An Estimate on the Heat Kernel of Magnetic Schrödinger Operators and Uniformly Elliptic Operators with Non-negative Potentials

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Abstract

In this paper we show an estimate of the heat kernel to the Schrödinger operator with magnetic fields and to uniformly elliptic operators with non-negative potentials which belongs to the reverse Hölder class. We also give a weighted smoothing estimates for the semigroup generated by the operators above.

1 Introduction and Main Results

We consider the uniformly elliptic operator \( L_E = -\nabla(A(x)\nabla) + V(x) \) with certain non-negative potential \( V \) and the Schrödinger operator \( L_M = (i^{-1}\nabla - a(x))^2 + V(x) \) with a magnetic field \( a(x) = (a_1(x), \cdots, a_n(x)), n \geq 2 \). We use the notation \( L_J \) for \( J = E \) or \( J = M \). The purpose of this paper is to give an estimate of the fundamental solution (or heat kernel) \( \Gamma_J(x, t ; y, s) \) to

\[
(\partial_t + L_J)u(x, t) = 0, \quad (x, t) \in \mathbb{R}^n \times (0, \infty),
\]

namely \( \Gamma_J(x, t ; y, s) \) satisfies

\[
(\partial_t + L_J)\Gamma_J(x, t ; y, s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \quad (2)
\]

\[
\lim_{t \to s} \Gamma_J(x, t ; y, s) = \delta(x - y). \quad (3)
\]
For the elliptic operator $L_E$, we assume the following conditions for $A(x) = (a_{ij}(x))$.

**Assumption (A.1):** $a_{ij}(x)$ is a real-valued measurable function and satisfies $a_{ij}(x) = a_{ji}(x)$ for every $i, j = 1, \ldots, n$ and $x \in \mathbb{R}^n$.

**Assumption (A.2):** There exists a constant $\lambda > 0$ such that
\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi^i \xi^j \leq \lambda |\xi|^2,
\]
for every $i, j = 1, \cdots, n$ and $x \in \mathbb{R}^n$.

(4)

To state our assumptions on $V$ and $a$, we prepare some notations. We say $U \in (RH)_{\infty}$ if $U \in L_{\text{loc}}^{\infty}(\mathbb{R}^n)$ and satisfies
\[
\sup_{y \in B(x,r)} |U(y)| \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| dy,
\]
and say $U \in (RH)_q$ if $U \in L_{\text{loc}}^{q}(\mathbb{R}^n)$ and satisfies
\[
\left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)|^q dy \right)^{1/q} \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| dy,
\]
for some constant $C$ and for every $x \in \mathbb{R}^n$ and $r > 0$, respectively. We can define the function $m(x, U)$ for $U \in (RH)_q$ with $q > n/2$ as follows:
\[
\frac{1}{m(x, U)} = \sup \{r > 0; \frac{r^2}{|B(x,r)|} \int_{B(x,r)} U(y) dy \leq 1 \}.
\]

(7)

We note that if there exist positive constants $K_1$ and $K_2$ such that $K_1 U_1(x) \leq U_2(x) \leq K_2 U_1(x)$, then it is easy to see that there exist positive constants $K'_1$ and $K'_2$ such that
\[
K'_1 m(x, U_1) \leq m(x, U_2) \leq K'_2 m(x, U_1).
\]

When $n \geq 3$, since it is known $U \in (RH)_{n/2}$ actually belongs to $(RH)_{n/2+\epsilon}$ for some $\epsilon > 0$, $m(x, U)$ can be defined for $U \in (RH)_{n/2}$ ([Sh1]). For other properties of the class $(RH)_q$, see, e.g., [KS]. We denote by $B(x) = (B_{jk}(x))$ the magnetic field defined by $B_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)$. We use the notation $m_J(x)$:
\[
m_E(x) = m(x, V), \quad m_M(x) = m(x, |B| + V)
\]
for the operator $L_J$, $J = E$ or $M$, respectively. We assume the following conditions for $V(x)$ and $a(x) = (a_1(x), \cdots, a_n(x))$. 

(113)
ASSUMPTION\((V, a, B)\): For each \(j = 1, \ldots, n\), \(a_j(x)\) is a real-valued \(C^1(\mathbb{R}^n)\)-function, \(V\) is non-negative.

(i) For \(n \geq 3\), we assume \(V(x)\) and \(a(x)\) satisfy

\[
V + |B| \in (RH)_{n/2}, \quad |\nabla B(x)| \leq C m(x, V + |B|)^3.
\]

(ii) For \(n = 2\), we assume \(V(x)\) and \(a(x)\) satisfy

\[
V + |B| \in (RH)_q, \quad |\nabla B(x)| \leq C m(x, V + |B|)^3
\]

for some \(q > 1\).

**Remark**

For \(n = 2\), we may assume the condition (ii') instead of (ii) by employing Lemma 1 (b).

