{+})^{c}\) is not always regular, it is not always true that \(\mu_{I}^{-} = (\mu_{I}^{+})^{c}\). That is, \(I\) is not always Choquet functional. See the example in [8].

4 Conjugate condition for compact sets

Definition 4.1. Let \(I\) be a c.m. functional and \(C \in \mathcal{C}\).

(1) We say that \(I\) satisfies the positive conjugate condition for \(C\) if there exists a positive real number \(M\) such that for any \(\epsilon > 0\) there exist \(f_{1}, f_{2} \in K_{1}\) satisfying the next condition.

\[1_{C} \leq g_{1} \leq f_{1} \text{ and } f_{2} \leq g_{2} \leq 1_{C^{c}} \text{ with } supp(f_{2}) \subset supp(g_{2}) \subset C^{c} \text{ imply} \]

\[|I(-g_{1}) - I(g_{2}) + M| < \epsilon\]

for \(g_{1}, g_{2} \in K_{1}\).

(2) We say that \(I\) satisfies the negative conjugate condition for \(C\) if there exists a positive real number \(M\) such that for any \(\epsilon > 0\) there exist \(f_{1}, f_{2} \in K_{1}\) satisfying the next condition.
\[ 1_C \leq g_1 \leq f_1 \text{ and } f_2 \leq g_2 \leq 1_{C^c} \text{ with } \text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c \text{ imply} \]

\[ |-I(g_1) + I(-g_2) + M| < \epsilon. \]

for \( g_1, g_2 \in K_1 \).

Suppose that a c.m. functional \( I \) satisfies the positive conjugate condition for \( \emptyset \). Let \( g_1(x) = 0 \) for all \( x \in X \). Since \( \emptyset \subset \text{supp}(g_1) \) and \( I(g_1) = 0 \), there exists \( M > 0 \) and for any \( \epsilon > 0 \) there exists \( f_2 \in K_1^+ \) such that \( \text{supp}(f_2) \subset \text{supp}(g_2) \subset X \) implies

\[ |-I(g_2) + M| < \epsilon. \]

Therefore we have the next proposition.

**Proposition 4.2.** Let \( I \) be a c.m. functional.

1. If \( I \) satisfies the positive conjugate condition for \( \emptyset \), then \( I \) is bounded above.
2. If \( I \) satisfies the negative conjugate condition for \( \emptyset \), then \( I \) is bounded below.

The next lemma follows from the definition of the induced regular fuzzy measure.

**Lemma 4.3.** Let \( A \in B \) and \( f \in K^+ \). Suppose that \( A \subset \{x|f \geq 1\} \), then we have

\[ \mu_I^+(A) \leq I(f) \text{ and } \mu_I^-(A) \leq -I(-f). \]

Applying this lemma, we have the next theorem. The detail of the proof is in Section 5.

**Theorem 4.4.** Let \( C \in \mathcal{C} \), \( I \) be a c.m. functional and \( \mu_I^+ \) and \( \mu_I^- \) the regular fuzzy measure induced by \( I \).

1. \( I \) satisfies the positive conjugate condition for every \( C \in \mathcal{C} \) if and only if

\[ \mu_I^-(C) = (\mu_I^+)^c(C) \]

for every \( C \in \mathcal{C} \).
(2) *I satisfies the negative conjugate condition for C if and only if*

\[ \mu_I^+(C) = (\mu_I^-)^c(C) \]

*for every C \in \mathcal{C}.*

Suppose that a c.m. functional \( I \) satisfies the positive conjugate condition for all \( C \in \mathcal{C} \). It follows from Theorem 4.4 that

\[
\mu_I^-(X) = \sup\{\mu_I^-(C) | C \subset X\} \\
= \sup\{(\mu_I^+)^c(C) | C \subset X\} \\
= \sup\{\mu_I^+(X) - \mu_I^+(C^c) | C \subset X\} \leq \mu_I^+(X).
\]

Therefore we have the next corollary.

**Corollary 4.5.** If a c.m. functional \( I \) satisfies the positive or negative conjugate condition for all \( C \in \mathcal{C} \), then \( I \) is bounded.

It follow from Theorem 4.4 that

\[
\mu_I^-(\{x|f(x) \geq r\}) = (\mu_I^+)^c(\{x|f(x) \geq r\})
\]

for all \( f \in K \) and \( r > 0 \). Therefore we have the next theorem.

**Theorem 4.6.** Let \( I \) be a c.m. functional.

(1) If \( I \) satisfies the positive conjugate condition for all \( C \in \mathcal{C} \), we have

\[
I(f) = (C) \int f \, d\mu_I^+
\]

for all \( f \in K \).
(2) If $I$ satisfies the negative conjugate condition for all $C \in C$, we have

$$I(f) = -(C) \int -f \, d\mu_I^-$$

for all $f \in K$.

The next theorem follows from Proposition 3.6

**Theorem 4.7.** Let $X$ be separable and $I$ be a c.m. functional on $K$ that is bounded, and $\mu_I^+$ and $\mu_I^-$ the regular fuzzy measure induced by $I$.

1. If $I(f) = (C) \int f \, d\mu_I^+$ for all $f \in K$, then $I$ satisfies the positive conjugate condition for all $C \in C$.

2. If $I(f) = -(C) \int -f \, d\mu_I^-$ for all $f \in K$, then $I$ satisfies the negative conjugate condition for all $C \in C$.

