An Estimate on the Heat Kernel of Magnetic Schrödinger Operators and Uniformly Elliptic Operators with Non-negative Potentials (Harmonic Analysis and Nonlinear Partial Differential Equations)

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An Estimate on the Heat Kernel of Magnetic Schrödinger Operators and Uniformly Elliptic Operators with Non-negative Potentials

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Abstract

In this paper we show an estimate of the heat kernel to the Schrödinger operator with magnetic fields and to uniformly elliptic operators with non-negative potentials which belongs to the reverse Hölder class. We also give a weighted smoothing estimates for the semigroup generated by the operators above.

1 Introduction and Main Results

We consider the uniformly elliptic operator $L_E = -\nabla(A(x)\nabla)+V(x)$ with certain non-negative potential $V$ and the Schrödinger operator $L_M = (i^{-1}\nabla-a(x))^2+V(x)$ with a magnetic field $a(x) = (a_1(x), \cdots, a_n(x)), n \geq 2$. We use the notation $L_J$ for $J = E$ or $J = M$. The purpose of this paper is to give an estimate of the fundamental solution (or heat kernel) $\Gamma_J(x,t:y,s)$ to

$$(\partial_t + L_J)u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty),$$  \hfill (1)

namely $\Gamma_J(x,t:y,s)$ satisfies

$$(\partial_t + L_J)\Gamma_J(x,t;y,s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \hfill (2)$$

$$\lim_{t \to s}\Gamma_J(x,t;y,s) = \delta(x-y). \hfill (3)$$
For the elliptic operator $L_E$, we assume the following conditions for $A(x) = (a_{ij}(x))$.

**Assumption (A.1):** $a_{ij}(x)$ is a real-valued measurable function and satisfies $a_{ij}(x) = a_{ji}(x)$ for every $i, j = 1, \cdots, n$ and $x \in \mathbb{R}^n$.

**Assumption (A.2):** There exists a constant $\lambda > 0$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi^i \xi^j \leq \lambda |\xi|^2,$$

$x = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$. (4)

To state our assumptions on $V$ and $a$, we prepare some notations. We say $U \in (RH)_\infty$ if $U \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\sup_{y \in B(x,r)} |U(y)| \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| \, dy,$$

and say $U \in (RH)_q$ if $U \in L^q_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)|^q \, dy \right)^{1/q} \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| \, dy,$$

for some constant $C$ and for every $x \in \mathbb{R}^n$ and $r > 0$, respectively. We can define the function $m(x, U)$ for $U \in (RH)_q$ with $q > n/2$ as follows:

$$\frac{1}{m(x, U)} = \sup \{ r > 0 ; \frac{r^2}{|B(x,r)|} \int_{B(x,r)} U(y) \, dy \leq 1 \}.$$ (7)

We note that if there exist positive constants $K_1$ and $K_2$ such that $K_1 U_1(x) \leq U_2(x) \leq K_2 U_1(x)$, then it is easy to see that there exist positive constants $K'_1$ and $K'_2$ such that

$$K'_1 m(x, U_1) \leq m(x, U_2) \leq K'_2 m(x, U_1).$$

When $n \geq 3$, since it is known $U \in (RH)_{n/2}$ actually belongs to $(RH)_{n/2+\epsilon}$ for some $\epsilon > 0$, $m(x, U)$ can be defined for $U \in (RH)_{n/2}$ ([Sh1]). For other properties of the class $(RH)_q$, see, e.g., [KS]. We denote by $B(x) = (B_{jk}(x))$ the magnetic field defined by $B_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)$. We use the notation $m_J(x)$:

$$m_J(x) = m(x, V), \quad m_M(x) = m(x, |B| + V)$$

for the operator $L_J$, $J = E$ or $M$, respectively. We assume the following conditions for $V(x)$ and $a(x) = (a_1(x), \cdots, a_n(x))$. 

ASSUMPTION($V, a, B$): For each $j = 1, \ldots, n$, $a_j(x)$ is a real-valued $C^1(\mathbb{R}^n)$-function, $V$ is non-negative.

(i) For $n \geq 3$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_{n/2}, \quad |\nabla B(x)| \leq C m(x, V + |B|)^3.$$

(ii) For $n = 2$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_q, \quad |\nabla B(x)| \leq C m(x, V + |B|)^3,$$

for some $q > 1$.

Remark 1 For $n = 2$, we may assume the condition (ii') instead of (ii) by employing Lemma 1 (b).

(ii') $V \in L^\infty_{\text{loc}}(\mathbb{R}^2), B(x) \geq 0$ and that $m_J(x)$ satisfies

$$C_1 \frac{m_J(x)}{(1 + |x - y|m_J(x))^{k_0/k_0 + 1}} \leq m_J(y) \leq C_2 (1 + |x - y|m_J(x))^{k_0} m_J(x) \quad (8)$$

for some positive constants $C_1, C_2, k_0$ and for every $x, y \in \mathbb{R}^2$, where $m_E(x) = \sqrt{V(x)}$ and $m_M(x) = \sqrt{V(x) + B(x)}$.