(iii') \(V \in L^\infty_{loc}(\mathbb{R}^2), B(x) \geq 0\) and that \(m_J(x)\) satisfies

\[
C_1 \frac{m_J(x)}{(1 + |x - y|m_J(x))^{k_0/(k_0 + 1)}} \leq m_J(y) \leq C_2 (1 + |x - y|m_J(x))^{k_0} m_J(x) \tag{8}
\]

for some positive constants \(C_1, C_2, k_0\) and for every \(x, y \in \mathbb{R}^2\), where \(m_E(x) = \sqrt{V(x)}\) and \(m_M(x) = \sqrt{V(x) + B(x)}\).

We remark that it is known that \(m_J(x)\) satisfies (8) under the assumption \((V, a, B)\) for \(n \geq 3([\text{Sh}1])\) and even for \(n = 2\) in the same way. We also note that if \(|B| + V \in (RH)_{\infty}\), then it is easy to see that \(|B(x)| + V(x) \leq C m(x, |B| + V)^2\) holds. For example, the condition \(|B| + V \in (RH)_{\infty}\) is satisfied for any \(a_j(x) = Q_j(x), V(x) = |P(x)|^\alpha\), where \(P(x)\) and \(Q_j(x), j = 1, \ldots, n\), are polynomials and \(\alpha\) is a positive constant. In this case, under the assumption \((V, a, B)\) (i) or (ii), we see that there exists a positive constant \(m_0\) such that \(m_J(x) \geq m_0\), although in general we cannot say \(|B| + V\) is strictly positive for inhomogeneous polynomials.

To state our main result, we introduce the notation:

\[
\Gamma_{C_0}(x, t; y, s) = \frac{1}{(t - s)^{n/2}} \exp(-C_0 \frac{|x - y|^2}{t - s})
\]

for some positive constant \(C_0\).
Theorem 1 (a) Suppose $A(x)$ and $V(x)$ satisfy the assumptions (A.1), (A.2) and $(V,0,0)$. Then, there exist positive constants $\alpha_0$ and $C_j$ ($j = 0,1,2$) such that

$$(0 \leq) \Gamma_E(x,t;y,s) \leq C_1 \exp\left(-C_2(1 + m_E(x)(t-s)^{1/2})^{\alpha_0/2}\right) \Gamma_{C_0}(x,t;y,s)$$

(9)

for $x,y \in \mathbb{R}^n$ and $t > s > 0$.

(b) Suppose $V(x)$ and $a(x)$ satisfy the assumption $(V,a,B)$. Then, there exist positive constants $\alpha_0$ and $C_j$ ($j = 0,1,2$) such that

$$|\Gamma_M(x,t;y,s)| \leq C_1 \exp\left(-C_2(1 + m_M(x)(t-s)^{1/2})^{\alpha_0/2}\right) \Gamma_{C_0}(x,t;y,s)$$

(10)

for $x,y \in \mathbb{R}^n$ and $t > s > 0$.

The number $\alpha_0$ is actually defined by $\alpha_0 = 2/(k_0 + 1)$, where $k_0$ is the constant in (8). The exponent $\alpha_0/2$ would not be sharp. If we restrict for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, the following sharp estimate is known ([Ma], [Er1,2] for $n \geq 3$ and [LT] for $n = 2$):

$$|\Gamma_M(x,t;y,s)| \leq D_1 \exp(-D_2B_0t)\Gamma_{D_0}(x,t;y,s).$$

More detail informations on the constants $D_j$ ($j = 0,1,2$) can be seen in those papers. By using the parabolic distance:

$$d_P((x,t),(y,s)) = \max(|x-y|, |t-s|^{1/2}),$$

we have the following decay estimate.

Corollary 1 (a) Under the same assumptions as in Theorem 1, there exist positive constants $C_j$ ($j = 1,2$) and $C_0$ such that

$$|\Gamma_J(x,t;y,s)| \leq C_1 \exp\left(-C_2(1 + m_J(x)d_P((x,t),(y,s)))^{2\alpha_0/(\alpha_0+4)}\right) \Gamma_{C_0}(x,t;y,s)$$

for $J = E$ and $M$, for every $x,y \in \mathbb{R}^n$ and $t > s > 0$.

(b) Under the same assumptions as in Theorem 1, for each $k > 0$ there exist positive constants $C_k$ and $C_0$ such that

$$|\Gamma_J(x,t;y,s)| \leq \frac{C_k}{(1 + m_J(x)d_P((x,t),(y,s)))^k} \Gamma_{C_0}(x,t;y,s)$$

for $J = E$ and $M$. 
Remark 2 Actually we can show the estimate in Theorem 1 for the operators \( L_E = -\nabla(A(x,t)\nabla) + V(x,t) \) with time-dependent coefficients, if we assume the uniform ellipticity (4) of \( A(x,t) \) and the existence of constants \( C_j, j = 1, 2, \) such that \( C_1 U(x) \leq V(x,t) \leq C_2 U(x) \) and \( U \) satisfies the condition \((U,0,0)\). For the magnetic Schrödinger operator \( L_M = (i^{-1}\nabla - a(x,t))^2 + V(x,t) \), the estimate in Theorem 1 still holds, if there exists positive constants \( C_j, j = 1, \cdots, 5, \) such that \( C_1 U(x) \leq V(x,t) \leq C_2 U(x), \) \( C_3 |B'(x)| \leq |B(x,t)| \leq C_4 |B'(x)|, \) and \( |\nabla B(x,t)| \leq C_5 m(x,|B'| + U)^3 \), where \( a(x,t) \) is \( C^1 \) and \( B_{jk}(x,t) = \partial_j a(x,t) - \partial_k a_j(x,t) \) and \( U(x) \) and \( B'(x) \) satisfy the assumption \((U,a,B')(except |\nabla B'(x)| \leq Cm(x,|B'| + U)|^3)) \), and if the upper bound:

\[
|\Gamma_M(x,t;y,s)| \leq C\Gamma_{C_0}(x,t;y,s)
\]

holds for some constants \( C \) and \( C_0 \).

