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A scale-invariant form of Trudinger-Moser inequality and its best exponent

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0. Introduction

In this note, we study the limit case of Sobolev's inequalities; suppose $N \geq 2$ and let $D \subset \mathbb{R}^N$ be an open set. We denote by $W_0^{1,N}(D)$ the usual Sobolev space with the norm $\|u\|_{W_0^{1,N}(D)} = \|
abla u\|_p + \|u\|_p$. Here

$$\|u\|_p = \left( \int |u|^p \, dx \right)^{1/p}.$$  

The case $p = N$ is the limit case of Sobolev imbeddings and it is known that

$$W_0^{1,N}(D) \subset L^q(D) \quad \text{for} \quad N \leq q < \infty,$$
$$W_0^{1,N}(D) \not\subset L^\infty(D).$$

This case is studied by Trudinger [8] more precisely and he showed for bounded domains $D \subset \mathbb{R}^N$

$$\int_D \exp \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^{N/(N-1)} \right) \, dx \leq C |D| \quad (0.1)$$

for $u \in W_0^{1,N}(D) \setminus \{0\}$, where the constants $\alpha$, $C$ are independent of $u$ and $D$.

Trudinger's result is extended into two directions; the first one is to find the best exponents in (0.1). Moser [4] proved that (0.1) holds for $\alpha \leq \alpha_N$ but not for $\alpha > \alpha_N$, where

$$\alpha_N = N \omega_{N-1}^{1/(N-1)} \quad (0.2)$$

and $\omega_{N-1}$ is the surface area of the unit sphere in $\mathbb{R}^N$. See also D. R. Adams [2]. The second direction is to extend Trudinger's result for unbounded domains and for Sobolev
spaces of higher order and fractional order. We refer to R. A. Adams [3], Ogawa [5], Ogawa-Ozawa [6], Ozawa [7].

Here, we study a version of Trudinger inequalities in $\mathbb{R}^N$ and their best exponents; we show

$$\int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) \, dx \leq C \frac{\|u(x)\|_N^{N}}{\|\nabla u\|_N^{N}}$$

(0.3)

for $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, where

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!} \xi^{N-2j}$$

and $\alpha, C > 0$ is independent of $u$. This type of inequality was first introduced in [5] for $N = 2$ and extended in [7] for $N \geq 3$ and for Sobolev spaces of fractional order. As to the proof of the inequality (0.3), following the original idea of Trudinger, [5, 6, 7] made use of a combination of the power series expansion of the exponential function and sharp multiplicative inequalities:

$$\|u\|_q \leq C(N,q) \|u\|_N^{\frac{N}{N/q}} \|\nabla u\|_N^{1-N/q}.$$

Our aim is to give a simplified proof of (0.3) and the best exponents $\alpha$ for (0.3).

One of the virtue of the inequality (0.3) is its scale-invariance; for $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$ and $\lambda > 0$, we set

$$u_{\lambda}(x) = u(\lambda x).$$

(0.4)

We can easily see that

$$\|\nabla u_{\lambda}\|_N = \|\nabla u\|_N,$$

$$\|u_{\lambda}\|_N^N = \lambda^{-N} \|u\|_N^N,$$

$$\int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{|u_{\lambda}(x)|}{\|\nabla u_{\lambda}\|_N} \right)^{\frac{N}{N-1}} \right) \, dx = \lambda^{-N} \int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) \, dx.$$

Thus (0.3) is invariant under the scaling (0.4).

Our main result is the following.

**Theorem 0.1 ([1]).** Suppose $N \geq 2$. Then for any $\alpha \in (0, \alpha_N)$, where $\alpha_N$ is given in (0.2), there exists a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) \, dx \leq C_\alpha \frac{\|u(x)\|_N^{N}}{\|\nabla u\|_N^{N}}$$

for $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}.$

(0.5)
Next we show that the restriction $\alpha < \alpha_N$ is optimal. The limit exponent $\alpha_N$ is excluded for (0.5). It is quite different from Moser's result for (0.1).