We remark that it is known that $m_J(x)$ satisfies (8) under the assumption $(V, a, B)$ for $n \geq 3$([Sh1]) and even for $n = 2$ in the same way. We also note that if $|B| + V \in (RH)_\infty$, then it is easy to see that $|B(x)| + V(x) \leq C m(x, |B| + V)^2$ holds. For example, the condition $|B| + V \in (RH)_\infty$ is satisfied for any $a_j(x) = Q_j(x), V(x) = |P(x)|^\alpha$, where $P(x)$ and $Q_j(x), j = 1, \ldots, n$, are polynomials and $\alpha$ is a positive constant. In this case, under the assumption $(V, a, B)$ (i) or (ii), we see that there exists a positive constant $m_0$ such that $m_J(x) \geq m_0$, although in general we cannot say $|B| + V$ is strictly positive for imhomogeneous polynomials.

To state our main result, we introduce the notation:

$$\Gamma_{C_0}(x, t; y, s) = \frac{1}{(t - s)^{n/2}} \exp\left(-C_0 \frac{|x - y|^2}{t - s}\right)$$

for some positive constant $C_0$. 


Theorem 1  (a) Suppose $A(x)$ and $V(x)$ satisfy the assumptions $(A.1), (A.2)$ and $(V, 0, 0)$. Then, there exist positive constants $\alpha_0$ and $C_j$ $(j = 0, 1, 2)$ such that

$$(0 \leq) \Gamma_E(x, t; y, s) \leq C_1 \exp \left(-C_2(1 + m_E(x)(t - s)^{1/2}/\alpha_0)\right) \Gamma_C_0(x, t; y, s)$$  \hfill (9)

for $x, y \in \mathbb{R}^n$ and $t > s > 0$.

(b) Suppose $V(x)$ and $a(x)$ satisfy the assumption $(V, a, B)$. Then, there exist positive constants $\alpha_0$ and $C_j$ $(j = 0, 1, 2)$ such that

$$|\Gamma_M(x, t; y, s)| \leq C_1 \exp \left(-C_2(1 + m_M(x)(t - s)^{1/2}/\alpha_0)\right) \Gamma_C_0(x, t; y, s)$$  \hfill (10)

for $x, y \in \mathbb{R}^n$ and $t > s > 0$.

The number $\alpha_0$ is actually defined by $\alpha_0 = 2/(k_0 + 1)$, where $k_0$ is the constant in (8). The exponent $\alpha_0/2$ would not be sharp. If we restrict for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, the following sharp estimate is known ([Ma], [Er1,2] for $n \geq 3$ and [LT] for $n = 2$):

$$|\Gamma_M(x, t; y, s)| \leq D_1 \exp(-D_2B_0t)\Gamma_D_0(x, t; y, s).$$

More detail informations on the constants $D_j$ $(j = 0, 1, 2)$ can be seen in those papers. By using the parabolic distance:

$$d_P((x, t), (y, s)) = \max(|x - y|, |t - s|^{1/2}),$$

we have the following decay estimate.

Corollary 1  (a) Under the same assumptions as in Theorem 1, there exist positive constants $C_j$ $(j = 1, 2)$ and $C_0$ such that

$$|\Gamma_J(x, t; y, s)| \leq C_1 \exp \left(-C_2(1 + m_J(x)d_P((x, t), (y, s))^{2\alpha_0/(\alpha_0 + 4)})\right) \Gamma_C_0(x, t; y, s)$$

for $J = E$ and $M$, for every $x, y \in \mathbb{R}^n$ and $t > s > 0$.

(b) Under the same assumptions as in Theorem 1, for each $k > 0$ there exist positive constants $C_k$ and $C_0$ such that

$$|\Gamma_J(x, t; y, s)| \leq \frac{C_k}{(1 + m_J(x)d_P((x, t), (y, s)))^k} \Gamma_C_0(x, t; y, s)$$

for $J = E$ and $M$. 

Remark 2 Actually we can show the estimate in Theorem 1 for the operators $L_E = -\nabla (A(x,t)\nabla) + V(x,t)$ with time-dependent coefficients, if we assume the uniform ellipticity (4) of $A(x,t)$ and the existence of constants $C_j, j = 1, 2$, such that $C_1 U(x) \leq V(x,t) \leq C_2 U(x)$ and $U$ satisfies the condition $(U,0,0)$. For the magnetic Schrödinger operator $L_M = (i^{-1}\nabla - a(x,t))^2 + V(x,t)$, the estimate in Theorem 1 still holds, if there exists positive constants $C_j, j = 1, \ldots, 5$, such that $C_1 U(x) \leq V(x,t) \leq C_2 U(x)$, $C_3 |B'(x)| \leq |B(x,t)| \leq C_4 |B'(x)|$, and $|\nabla B(x,t)| \leq C_5 m(x,|B'| + U)^3$, where $a(x,t)$ is $C^1$ and $B_{jk}(x,t) = \partial_j a(x,t) - \partial_k a_j(x,t)$ and $U(x)$ and $B'(x)$ satisfy the assumption $(U, a, B')$ (except $|\nabla B'(x)| \leq C m(x,|B'| + U)^3$), and if the upper bound:

$$\Gamma_M(x,t;y,s) \leq C \Gamma_{C_0}(x,t;y,s)$$

holds for some constants $C$ and $C_0$.