Remark 3 In particular, Corollary 1 (b) yields

\[
|\Gamma_J(x,t;y,s)| \leq \frac{C_k}{(1 + m_J(x)|x-y|)^k(1 + m_J(x)|t-s|)^k} \Gamma_{C_0}(x,t;y,s)
\]

\[
\leq \frac{C_k}{(1 + m_J(x)|x-y|)^k} \Gamma_{C_0}(x,t;y,s)
\]

(11)

for \( J = E \) or \( M \). Let \( n \geq 3 \). Then this implies

\[
|\Gamma_J(x,y)| = \int_s^{+\infty} \Gamma_J(x,t;y,s) \ dt \leq \frac{C_k}{(1 + m_J(x)|x-y|)^k|x-y|^{n-2}}
\]

where \( \Gamma_J(x,y) \) is the fundamental solution to \( L_J u = 0 \). This estimate for the elliptic operator was proved by Shen [Sh1,2]. Thus, Corollary 1 (b) is a generalization of his estimate.

Remark 4 Recently we are informed by Z. Shen that he obtained the following shape estimate [Sh3] for the elliptic operators: under the assumption \( V \in (RH)_{n/2} \) for \( n \geq 3 \) and \( V \in (RH)_q \) with \( q > 1 \) for \( n = 2 \),

\[
C_1 \exp(-C_2 d(x,y)) |x-y|^{-n} \leq \Gamma_E(x,y) \leq C_3 \exp(-C_4 d(x,y)) |x-y|^{-n}
\]

holds for some positive constants \( C_j, (j = 1, 2, 3, 4), \) where \( d(x,y) \) is defined by

\[
d(x,y) = \inf_{\gamma} \int_0^1 m(\gamma(t),V)\left|\frac{d\gamma}{dt}(t)\right| \ dt.
\]
Here the infimum is taken over all curves \( \gamma \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). Moreover, he gave the following estimate:

\[
C_1(1 + m(x)|x - y|)^{\alpha_0/2} \leq d(x, y) \leq C_2(1 + m(x)|x - y|)^{\beta_0}
\]

for some positive constants \( C_j (j = 1, 2) \) and \( \beta_0 \). In particular, it follows

\[
\Gamma_E(x, y) \leq C_5 \exp(-C_6(1 + m_E(x)|x - y|)^{\alpha_0/2})|x - y|^{2-n}
\]

for some positive constants \( C_5 \) and \( C_6 \). We remark that this decay estimate also can be shown for the fundamental solution \( \Gamma_M(x, y) \) to \( L_M \) in a similar way. On the other hand, it follows from Corollary 1 (a) a somewhat weaker decay estimate:

\[
|\Gamma_J(x, y)| \leq C \exp(-C(1 + m_J(x)|x - y|)2^{\alpha_0/(\alpha_0 + 4)})|x - y|^{2-n}
\]

for \( J = E \) or \( M \). We do not know whether his sharp estimate can be generalized to heat kernel estimates or not.

We denote by \( e^{-tL_J} \) the semigroup generated by \( L_J \). Here we also denote by \( L_J \) the self-adjoint operator determined from the form associated with \( L_J \) (see, e.g., [Si], [LS]). We obtain the following weighted smoothing estimate by using Corollary 1 (b).

**Theorem 2** Assume the same assumptions as in Theorem 1. Let \( J = E \) or \( M \). Suppose \( 1 < p \leq q \leq +\infty \) and \( 1/p - 1/q < 1 \) and put \( \gamma = n(1/p - 1/q) \). Then for each \( l \in [0, (n - \gamma)/2] \) there exists a constant \( C_l \) such that

\[
\|m_J(x)^{2l}e^{-tL_J}f\|_{L^q(\mathbb{R}^n)} \leq \frac{C_l}{t^{l+(\gamma/2)}}\|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0.
\]  

(12)

**Corollary 2** Suppose the additional condition \( |B| + V \in (RH)_{\infty} \). Then we have the following estimates:

\[
\|(|B| + V)^l e^{-tL_J}f\|_{L^p(\mathbb{R}^n)} \leq \frac{C_l}{t^l}\|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0
\]

(13)

holds for \( 1 < p < +\infty \) and \( l \in [0, n/2] \), and

\[
\|(|B| + V)^l e^{-tL_J}f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_l}{t^{l+(n/2p')}}\|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0
\]

(14)

holds for \( 1 \leq p < +\infty \) and \( l \in [0, n/(2p')] \). Here \( 1/p' = 1 - 1/p \) and \( C_l \) is a constant depending on \( l \) and \( p \).
Corollary 2 is an easy consequence of Theorem 2 by using the inequality $|B| + V(x) \leq C m_J(x)^2$. Note that (14) for the case $l = 0$ is a classical result.