**Theorem 0.2 ([1]).** For $\alpha \geq \alpha_N$, there exists a sequence $(u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbb{R}^N)$ such that

$$\left\| \nabla u_k \right\|_N = 1$$

and

$$\frac{1}{\left\| u_k \right\|_N^N} \int_{\mathbb{R}^N} \Phi_N \left( \alpha \left| u_k(x) \right|^{\frac{N}{N-1}} \right) \, dx \geq \frac{1}{\left\| u_k \right\|_N^N} \int_{\mathbb{R}^N} \Phi_N \left( \alpha_N \left| u_k(x) \right|^{\frac{N}{N-1}} \right) \, dx$$

as $k \to \infty$.

1. **Proof of Theorem 0.1**

To prove Theorem 0.1, we use an idea of Moser [4]. By means of symmetrization, it suffices to show the desired inequality (0.5) for functions $u(x) = u(|x|)$, which are non-negative, compactly supported, radially symmetric, and $u(|x|) : [0, \infty) \to \mathbb{R}$ are decreasing.

Following Moser's argument, we set

$$w(t) = N^\frac{N-1}{N} \omega_{N-1}^\frac{1}{N} u \left( e^{-\frac{t}{N}} \right), \quad |x|^N = e^{-t}. \quad (1.1)$$

Then $w(t)$ is defined on $(\mathbb{R})$ and satisfies

$$w(t) \geq 0 \quad \text{for } t \in \mathbb{R}, \quad (1.2)$$

$$\dot{w}(t) \geq 0 \quad \text{for } t \in \mathbb{R}, \quad (1.3)$$

$$w(t_0) = 0 \quad \text{for some } t_0 \in \mathbb{R}. \quad (1.4)$$

Moreover we have

$$\int_{\mathbb{R}^N} |\nabla u|^N \, dx = \int_{-\infty}^{\infty} |\dot{w}(t)|^N \, dt, \quad (1.5)$$

$$\int_{\mathbb{R}^N} \Phi_N \left( \alpha u^{\frac{N}{N-1}} \right) \, dx = \frac{\omega_{N-1}}{N} \int_{-\infty}^{\infty} \Phi_N \left( \frac{\alpha}{\alpha_N} w(t)^{\frac{N}{N-1}} \right) e^{-t} \, dt, \quad (1.6)$$

$$\int_{\mathbb{R}^N} |u(x)|^N \, dx = \frac{1}{N^N} \int_{-\infty}^{\infty} |w(t)|^N e^{-t} \, dt. \quad (1.7)$$

Thus, to prove Theorem 0.1, it suffices to show that for any $\beta \in (0,1)$ there exists a constant $C_\beta > 0$ such that

$$\int_{-\infty}^{\infty} \Phi_N \left( \beta w(t)^{\frac{N}{N-1}} \right) e^{-t} \, dt \leq C_\beta \int_{-\infty}^{\infty} |w(t)|^N e^{-t} \, dt \quad (1.8)$$
for all functions \( w(t) \) satisfying (1.2)-(1.4) and
\[
\int_{-\infty}^{\infty} \left| \dot{w}(t) \right|^N dt = 1. \tag{1.9}
\]