Remark 3 In particular, Corollary 1 (b) yields

$$|\Gamma_J(x,t;y,s)| \leq \frac{C_k}{(1 + m_J(x)|x-y|)^k(1 + m_J(x)|t-s|)^k} \Gamma_{C_0}(x,t;y,s)$$

for $J = E$ or $M$. Let $n \geq 3$. Then this implies

$$|\Gamma_J(x,y)| \leq \frac{C_k}{(1 + m_J(x)|x-y|)^k} \Gamma_{C_0}(x,t;y,s)$$

where $\Gamma_J(x,y)$ is the fundamental solution to $L_J u = 0$. This estimate for the elliptic operator was proved by Shen [Sh1,2]. Thus, Corollary 1 (b) is a generalization of his estimate.

Remark 4 Recently we are informed by Z. Shen that he obtained the following shape estimate [Sh3] for the elliptic operators: under the assumption $V \in (RH)_{n/2}$ for $n \geq 3$ and $V \in (RH)_q$ with $q > 1$ for $n = 2$,

$$C_1 \exp(-C_2 d(x,y))|x-y|^{2-n} \leq \Gamma_E(x,y) \leq C_3 \exp(-C_4 d(x,y))|x-y|^{2-n}$$

holds for some positive constants $C_j (j = 1, 2, 3, 4)$, where $d(x,y)$ is defined by

$$d(x,y) = \inf_{\gamma} \int_0^1 m(\gamma(t),V)\left| \left(\frac{d\gamma}{dt}\right)(t) \right| dt.$$
Here the infimum is taken over all curves \( \gamma \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). Moreover, he gave the following estimate:

\[
C_1 (1 + m(x)|x - y|)^{\alpha_0/2} \leq d(x, y) \leq C_2 (1 + m(x)|x - y|)^{\beta_0}
\]

for some positive constants \( C_j (j = 1, 2) \) and \( \beta_0 \). In particular, it follows

\[
\Gamma_E(x, y) \leq C_5 \exp(-C_6 (1 + m_E(x)|x - y|)^{\alpha_0/2})|x - y|^{2-n}
\]

for some positive constants \( C_5 \) and \( C_6 \). We remark that this decay estimate also can be shown for the fundamental solution \( \Gamma_M(x, y) \) to \( L_M \) in a similar way. On the other hand, it follows from Corollary 1 (a) a somewhat weaker decay estimate:

\[
|\Gamma_J(x, y)| \leq C \exp(-C(1 + m_J(x)|x - y|)^{2\alpha_0/(\alpha_0 + 4)})|x - y|^{2-n}
\]

for \( J = E \) or \( M \). We do not know whether his sharp estimate can be generalized to heat kernel estimates or not.

We denote by \( e^{-tL_J} \) the semigroup generated by \( L_J \). Here we also denote by \( L_J \) the self-adjoint operator determined from the form associated with \( L_J \) (see, e.g., [Si], [LS]). We obtain the following weighted smoothing estimate by using Corollary 1 (b).

**Theorem 2** Assume the same assumptions as in Theorem 1. Let \( J = E \) or \( M \). Suppose \( 1 < p \leq q \leq +\infty \) and \( 1/p - 1/q < 1 \) and put \( \gamma = n(1/p - 1/q) \). Then for each \( l \in [0, (n - \gamma)/2] \) there exists a constant \( C_l \) such that

\[
\left\| m_J(x)^{2l} e^{-tL_J}f \right\|_{L^q(\mathbb{R}^n)} \leq \frac{C_l}{t^{l+(\gamma/2)}} \left\| f \right\|_{L^p(\mathbb{R}^n)}, \quad t > 0.
\]

**Corollary 2** Suppose the additional condition \( |B| + V \in (RH)_\infty \). Then we have the following estimates:

\[
\left\| (|B| + V)^l e^{-tL_J} f \right\|_{L^p(\mathbb{R}^n)} \leq \frac{C_l}{t^{l}} \left\| f \right\|_{L^p(\mathbb{R}^n)}, \quad t > 0
\]

holds for \( 1 < p < +\infty \) and \( l \in [0, n/2] \), and

\[
\left\| (|B| + V)^l e^{-tL_J} f \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_l}{t^{l+(n/2p)}} \left\| f \right\|_{L^p(\mathbb{R}^n)}, \quad t > 0
\]

holds for \( 1 \leq p < +\infty \) and \( l \in [0, n/(2p')] \). Here \( 1/p' = 1 - 1/p \) and \( C_l \) is a constant depending on \( l \) and \( p \).
Corollary 2 is an easy consequence of Theorem 2 by using the inequality $(|B| + V)(x) \leq C m_J(x)^2$. Note that (14) for the case $l = 0$ is a classical result.

Theorem 1 yields a weighted smoothing estimate with an exponential decay in time.