Theorem 1 yields a weighted smoothing estimate with an exponential decay in time.

**Theorem 3** Assume the same assumptions as in Theorem 1 and the additional assumption $m_J(x) \geq m_0 > 0$.

(a) Let $1 \leq p < +\infty$ and $l \in [0, n/(2p')]$. Then we have

$$
\| m_J(x)^{2l} e^{-tL_J} f \|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(-C\left(1 + m_0^{t/2}\right)^{3/2}\right) \frac{1}{t^{l+(n/2p)}} \| f \|_{L^p(\mathbb{R}^n)}, \quad t > 0.
$$

(b) Let $1 \leq p \leq 2$ and $l \in [0, n/(2p')]$. Then we have

$$
\| m_J(x)^{2l} e^{-tL_J} f \|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(-C\left(1 + m_0^2 t\right)\right) \frac{1}{t^{l+(n/2p)}} \| f \|_{L^p(\mathbb{R}^n)}, \quad t > 0.
$$

Especially, for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, Theorem 3 (b) yields an exponential decay estimate in time:

$$
\| e^{-tL_M} f \|_{L^\infty(\mathbb{R}^n)} \leq C_1 \frac{1}{t^{n/2}} \exp(-C_2 B_0 t) \| f \|_{L^p(\mathbb{R}^n)}, \quad t > 0
$$

for some positive constant $C_1$ and $C_2$, which is known (see, e.g., [Ma], [Er1,2], [Ue], [LT]). Indeed, in this case $m_M(x) \sim \sqrt{B_0}$ holds. Note that Theorem 3 (a) gives weaker decay rate $e^{-C\sqrt{B_0} t}$, since $k_0 = 0$ and $\alpha_0 = 2$. We also emphasize that Theorem 3 can be applied to any polynomial like magnetic field $B(x)$ which may be zero somewhere.

**Definition 1** We say $u(x, t)$ is a complex-valued weak solution to

$$(\partial_t + L_M) u = 0 \quad \text{in} \quad Q_r(x_0, t_0),$$

if $u \in L^\infty((t_0 - r^2, t_0); L^2(B(x_0, r); \mathbb{C})) \cap L^2((t_0 - r^2, t_0); H^1(B(x_0, r); \mathbb{C}))$ and satisfies

$$
\int_{B(x_0, r^2)} u(x, t) \overline{\phi(x, t)} dx - \int_{t_0 - r^2}^t \int_{B(x_0, r^2)} u(x, s) \partial_x \overline{\phi(x, s)} dx ds
+ \int_{t_0 - r^2}^t \int_{B(x_0, r^2)} \sum_{j=1}^n D_j^2 u(x, s) D_j^2 \overline{\phi(x, s)} dx ds
+ \int_{t_0 - r^2}^t \int_{B(x_0, r^2)} V(x) u(x, s) \overline{\phi(x, s)} dx ds = 0 \quad (16)
$$
for every $\phi \in C \equiv \{ \phi \in L^2((t_0-r^2, t_0); H^1(B(x_0, r); \mathbb{C})); \partial_s \phi \in L^2((t_0-r^2, t_0); L^2(B(x_0, r); \mathbb{C})), \phi(x, t_0 - r^2) = 0 \}$, where $\overline{\phi}$ is the complex conjugate of $\phi$.

Here, we used the notation $D_j^a = i^{-1} \partial_{x_j} - a_j(x)$ and

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0 \}.$$  

A real-valued weak solution $u$ to $(\partial_t + L_E)u = 0$ in $Q_r(x_0, t_0)$ can be defined in a similar way. Our proof of Theorem 1 is based on the following subsolution estimate.

**Theorem 4** Let $u(x, t)$ be a weak solution to $\partial_t u + L_J u = 0$ in $Q_{2r}(x_0, t_0)$. Then there exist positive constants $C_j, j = 1, 2$, such that

$$\sup_{(x,t) \in Q_{r/2}(x_0,t_0)} |u(x,t)| \leq C_1 \exp \left( -C_2(1+rm_J(x_0))^{\alpha/2} \right) \left( \frac{1}{r^{n+2}} \int_{Q_{r}(x_0,t_0)} |u|^2 dx dt \right)^{1/2}. \tag{17}$$

Throughout this paper, we use the following notation: $D = i^{-1} \nabla - a$,

$$B(x_0, r) = \{ y \in \mathbb{R}^n; |y - x_0| < r \}, \quad \langle A \nabla u, \nabla u \rangle = \sum_{j,k=1}^n a_{jk} \partial_{x_j} u \partial_{x_k} u,$$

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0 \}.$$

## 2 Proof of Theorem 4

We use the following inequalities.

**Lemma 1**

(a) ([Sh2]) Suppose $n \geq 2$ and $V(x)$ and $a(x)$ satisfy the condition $(V, a, B)$. Then there exists a constant $C_0$ such that

$$\int m(x, |B| + V)^2 |u|^2 dx \leq C_0 \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx$$

for $u \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$.