## 5 Proof of Theorem 4.4

In this section, the proof of Theorem 4.4 (1) is shown. The proof of Theorem 4.4 (2) is much the same.

Let $\epsilon > 0$ and $C \in C$.

First suppose that a c.m. functional $I$ satisfies the positive conjugate condition for every compact set $C$. That is, there exists a positive real number $M$ such that $\forall \epsilon > 0$, $\exists f_1, f_2 \in K_1$ , $1_C \leq g_1 \leq f_1$ and $f_2 \leq g_2 \leq 1_{C^c}$ with $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$ imply

$$M - I(g_2) - \epsilon < -I(-g_1) < M - I(g_2) + \epsilon$$

(1)

for $g_1, g_2 \in K_1$. 
Since $\mu^-_I$ is regular, there exists an open set $O$ such that $C \subset O$ and
\[ \mu^-_I(C) + \epsilon \geq \mu^-_I(O). \tag{2} \]

Using Uryson's lemma, there exists $h_1 \in K^+_1$ such that $1_C \leq h_1 \leq 1_O$. Since $1_C \leq f_1$, we may suppose that $f_1 \geq h_1$. It follows from Lemma 4.3 that
\[ \mu^-_I(C) \leq -I(-h_1). \tag{3} \]

Since $\text{supp}(h_1) \subset O$, we have
\[ \mu^-_I(O) \geq -I(-h_1) \tag{4} \]
from the definition of $\mu^-_I$. Then it follows from (2) and (4) that
\[ \mu^-_I(C) + \epsilon \geq -I(-h_1). \tag{5} \]

Since $C^c$ is an open set, it follows from the definition of the induced regular fuzzy measure $\mu^+_I$ that there exists $h_2 \in K^+_1$ such that $\text{supp}(h_2) \subset C^c$ and
\[ I(h_2) \geq \mu^+_I(C^c) - \epsilon. \tag{6} \]

We may suppose that $f_2 \leq h_2 \leq 1_{C^c}$. Then applying (5) and (6), we have
\[ \mu^-_I(C) + \epsilon \geq M - I(h_2) - \epsilon. \tag{7} \]

Since we have $I(h_2) \leq \mu^+_I(C^c)$ from $\text{supp}(h_2) \subset C^c$, we have
\[ \mu^-_I(C) + \epsilon \geq M - \mu^+_I(C^c) - \epsilon. \tag{8} \]

Since $I$ satisfies the conjugate condition for $\emptyset$, we have $M = \mu^+_I(X)$. Therefore we have
\[ 2\epsilon \geq (\mu^+_I)^c(C) - \mu^-_I(C) \tag{9} \]
from (8).

On the other hand, it follows from (1),(2) and (6) that

\[-I(-h_1) \leq M - I(h_0) + \epsilon\]
\[\leq M - (\mu_I^+(C^c) - \epsilon) + \epsilon\]
\[\leq (\mu_I^+)^c(C) + 2\epsilon.\]

Therefore we have

\[|\mu_I^-(C) - (\mu_I^+)^c(C)| \leq 2\epsilon.\]

Since \(\epsilon\) is an arbitrary, we have \(\mu_I^-(C) = (\mu_I^+)^c(C)\).

Next suppose that \(\mu_I^-(C) = (\mu_I^+)^c(C)\). Define \(M = \mu_I^+(X)\). Then it follows from the definition of the conjugate of \(\mu_I^-\) that

\[\mu_I^-(C) = M - \mu_I^-(C^c).\] (10)

Since \(\mu_I^-\) is regular, there exists an open set \(O\) such that \(O \supset C\) and

\[\mu_I^-(C) + \epsilon \geq \mu_I^-(O).\] (11)

Using Uryson's lemma, there exists \(f_1 \in K_1^+\) such that \(1_C \leq f_1 \leq 1_O\). Then for every \(g_1 \in K_1^+\) such that \(1_C \leq g_1 \leq f_1\), we have

\[\mu_I^-(O) \geq -I(-g_1) \geq \mu_I^-(C)\] (12)

from Lemma 4.3. It follows from the definition of the induced regular fuzzy measure \(\mu_I^+\) that there exists \(f_2 \in K_1^+\) such that \(supp(f_2) \subset C^c\) and

\[\mu_I^+(C^c) - \epsilon \leq I(f_2).\] (13)
Therefore for every $g_2 \in K_1^+$ such that $f_2 \leq g_2 \leq C^c$ and $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$, we have

$$\mu_f^+(C^c) - \epsilon \leq I(f_2) \leq I(g_2) \leq \mu_f^+(C^c).$$  \hfill (14)

It follows from (10), (11) and (12) that

$$M - \mu_f^+(C^c) + \epsilon \geq -(-g_2).$$

Then we have

$$\epsilon \geq -M - I(-g_1) + I(g_2)$$ \hfill (15)

from (14). On the other hand, it follows from (10) and (14) that

$$I(g_2) + \epsilon \geq M - \mu_f^{-}(C).$$ \hfill (16)

Then we have

$$\epsilon \geq M - I(g_2) + I(-g_1)$$ \hfill (17)

from (12). Therefore we have

$$|I(-g_1) - I(g_2) + M| < \epsilon$$

from (15) and (17).

\[ \square \]

**References**