**Theorem 3** Assume the same assumptions as in Theorem 1 and the additional assumption $m_J(x) \geq m_0 > 0$.

(a) Let $1 \leq p < +\infty$ and $l \in [0, n/(2p')]$. Then we have

$$
\|m_J(x)^{2l} e^{-tLM} f\|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(-C(1 + m_0 t^{1/2})^{2p/2}\right) \frac{1}{t^{l+(n/p)}2} \|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0.
$$

(b) Let $1 \leq p \leq 2$ and $l \in [0, n/(2p')]$. Then we have

$$
\|m_J(x)^{2l} e^{-tLM} f\|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(-C(1 + m_0^2 t)\right) \frac{1}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0.
$$

Especially, for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, Theorem 3 (b) yields an exponential decay estimate in time:

$$
\|e^{-tLM} f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_1}{t^{n/2}} \exp\left(-C_2 B_0 t\right) \|f\|_{L^1(\mathbb{R}^n)}, \quad t > 0
$$

for some positive constant $C_1$ and $C_2$, which is known (see, e.g., [Ma], [Er1,2], [Ue], [LT]). Indeed, in this case $m_M(x) \sim \sqrt{B_0}$ holds. Note that Theorem 3 (a) gives weaker decay rate $e^{-C\sqrt{B_0} t}$, since $k_0 = 0$ and $\alpha_0 = 2$. We also emphasize that Theorem 3 can be applied to any polynomial like magnetic field $B(x)$ which may be zero somewhere.

**Definition 1** We say $u(x, t)$ is a complex-valued weak solution to

$$
(\partial_t + L_M) u = 0 \quad \text{in} \quad Q_r(x_0, t_0),
$$

if $u \in L^\infty((t_0 - r^2, t_0); L^2(B(x_0, r); \mathbb{C})) \cap L^2((t_0 - r^2, t_0); H^1(B(x_0, r); \mathbb{C}))$ and satisfies

$$
\int_{B(x_0, r)} u(x, t) \overline{\phi(x, t)} \, dx - \int_{t_0 - r^2}^{t} \int_{B(x_0, r)} u(x, s) \partial_s \overline{\phi(x, s)} \, dxds
$$

$$
+ \int_{t_0 - r^2}^{t} \int_{B(x_0, r)} \sum_{j=1}^{n} D_j^2 u(x, s) \overline{D_j^2 \phi(x, s)} \, dxds
$$

$$
+ \int_{t_0 - r^2}^{t} \int_{B(x_0, r)} V(x) u(x, s) \overline{\phi(x, s)} \, dxds = 0 \quad (16)
$$
for every \( \phi \in C \equiv \{ \phi \in L^2(\langle t_0-r^2, t_0 \rangle; H^1(B(x_0, r); \mathbb{C})); \partial_s \phi \in L^2(\langle t_0-r^2, t_0 \rangle; L^2(B(x_0, r); \mathbb{C})), \phi(x, t_0-r^2) = 0 \} \), where \( \overline{\phi} \) is the complex conjugate of \( \phi \).

Here, we used the notation \( D_j^a = i^{-1} \partial_{x_j} - a_j(x) \) and

\[
Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0 \}.
\]

A real-valued weak solution \( u \) to \( (\partial_t + L_E)u = 0 \) in \( Q_r(x_0, t_0) \) can be defined in a similar way. Our proof of Theorem 1 is based on the following subsolution estimate.

**Theorem 4** Let \( u(x, t) \) be a weak solution to \( \partial_t + L_J u = 0 \) in \( Q_{2r}(x_0, t_0) \). Then there exists positive constants \( C_j, j = 1, 2 \), such that

\[
\sup_{(x,t) \in Q_{r/2}(x_0,t_0)} |u(x,t)| \leq C_1 \exp\left(-C_2(1+rm_J(x_0))^{\alpha/2}\right) \left( \frac{1}{r^{n+2}} \int_{Q_{r}(x_0,t_0)} |u|^2 dxdt \right)^{1/2}.
\]

(17)

Throughout this paper, we use the following notation: \( D = i^{-1} \nabla - a \),

\[
B(x_0, r) = \{y \in \mathbb{R}^n; |y - x_0| < r \}, \quad \langle A \nabla u, \nabla u \rangle = \sum_{j,k=1}^n a_{jk} \partial_{x_j} u \partial_{x_k} u,
\]

\[
Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0 \}.
\]

## 2 Proof of Theorem 4

We use the following inequalities.

**Lemma 1** (a) ([Sh2]) Suppose \( n \geq 2 \) and \( V(x) \) and \( a(x) \) satisfy the condition \( (V, a, B) \). Then there exists a constant \( C_0 \) such that

\[
\int m(x, |B| + V)^2 |u|^2 dx \leq C_0 \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx
\]

for \( u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}) \).

(b) ([AHS]) Suppose \( n = 2, V \geq 0, V \in L_\text{loc}^\infty(\mathbb{R}^2), a \in C^1(\mathbb{R}^2) \), and \( B(x) \geq 0 \). Then the inequality

\[
\int (B(x) + V(x))|u|^2 dx \leq \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx
\]

holds for \( u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}) \).
We also prepare the following Caccioppoli-type inequality.