(b) ([AHS]) Suppose $n = 2, V \geq 0, V \in L^\infty_{loc}(\mathbb{R}^2), a \in C^1(\mathbb{R}^2)$, and $B(x) \geq 0$. Then the inequality

$$\int (B(x) + V(x))|u|^2 dx \leq \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx$$

holds for $u \in C^\infty_0(\mathbb{R}^n, \mathbb{C})$. 
We also prepare the following Caccioppoli-type inequality.

**Lemma 2** Let $0 < \sigma < 1$. Let $u$ be a weak solution to $(\partial_s + L_J)u = 0$ in $Q_{2r}(x_0, t_0)$ for $J = E$ or $J = M$. Then there exists a constant $C$ such that

$$
\sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, \sigma r)} |u(x, t)|^2 \, dx + \int \int_{Q_{\sigma r}(x_0, t_0)} |(i^{-1} \nabla - a)u|^2 + V|u|^2 \, dx \, ds \leq \frac{C}{(1 - \sigma)^2 r^2} \int \int_{Q_r(x_0, t_0)} |u|^2 \, dx \, dt.
$$

**Proof:** Although the proof is standard, we give it here for the sake of completeness. We show the estimate for a weak solution $u$ to $(\partial_t + L_E)u = 0$ in $Q_{2r}(x_0, t_0)$. Since we can show the estimate for a weak solution to $(\partial_t + L_M)u = 0$ in the similar way, we just mention some modifications we need at the end of this proof. Take functions $\chi(x) \in C_0^\infty(B(x_0, r))$ and $\eta(t) \in C^\infty(\mathbb{R}^1)$ satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) \equiv 1$ on $B(x_0, \sigma r)$ and $|\nabla \chi(x)| \leq C/(1 - \sigma)r$, and $0 \leq \eta(t) \leq 1$, $\eta(t) \equiv 1$ on $t \geq t_0 - (\sigma r)^2$, $\eta(t) \equiv 0$ on $t \leq t_0 - r^2$, $|\partial_t \eta(t)| \leq C/r^2(1 - \sigma^2)$. For the sake of simplicity, we also assume $\partial_t u \in L^2(Q_{2r}(x_0, t_0))$. Actually, we can remove this additional assumption by using the argument as in [AS]. Fix $t \in [t_0 - (\sigma r)^2, t_0]$. Multiplying $\eta^2(t) \chi^2(x) u(x, t)$ to the equation and integrating over $B(x_0, r) \times [t_0 - r^2, t]$, we have

$$
\frac{1}{2} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 \, dx \\
+ \int_{t_0 - r^2}^t \int_{B(x_0, r)} \langle A(x) \nabla u(x, s), \nabla u(x, s) \rangle \eta(s)^2 \chi(x)^2 \, dx \, ds \\
+ \int_{t_0 - r^2}^t \int_{B(x_0, r)} V(x) u(x, s)^2 \eta(s)^2 \chi(x)^2 \, dx \, ds \\
= \int_{t_0 - r^2}^t \int_{B(x_0, r)} u(x, s)^2 \chi(x)^2 \eta(s) \partial_s \eta(s) \, dx \, ds \\
- \int_{t_0 - r^2}^t \int_{B(x_0, r)} \langle A(x) \nabla u(x, s), \nabla (\chi^2(x)) \rangle \eta(s)^2 u(x, s) \, dx \, ds.
$$

Because of the ellipticity of $A(x)$ and the positivity of $V$, we obtain by (18)

$$
\sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 \, dx \leq \int \int_{Q_r(x_0, t_0)} u^2 |\partial_s \eta| \, dx \, ds.
$$
\[ + \int Q_r(x_0, t_0) |\nabla u| u |\nabla^2 \chi| \nabla \chi| \, dx \, ds \leq \frac{C}{(1 - \sigma)} \left\{ \frac{1}{r^2} \int Q_r(x_0, t_0) u^2 \, dx \, ds + \int Q_r(x_0, t_0) \chi^2 \eta^2 |\nabla u|^2 \, dx \, ds \right\}. \]

By using (18) again, we have

\[ \lambda \int Q_r(x_0, t_0) |\nabla u|^2 \chi^2 \eta^2 \, dx \, ds + \int Q_r(x_0, t_0) V u^2 \chi^2 \eta^2 \, dx \, ds \leq \frac{C}{(1 - \sigma) r^2} \int Q_r(x_0, t_0) u^2 \, dx \, ds + \frac{\lambda}{2} \int Q_r(x_0, t_0) |\nabla u| \nabla \chi \chi |\eta^2| \eta \, dx \, ds. \] (20)

It follows

\[ \frac{\lambda}{2} \int Q_r(x_0, t_0) |\nabla u|^2 \chi^2 \eta^2 \, dx \, ds + \int Q_r(x_0, t_0) V u^2 \chi^2 \eta^2 \, dx \, ds \leq \frac{C}{(1 - \sigma) r^2} \int Q_r(x_0, t_0) u^2 \, dx \, ds. \] (21)

(19) and (21) yield the desired result. For \( L_M \), we can prove in a similar way by noting the following identities:

\[ D_j^a (u \chi) = (D_j^a u) \chi + u (i^{-1} \nabla \chi), \quad \int D_j^a u \overline{v} \, dx = \int u \overline{D_j^a v} \, dx. \]