**Proof of Theorem 0.1.** Let \( w(t) \) be a function satisfying (1.2)-(1.4) and (1.9). We set
\[
T_0 = \sup \{ t \in \mathbb{R}; w(t) \leq 1 \} \in (-\infty, \infty].
\]
We decompose the integral in the left hand side of (1.8) according to the decomposition
\((-\infty, \infty) = (-\infty, T_0] \cup [T_0, \infty)\).
For \( t \in (-\infty, T_0] \), we have \( w(t) \in [0,1] \). We can find a constant \( m_N > 0 \) such that
\[
\Phi_N(\xi) \leq m_N \xi^{N-1} \quad \text{for} \quad \xi \in [0,1].
\]
Thus we have
\[
\int_{-\infty}^{T_0} \Phi_N \left( \beta w(t)^{N-1} \right) e^{-t} dt \leq m_N \int_{-\infty}^{T_0} w(t)^N e^{-t} dt. \tag{1.10}
\]
Next we consider the integral over \([T_0, \infty)\). Since \( w(T_0) = 1 \), we have for \( t \geq T_0 \)
\[
w(t) = w(T_0) + \int_{T_0}^{t} \dot{w}(\tau) d\tau
\leq w(T_0) + (t - T_0)^{N-1} \left( \int_{T_0}^{\infty} \dot{w}(\tau)^N d\tau \right)^{\frac{1}{N}}
\leq 1 + (t - T_0)^{N-1}.
\]
We remark that for any \( \epsilon > 0 \) there exists a constant \( C_\epsilon > 0 \) such that
\[
1 + s^{\frac{N-1}{N}} \leq ((1 + \epsilon)s + C_\epsilon)^{\frac{N-1}{N}} \quad \text{for all} \quad s \geq 0.
\]
Thus, we have
\[
|w(t)|^{\frac{N}{N-1}} \leq (1 + \epsilon)(t - T_0) + C_\epsilon \quad \text{for} \quad t \geq T_0.
\]
Since \( \beta \in (0,1) \), we can choose \( \epsilon > 0 \) small so that \( \beta(1 + \epsilon) < 1 \). Thus we have
\[
\int_{T_0}^{\infty} \Phi_N \left( \beta w(t)^{N-1} \right) e^{-t} dt \leq \int_{T_0}^{\infty} \exp \left( \beta w(t)^{N-1} - t \right) dt
\leq \int_{T_0}^{\infty} \exp \left( (\beta(1 + \epsilon) - 1)(t - T_0) + \beta C_\epsilon - T_0 \right) dt
= \frac{1}{1 - \beta(1 + \epsilon)} e^{\beta C_\epsilon} e^{-T_0}. \tag{1.11}
\]
On the other hand,
\[ \int_{T_0}^{\infty} |w(t)|^N e^{-t} \, dt \geq \int_{T_0}^{\infty} e^{-t} \, dt = e^{-T_0}. \]  
(1.12)

Therefore it follows from (1.11) and (1.12) that
\[ \int_{T_0}^{\infty} \Phi_N \left( \beta w(t)^{\frac{N}{N-1}} \right) e^{-t} \, dt \leq \frac{e^{\beta G_e}}{1 - \beta(1 + \epsilon)} \int_{T_0}^{\infty} |w(t)|^N e^{-t} \, dt. \]  
(1.13)

Thus, setting \( C_{\beta} = \max \{ m_N, \frac{e^{\beta G_e}}{1 - \beta(1 + \epsilon)} \} \), we obtain (1.8).

2. Proof of Theorem 0.2

It suffices to show Theorem 0.2 for \( \alpha = \alpha_N \). We use the idea of Moser again. Repeating the argument of the previous section, it suffices to find a sequence of functions \( w_k(t) : \mathbb{R} \to \mathbb{R} \) which satisfies (1.1)-(1.4), (1.9) and
\[ \int_{-\infty}^{\infty} |w_k(t)|^N e^{-t} \, dt \to 0 \quad \text{as } k \to \infty, \]  
(2.1)
\[ \int_{-\infty}^{\infty} \Phi_N \left( w_k(t)^{\frac{N}{N-1}} \right) e^{-t} \, dt \geq \frac{1}{2} \quad \text{for large } k. \]  
(2.2)

If we define a sequence of functions \( (u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbb{R}^N) \) from \( (w_k(t))_{k=1}^{\infty} \) through the relation (1.1), it satisfies (0.6) and (0.7).

Here we give an example of \( (w_k(t))_{k=1}^{\infty} \) explicitly:
\[
 w_k(t) = \begin{cases} 
 0 & \text{for } t \leq 0, \\
 k^{\frac{N-1}{N}} \frac{t}{k} & \text{for } 0 \leq t \leq k, \\
 k^{\frac{N-1}{N}} & \text{for } k \leq t.
\end{cases}
\]

Such functions appeared in [4] to show that the integral in the left hand side of (0.1) can be made arbitrarily large for \( \alpha > \alpha_N \).

References