**Lemma 2** Let $0 < \sigma < 1$. Let $u$ be a weak solution to $(\partial_t + L_J)u = 0$ in $Q_{2r}(x_0, t_0)$ for $J = E$ or $J = M$. Then there exists a constant $C$ such that

$$\sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, \sigma r)} |u(x, t)|^2 \, dx + \int \int_{Q_{\sigma r}(x_0, t_0)} |(i^{-1} \nabla - a)u|^2 + V|u|^2 \, dx \, ds \leq \frac{C}{(1 - \sigma)^2 r^2} \int \int_{Q_r(x_0, t_0)} |u|^2 \, dx \, dt.$$

**Proof:** Although the proof is standard, we give it here for the sake of completeness. We show the estimate for a weak solution $u$ to $(\partial_t + L_E)u = 0$ in $Q_{2r}(x_0, t_0)$. Since we can show the estimate for a weak solution to $(\partial_t + L_M)u = 0$ in the similar way, we just mention some modifications we need at the end of this proof. Take functions $\chi(x) \in C^\infty_0(B(x_0, r))$ and $\eta(t) \in C^\infty(\mathbb{R}^1)$ satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) \equiv 1$ on $B(x_0, \sigma r)$ and $|\nabla \chi(x)| \leq C/(1 - \sigma)r$, and $0 \leq \eta(t) \leq 1$, $\eta(t) \equiv 1$ on $t \geq t_0 - (\sigma r)^2$, $\eta(t) \equiv 0$ on $t \leq t_0 - r^2$, $|\partial_t \eta(t)| \leq C/r^2(1 - \sigma^2)$. For the sake of simplicity, we also assume $\partial_t u \in L^2(Q_{2r}(x_0, t_0))$. Actually, we can remove this additional assumption by using the argument as in [AS]. Fix $t \in [t_0 - (\sigma r)^2, t_0]$. Multiplying $\eta^2(t)\chi^2(x)u(x, t)$ to the equation and integrating over $B(x_0, r) \times [t_0 - r^2, t]$, we have

$$\frac{1}{2} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 \, dx + \int_{t_0 - r^2}^t \int_{B(x_0, r)} \left\langle A(x) \nabla u(x, s), \nabla u(x, s) \right\rangle \eta(s)^2 \chi(x)^2 \, dx \, ds$$

$$+ \int_{t_0 - r^2}^t \int_{B(x_0, r)} V(x)u(x, s)^2 \eta(s)^2 \chi(x)^2 \, dx \, ds = \int_{t_0 - r^2}^t \int_{B(x_0, r)} u(x, s)^2 \chi(x)^2 \eta(s) \partial_s \eta(s) \, dx \, ds$$

$$- \int_{t_0 - r^2}^t \int_{B(x_0, r)} \left\langle A(x) \nabla u(x, s), \nabla \chi^2(x) \right\rangle \eta(s)^2 u(x, s) \, dx \, ds. \tag{18}$$

Because of the ellipticity of $A(x)$ and the positivity of $V$, we obtain by (18)

$$\sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 \, dx \leq \int \int_{Q_r(x_0, t_0)} u^2 |\partial_s \eta| \, dx \, ds \tag{19}$$
\[ + \int \int_{Q_{r}(x_{0},t_{0})} |\nabla u|||u^{2}|\nabla \chi| \, dx \, ds \]
\[ \leq \frac{C}{(1-\sigma)} \left\{ \frac{1}{r^{2}} \int \int_{Q_{r}(x_{0},t_{0})} u^{2} \, dx \, ds + \int \int_{Q_{r}(x_{0},t_{0})} \chi^{2} \eta^{2}|\nabla u|^{2} \, dx \, ds \right\}. \]

By using (18) again, we have

\[ \lambda \int \int_{Q_{r}(x_{0},t_{0})} |\nabla u|^{2} \chi^{2} \eta^{2} \, dx \, ds + \int \int_{Q_{r}(x_{0},t_{0})} V u^{2} \chi^{2} \eta^{2} \, dx \, ds \]
\[ \leq \int \int_{Q_{r}(x_{0},t_{0})} (A \nabla u, \nabla u) \partial_{k} w \chi^{2} \eta^{2} \, dx \, ds + \int \int_{Q_{r}(x_{0},t_{0})} V u^{2} \chi^{2} \eta^{2} \, dx \, ds \]
\[ \leq \frac{C}{(1-\sigma)^{2}} \int \int_{Q_{r}(x_{0},t_{0})} u^{2} \, dx \, ds + \frac{\lambda}{2} \int \int_{Q_{r}(x_{0},t_{0})} |\nabla u||\nabla \chi| \eta^{2}|u| \, dx \, ds \]
\[ \leq \frac{C}{(1-\sigma)^{2}} \int \int_{Q_{r}(x_{0},t_{0})} u^{2} \, dx \, ds. \quad (20) \]