\( \square \)

**Proof of Theorem 3:** Let \( k \in \mathbb{N} \) and define \( p_j (j = 1, 2, \ldots, k + 1) \) by \( p_j = 2/3 + ((j - 1)/k)(1 - (2/3)) \). Let \( \chi_j(x) \in C_0^\infty(B(x_0, p_j r)) \) and \( \eta_j(t) \in C^\infty(\mathbb{R}) \) be the functions satisfying \( 0 \leq \chi_j \leq 1, \chi_j(x) \equiv 1 \) on \( B(x_0, p_j r), |\nabla \chi_j(x)| \leq Ck/r \), and \( 0 \leq \eta_j \leq 1, \eta_j(t) \equiv 1 \) on \( t \geq t_0 - (p_j - 1)r^2, \eta_j(t) \equiv 0 \) on \( t \leq t_0 - (p_j r)^2, |\nabla \eta_j(t)| \leq Ck/r^2 \). By Lemma 2 (see also (21)) , we have

\[ \int Q_{p_j+1 r}(x_0, t_0) \left( (i^{-1} \nabla - a) u |^2 \chi_j^2 \eta_j^2 + V |u|^2 \chi_j^2 \eta_j^2 \right) \, dx \, ds \leq \frac{Ck^2}{r^2} \int Q_{p_j+1 r}(x_0, t_0) |u|^2 \, dx \, ds. \]
We write just $\chi = \chi_{j+1}$ and $\eta = \eta_{j+1}$, for simplicity. Since $|(i^{-1}\nabla - a)(u\eta\chi)|^2 \leq 2|(i^{-1}\nabla - a)u|^2 \chi^2 \eta^2 + 2u^2|\nabla \chi|^2 \eta^2$, it follows that

$$\int \int_{Q_{p_{j+1}}(x_0, t_0)} (|(i^{-1}\nabla - a)(\eta \chi u)|^2 \chi \eta^2 + V|u|^2 \chi \eta^2) \, dx \, ds \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}}(x_0, t_0)} |u|^2 \, dx \, ds$$

for $j = 1, \cdots, k$. By using Lemma 1, we obtain

$$\int_{t_0}^{t_0} \left( \int_{B(x_0, p_{j+1}r)} m_J(x)^2 |\eta \chi u|^2 \, dx \right) \, dt \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}}(x_0, t_0)} |u|^2 \, dx \, ds.$$ 

By using $m_J(x) \geq C(1 + p_{j+1}r m_J(x_0))^{-k_0/(1+k_0)} m_J(x_0)$ on $|x - x_0| < p_{j+1}r$ and noting $2/3 \leq p_{j+1} \leq 1$ (see (8) and the remark after that), we have

$$\int \int_{Q_{p_{j+1}}(x_0, t_0)} |u|^2 \, dx \, dt \leq \int_{t_0}^{t_0} \left( \int_{B(x_0, p_{j+1}r)} |\eta \chi u|^2 \right) \, dx \, dt \leq \frac{Ck^2}{r^2 \, m_J(x_0)^2} (1 + rm_J(x_0))^{2/(k_0+1)} \int \int_{Q_{p_{j+1}}(x_0, t_0)} |u|^2 \, dx \, dt.$$

for each $j = 1, 2, \cdots, k$. Here we used a trivial inequality $\int \int_{Q_{p_{j+1}}(x_0, t_0)} (\cdots) \, dx \, dt \leq \int \int_{Q_{p_{j+1}}(x_0, t_0)} (\cdots) \, dx \, dt$ for the case $rm_J(x_0) \leq 1$. By this procedure, we can obtain the following: there exists a constant $C$ such that for every $k \in \mathbb{N}$

$$\int \int_{Q_{2r/3}(x_0, t_0)} |u|^2 \, dx \, dt \leq \frac{C(k^2)^k}{(1 + rm_J(x_0))^{k_0}} \int \int_{Q_{r}(x_0, t_0)} |u|^2 \, dx \, dt,$$

(23)

where $\alpha_0 = 2/(k_0 + 1)$. Since $V(x) \geq 0$, the well-known subsolution estimate (see, e.g., [AS]) yields

$$\sup_{Q_{r/2}(x_0, t_0)} |u| \leq C \left( \frac{1}{r^{n+2}} \int \int_{Q_{r/2}(x_0, t_0)} |u|^2 \, dx \, dt \right)^{1/2}$$

(24)

for some constant $C$. For the magnetic Schrödinger operator case, we have used Kato's inequality. Combining (23) and (24), we arrive at

$$\sup_{Q_{r/2}(x_0, t_0)} |u| \leq C \left( \frac{C^{k/2}k^k}{(1 + rm_J(x_0))^{k_0/2}} \left( \frac{1}{r^{n+2}} \int \int_{Q_{r}(x_0, t_0)} |u|^2 \, dx \, dt \right)^{1/2} \right)^{1/2}$$

(25)
for every \( k \in \mathbb{N} \). Note that, by Stirling’s formula \( k^k \sim e^k k!(1/\sqrt{2\pi k}) \) as \( k \to \infty \), there exists a constant \( C_0 \) such that \( k^k \leq C_0 e^k k! \) for \( k \geq 1 \). Multiplying \( e^k / k! \) and taking the summation, we obtain

\[
\left( \sup_{Q_{r/2}(x_0,t_0)} |u| \right) \sum_{k=1}^{\infty} \frac{(e(1 + r m_J(x_0)))^{\alpha_0/2})^k}{k!} 
\leq CC_0 \sum_{k=1}^{\infty} (e e \sqrt{C})^k \left( \frac{1}{r^{n+2}} \int \int_{Q_r(x_0,t_0)} |u|^2 \, dx \, dt \right)^{1/2}.
\]

Take \( \epsilon > 0 \) so that \( ee \sqrt{C} < 1 \). Then we have

\[
\sup_{Q_{r/2}(x_0,t_0)} |u| \leq C \exp(-c(1 + r m_J(x_0)))^{\alpha_0/2}) \left( \frac{1}{r^{n+2}} \int \int_{Q_r(x_0,t_0)} |u|^2 \, dx \, dt \right)^{1/2}.
\]

This complete the proof. \( \square \)

3 Proof of Theorem 1

To show Theorem 1 we prove the following proposition.

**Proposition 1** Under the assumptions as in Theorem 1, there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
|\Gamma_J(x,t;y,s)| \leq C_1 \exp(-C_2(1 + m_J(x)) |t-s|^{1/2}) \frac{1}{(t-s)^{n/2}}
\]

for \( x, y \in \mathbb{R}^n \) and \( t > s > 0 \).

**Proof**: Assume \( t-s \geq 2|y-x|^2 \). Take \( r^2 = |t-s|/8 \). Then \( u(z,u) = \Gamma_J(z,u;y,s) \) satisfies \((\partial_t + L_J)u(z,u) = 0 \) in \( Q_{2r}(x,t) \). Hence, by applying Theorem 4 to \( u(z,u) \), we obtain

\[
|\Gamma_J(x,t;y,s)| \leq \sup_{Q_{r/2}(x,t)} |u|
\leq C \exp(-C(1 + m_J(x)) |t-s|^{1/2}) \frac{1}{(t-s)^{n/2}} \left( \int \int_{Q_r(x,t)} \left| \Gamma(z,u;y,s) \right|^2 \, dz \, du \right)^{1/2}.
\]

By using the maximum principle for \( L_E \) and the diamagnetic inequality (see, e.g., [AS], [LS], [AHS]) for \( L_M \), we have

\[
|\Gamma_J(z,u;y,s)| \leq \frac{C}{(u-s)^{n/2}} \exp\left(-C \frac{|z-y|^2}{(u-s)}\right)
\]

(27)
for some constant $C = C(n, \lambda)$. Since $t - s \geq u - s \geq 7r^2 \geq (7/8)(t - s)$ on $(z, u) \in Q_r(x, t)$, it is easy to see

$$
\left( \frac{1}{r^{n+2}} \int \int_{Q_r(x,t)} |\Gamma_J(z, u; y, s)|^2 \, dz \, du \right)^{1/2} \leq \frac{C}{(t - s)^{n/2}}.
$$

This yields the desired estimate. □

**Proof of Theorem 1:** The positivity of $\Gamma_E(x, t; y, s)$ is a consequence of $V \geq 0$ and the maximum principle. Hence Proposition 1 and (27) imply

$$
|\Gamma_J(x, t; y, s)|^2 \leq C \exp(-C(1 + |t - s|^{1/2}m_J(x))^{\alpha_0/2}) \frac{1}{(t - s)^{n}} \exp\left(-C\frac{|y - x|^2}{(t - s)}\right)
$$

for some constant $C$. This concludes the desired estimate. □

**Proof of Corollary 1:** Let $f(t) = (m_J(x)t^{1/2})^{\alpha_0/2} + |x - y|^2/t$ for $t > 0$. The, an easy computation shows that

$$
\inf_{t > 0} f(t) \geq C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0 + 4)}
$$

for some positive constant $C$. Thus, we obtain

$$
|\Gamma_J(x, t; y, s)| \leq C \frac{1}{(t - s)^{n/2}} \exp(-Cf(t - s)) \exp\left(-C\frac{|x - y|^2}{t}\right)
\times \exp(-C(m_J(x)(t - s)^{1/2})^{\alpha_0/2})
\leq CTc_0(x, t; y, s) \exp(-C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0 + 4)})
\times \exp(-C(m_J(x)t^{1/2})^{\alpha_0/2}).
$$

This proves the part (a) since $2\alpha_0/(\alpha_0 + 4) \leq \alpha_0/2$. The part (b) is an easy consequence of the part (a). □

### 4 Proof of Theorem 2, 3

To show Theorem 2, we prove the following inequality.