It follows

\[ \frac{\lambda}{2} \int \int_{Q_{r}(x_{0},t_{0})} |\nabla u|^{2} \chi^{2} \eta^{2} \, dx \, ds + \int \int_{Q_{r}(x_{0},t_{0})} V u^{2} \chi^{2} \eta^{2} \, dx \, ds \]
\[ \leq \frac{C}{(1-\sigma)^{2}r^{2}} \int \int_{Q_{r}(x_{0},t_{0})} u^{2} \, dx \, ds. \quad (21) \]

(19) and (21) yield the desired result. For \( L_{M} \), we can prove in a similar way by noting the following identities:

\[ D^{a}_{j}(u \chi) = (D^{a}_{j}u) \chi + u (i^{-1} \nabla \chi), \quad \int D^{a}_{j}u \overline{v} \, dx = \int u \overline{D^{a}_{j}v} \, dx. \]

\[ \square \]

Proof of Theorem 3: Let \( k \in \mathbb{N} \) and define \( p_{j} \ (j = 1, 2, \ldots, k + 1) \) by \( p_{j} = 2/3 + ((j - 1)/k)(1 - (2/3)) \). Let \( \chi_{j}(x) \in C_{0}^{\infty}(B(x_{0}, p_{j} r)) \) and \( \eta_{j}(t) \in C^{\infty}(\mathbb{R}) \) be the functions satisfying \( 0 \leq \chi_{j} \leq 1, \chi_{j}(x) \equiv 1 \) on \( B(x_{0}, p_{j-1} r) \), \( |\nabla \chi_{j}(x)| \leq Ck/r \), and \( 0 \leq \eta_{j} \leq 1, \eta_{j}(t) \equiv 1 \) on \( t \geq t_{0} - (p_{j-1} r)^{2}, \eta_{j}(t) \equiv 0 \) on \( t \leq t_{0} - (p_{j} r)^{2} \), \( |\nabla \eta_{j}(t)| \leq Ck/r^{2} \). By Lemma 2 (see also (21)), we have

\[ \int \int_{Q_{p_{j+1}r}(x_{0},t_{0})} \left( |i^{-1} \nabla - a|u|^{2} \chi_{j+1}^{2} \eta_{j+1}^{2} + V|u|^{2} \chi_{j+1}^{2} \eta_{j+1}^{2} \right) \, dx \, ds \]
\[ \leq \frac{Ck^{2}}{r^{2}} \int \int_{Q_{p_{j+1}r}(x_{0},t_{0})} |u|^{2} \, dx \, ds. \]
We write just $\chi_j$ and $\eta_j$, for simplicity. Since $|(i^{-1}\nabla - a)(u\eta \chi)|^2 \leq 2|(i^{-1}\nabla - a)u|^2 \chi^2 \eta^2 + 2u^2 |\nabla \chi|^2 \eta^2$, it follows that

$$\int \int_{Q_{p_j+1}(x_0,t_0)} \left( |(i^{-1}\nabla - a)(\eta \chi u)|^2 \chi^2 \eta^2 + 4|u|^2 |\nabla \chi| \eta \right) dx \, ds \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_j+1}(x_0,t_0)} |u|^2 \, dx \, ds$$

for $j = 1, \cdots, k$. By using Lemma 1, we obtain

$$\int_{t_0-(p_{j+1}r)^2}^{t_0} \left( \int_{B(x_0,p_{j+1}r^2)} m_j(x)^2 |\eta \chi u|^2 \, dx \right) dt \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}}(x_0,t_0)} |u|^2 \, dx \, ds.$$

By using $m_j(x) \geq C(1 + p_{j+1}r m_j(x_0))^{-\kappa_0/(1+\kappa_0)} m_j(x_0)$ on $|x - x_0| < p_{j+1}r$ and noting $2/3 \leq p_{j+1} \leq 1$ (see (8) and the remark after that), we have

$$\int \int_{Q_{p_{j+1}}(x_0,t_0)} |u|^2 \, dx \, dt \leq \int_{x_0-(p_{j+1}r)^2}^{x_0} \left( \int_{B(x_0,p_{j+1}r^2)} |\eta \chi u|^2 \, dx \right) dt \leq \frac{Ck^2}{r^2 m_j(x_0)^2(1 + rm_j(x_0))^{2/(k_0+1)}} \int \int_{Q_{p_{j+1}}(x_0,t_0)} |u|^2 \, dx \, dt.$$

(22)

for each $j = 1, 2, \cdots, k$. Here we used a trivial inequality $\int \int_{Q_{p_{j+1}}(x_0,t_0)} (\cdots) \, dx \, dt \leq \int \int_{Q_{p_{j+1}}(x_0,t_0)} (\cdots) \, dx \, dt$ for the case $rm_j(x_0) \leq 1$. By this procedure, we can obtain the following: there exists a constant $C$ such that for every $k \in \mathbb{N}$

$$\int \int_{Q_{r/2}(x_0,t_0)} |u|^2 \, dx \, dt \leq \frac{C(k^2)^k}{(1 + rm_j(x_0))^{\kappa_0}} \int \int_{Q_{r}(x_0,t_0)} |u|^2 \, dx \, dt,$$

(23)

where $\alpha_0 = 2/(k_0 + 1)$. Since $V(x) \geq 0$, the well-known subsolution estimate (see, e.g., [AS]) yields

$$\sup_{Q_{r/2}(x_0,t_0)} |u| \leq C \left( \frac{1}{r^{n+2}} \int \int_{Q_{r/2}(x_0,t_0)} |u|^2 \, dx \, dt \right)^{1/2}$$

(24)

for some constant $C$. For the magnetic Schrödinger operator case, we have used Kato's inequality. Combining (23) and (24), we arrive at

$$\sup_{Q_{r/2}(x_0,t_0)} |u| \leq C \left( \frac{C^{k/2} k^k}{(1 + rm_j(x_0))^{\kappa_0/2}} \left( \frac{1}{r^{n+2}} \int \int_{Q_{r}(x_0,t_0)} |u|^2 \, dx \, dt \right)^{1/2} \right.$$

(25)
for every $k \in \mathbb{N}$. Note that, by Stirling's formula $k^k \sim e^k k!(1/\sqrt{2\pi k})$ as $k \to \infty$, there exists a constant $C_0$ such that $k^k \leq C_0 e^k k!$ for $k \geq 1$. Multiplying $e^k/k!$ and taking the summation, we obtain

$$
(\sup_{Q_{r/2}(x_0,t_0)} |u|) \sum_{k=1}^{\infty} \frac{(e(1 + rm_{J}(x_0))^{\alpha_0/2})^k}{k!} \leq C C_0 \sum_{k=1}^{\infty} (e \sqrt{C})^k \left( \frac{1}{r^{n+2}} \int \int_{Q_{r}(x_0, t_0)} |u|^2 \, dx \, dt \right)^{1/2}.
$$

Take $\epsilon > 0$ so that $\epsilon e \sqrt{C} < 1$. Then we have

$$
\sup_{Q_{r/2}(x_0,t_0)} |u| \leq C \exp(-e^k k!(1/\sqrt{2\pi k})) \left( \frac{1}{r^{n+2}} \int \int_{Q_{r}(x_0, t_0)} |u|^2 \, dx \, dt \right)^{1/2}.
$$

This complete the proof. □

3 Proof of Theorem 1

To show Theorem 1 we prove the following proposition.

**Proposition 1** Under the assumptions as in Theorem 1, there exist positive constants $C_1$ and $C_2$ such that

$$
|\Gamma_{J}(x,t;y,s)| \leq C_1 \exp(-C_2(1 + m_{J}(x))|t - s|^{1/2}) \frac{1}{(t - s)^{n/2}}
$$

(26)

for $x, y \in \mathbb{R}^n$ and $t > s > 0$.

**Proof:** Assume $t - s \geq 2|y - x|^2$. Take $r^2 = |t - s|/8$. Then $u(z, u) = \Gamma_{J}(z, u; y, s)$ satisfies $(\partial_t + L_{J})u(z, u) = 0$ in $Q_{2r}(x, t)$. Hence, by applying Theorem 4 to $u(z, u)$, we obtain

$$
|\Gamma_{J}(x,t;y,s)| \leq \sup_{Q_{r/2}(x,t)} |u|
$$

$$
\leq C \exp(-C(1 + m_{J}(x))|t - s|^{1/2}) \frac{1}{(t - s)^{n/2}} \left( \int \int_{Q_{r}(x,t)} |\Gamma(z,u;y,s)|^2 \, dz \, du \right)^{1/2}.
$$

By using the maximum principle for $L_E$ and the diamagnetic inequality (see, e.g., [AS], [LS], [AHS]) for $L_M$, we have

$$
|\Gamma_{J}(z,u;y,s)| \leq \frac{C}{(u - s)^{n/2}} \exp\left(-C \frac{|z - y|^2}{(u - s)}\right)
$$

(27)
for some constant $C = C(n, \lambda)$. Since $t - s \geq u - s \geq 7r^2 \geq (7/8)(t - s)$ on $(z, u) \in Q_r(x, t)$, it is easy to see

$$\left(\frac{1}{r^{n+2}} \int \int_{Q_r(x, t)} |\Gamma_J(z, u; y, s)|^2 dz du\right)^{1/2} \leq \frac{C}{(t - s)^{n/2}}.$$  

This yields the desired estimate. □

**Proof of Theorem 1:** The positivity of $\Gamma_E(x, t; y, s)$ is a consequence of $V \geq 0$ and the maximum principle. Hence Proposition 1 and (27) imply

$$|\Gamma_J(x, t; y, s)|^2 \leq C \exp(-C(1 + |t - s|^{1/2} m_J(x)\alpha_0/2)) \frac{1}{(t - s)^n} \exp\left(-C \frac{|y - x|^2}{(t - s)}\right)$$

for some constant $C$. This concludes the desired estimate. □

**Proof of Corollary 1:** Let $f(t) = (m_J(x) t^{1/2}) \alpha_0/2 + |x - y|^2/t$ for $t > 0$. The, an easy computation shows that

$$\inf_{t > 0} f(t) \geq C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0 + 4)}$$

for some positive constant $C$. Thus, we obtain

$$|\Gamma_J(x, t; y, s)| \leq C \frac{1}{(t - s)^{n/2}} \exp(-Cf(t - s)) \exp\left(-\frac{C|x - y|^2}{t}\right) \times \exp\left(-C(m_J(x)(t - s)^{1/2}\alpha_0/2)\right) \leq C \Gamma_C(x, t; y, s) \exp\left(-C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0 + 4)}\right) \times \exp\left(-C(m_J(x)t^{1/2})\alpha_0/2\right).$$

This proves the part (a) since $2\alpha_0/(\alpha_0 + 4) \leq \alpha_0/2$. The part (b) is an easy consequence of the part (a). □

4 **Proof of Theorem 2, 3**

To show Theorem 2, we prove the following inequality.

**Theorem 5** Let $\gamma \in [0, n)$. Then there exists a constant $C$ such that

$$|m_J(x)^{2\gamma}(e^{-tL_J}f)(x)| \leq \frac{C}{t^{\gamma+\gamma/2}}(M_\gamma f)(x)$$  

(28)
holds for every $0 < l \leq (n - \gamma)/2$. Here $M_\gamma f$ is the fractional maximal function defined by

$$(M_\gamma f)(x) = \sup_{x \in B} \frac{1}{|B|^{1-\gamma/n}} \int_B |f| \, dy,$$

where the supremum is taken all balls $B$ containing $x$.

Theorem 2 is a consequence of Theorem 5 and the following lemma (see, e.g., [St]).

**Lemma 3** Let $0 \leq \gamma < n$. There exists a constant $C$ such that

$$||M_\gamma f||_q \leq C||f||_p$$

for $1 < p \leq q \leq +\infty$ and $1/q = 1/p - \gamma/n$.

**Proof of Theorem 5:** Let $r = 1/m_J(x)$. By Corollary 1 (b) we have

$$|m_J(x)^{2l} (e^{-tL_J f})(x)| \leq C m_J(x)^{2l} \int \frac{|f(y)|}{(1 + m_J(x)|x-y|)^{k\gamma/2}} \exp\left(-\frac{C|x-y|^2}{t}\right) dy$$

$$\leq \frac{C}{t^{2l(\gamma/2)}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r < |x-y| \leq 2jr\}} \frac{|f(y)|}{(1+2^{j-1})^k} \exp\left(-\frac{C(2^jr)^2}{t}\right) dy. \quad (29)$$

By the assumption on $l$, we take $\alpha \geq 0$ such that $2\alpha = n - \gamma - 2l$. Put $C_\alpha = \sup_{s>0} s^\alpha e^{-s} < +\infty$ for $\alpha \geq 0$. Then the right hand side of (29) is dominated by

$$C_\alpha C \frac{C}{t^{n/2}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r < |x-y| \leq 2jr\}} \frac{1}{(1+2^{j-1}r)^{k(2^j-1)^2\alpha}} \left(\frac{C(2^jr)^2}{t}\right)^{-\alpha} |f(y)| dy$$

$$\leq C_\alpha C \frac{C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1}r)^k(2^j-1)^{2\alpha}} \int_{\{|x-y| \leq 2jr\}} |f(y)| \, dy.$$
\[
\sum_{j=-\infty}^{0} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^j)^{2\alpha}} \leq \sum_{j=-\infty}^{0} C(2^j)^{2l} < +\infty.
\]

Thus, we obtain the desired result. \(\square\)

**Proof of Theorem 3:** First, the estimate for the case \(l = 0\) and \(p = 1\) is classical except the exponential factor in time. Under the assumption, by Corollary 1 (a) we have

\[
|\Gamma_j(x, t; y, s)| \leq C\mathcal{C}_0(x, t; y, s) \exp(-C(1 + m_J(x)|x - y|)^{2\alpha/(\alpha_0 + 4)}) \times \exp(-C(1 + m_0^{1/2})^{\alpha_0/2})
\]

denoting for some positive constants \(C\) and \(C_0\). Then by using this estimate we can prove the part (a) of Theorem 3 in a similar way as in the proof of Theorem 2. To show the part (b), we use the semigroup property and Theorem 2 and get

\[
\|m_J(x)^2 e^{-tL_J} f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{t^{l+(n/4)}} \|e^{-(2/3)tL_J} f\|_{L^2(\mathbb{R}^n)}
\]

for some constant \(C\). Note that under the assumption \(m_J(x) \geq m_0\), Lemma 1 yields \(\inf \sigma(L_J) \geq Cm_0^2\) for some positive constant \(C\). Here \(\sigma(L_J)\) is the spectrum of the operator \(L_J\). So, we have

\[
\|e^{-(1/3)tL_J} g\|_{L^2(\mathbb{R}^n)} \leq e^{-Cm_0^2 t} \|g\|_{L^2(\mathbb{R}^n)}.
\]

Using this estimate, we obtain

\[
\|m_J(x)^2 e^{-tL_J} f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \|e^{-(1/3)tL_J} f\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \frac{C}{t^{n/2(1/p-1/2)}} \|f\|_{L^p(\mathbb{R}^n)}.
\]

In the last inequality, we used \(p \leq 2\) and Theorem 2.

\(\square\)

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