**Theorem 5** Let $\gamma \in [0, n)$. Then there exists a constant $C$ such that

$$
|m_J(x)^{2l}(e^{-tL_J}f)(x)| \leq \frac{C}{t^{l+(\gamma/2)}}(M_{\gamma}f)(x)
$$

(28)
holds for every $0 < l \leq (n - \gamma)/2$. Here $M_\gamma f$ is the fractional maximal function defined by

$$(M_\gamma f)(x) = \sup_{x \in B} \frac{1}{|B|^{1-\gamma/n}} \int_B |f| \, dy,$$

where the supremum is taken all balls $B$ containing $x$.

Theorem 2 is a consequence of Theorem 5 and the following lemma (see, e.g., [St]).

**Lemma 3** Let $0 \leq \gamma < n$. There exists a constant $C$ such that

$$||M_\gamma f||_q \leq C||f||_p$$

for $1 < p \leq q \leq +\infty$ and $1/q = 1/p - \gamma/n$.

**Proof of Theorem 5:** Let $r = 1/m_J(x)$. By Corollary 1 (b) we have

\[
|m_J(x)^{2l}(e^{-iL_J} f)(x)| \leq C m_J(x)^{2l} \int \frac{|f(y)|}{(1 + m_J(x)|x-y|)^k r^{n/2}} \exp\left(-\frac{C|x-y|^2}{t}\right) dy
\]

\[
\leq \frac{C}{r^{2l}t^{n/2}} \sum_{j=-\infty}^{\infty} \int_{\{2^{-j}r < |x-y| \leq 2jr\}} \frac{|f(y)|}{(1+2^{-j-1})^k} \exp\left(-\frac{C(2^{j}r)^2}{t}\right) dy.
\] (29)

By the assumption on $l$, we take $\alpha \geq 0$ such that $2\alpha = n - \gamma - 2l$. Put $C_\alpha = \sup_{s>0} s^\alpha e^{-s} < +\infty$ for $\alpha \geq 0$. Then the right hand side of (29) is dominated by

\[
\frac{C}{r^{2l}t^{n/2}} \sum_{j=-\infty}^{\infty} \int_{\{2^{-j}r < |x-y| \leq 2jr\}} \frac{1}{r^{2l}(1+2^{j-1})^k} \left(\frac{C(2^{j}r)^2}{t}\right)^{-\alpha} |f(y)| dy
\]

\[
\leq \frac{C_\alpha C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{-j-1})^{k(2^j-1)2\alpha}} \left(\frac{1}{(2^j r)^{n-\gamma}} \int_{\{x-y| \leq 2jr\}} |f(y)| dy\right).
\] (30)

Now, since $n - \gamma - 2\alpha = 2l > 0$, by taking $k > 2l$ we have

\[
\sum_{j=1}^{\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^{k(2^j-1)2\alpha}} \leq \sum_{j=1}^{\infty} \frac{C}{2^{j(k-2l)}} < +\infty,
\]
and
\[ \sum_{j=-\infty}^{0} \frac{(2^{j})^{n-\gamma}}{(1 + 2^{j-1})^{k}(2^{j-1})^{2\alpha}} \leq \sum_{j=-\infty}^{0} C(2^{j})^{2l} < +\infty. \]

Thus, we obtain the desired result. \( \square \)

**Proof of Theorem 3:** First, the estimate for the case \( l = 0 \) and \( p = 1 \) is classical except the exponential factor in time. Under the assumption, by Corollary 1 (a) we have

\[ \left| \Gamma_{J}(x; t; y, s) \right| \leq C_{0} \gamma_{J}(x; t; y, s) \exp(-C(1 + m_{J}(x)|x - y|)^{2\alpha/(\alpha + 4)}) \times \exp(-C(1 + m_{0}t^{1/2})^{\alpha/2}) \]  

(31)

for some positive constants \( C \) and \( C_{0} \). Then by using this estimate we can prove the part (a) of Theorem 3 in a similar way as in the proof of Theorem 2. To show the part (b), we use the semigroup property and Theorem 2 and get

\[ \|m_{J}(x)^{2}e^{-tL_{J}f}\|_{L^{\infty}(\mathbb{R}^{n})} \leq \frac{C}{t^{l+(n/4)}}\|e^{-(c/3)tL_{J}}f\|_{L^{2}(\mathbb{R}^{n})} \]

for some constant \( C \). Note that under the assumption \( m_{J}(x) \geq m_{0} \), Lemma 1 yields \( \inf \sigma(L_{J}) \geq Cm_{0}^{2} \) for some positive constant \( C \). Here \( \sigma(L_{J}) \) is the spectrum of the operator \( L_{J} \). So, we have

\[ \|e^{-(1/3)tL_{J}}g\|_{L^{2}(\mathbb{R}^{n})} \leq e^{-Cm_{0}^{2}t}\|g\|_{L^{2}(\mathbb{R}^{n})}. \]

Using this estimate, we obtain

\[ \|m_{J}(x)^{2}e^{-tL_{J}f}\|_{L^{\infty}(\mathbb{R}^{n})} \leq \frac{C}{t^{l+(n/4)}}e^{-Cm_{0}^{2}t}\|e^{-(1/3)tL_{J}}f\|_{L^{2}(\mathbb{R}^{n})} \]

\[ \leq \frac{C}{t^{l+(n/4)}}e^{-Cm_{0}^{2}t} \frac{C}{t^{l/(2(1/p-1/2))}}\|f\|_{L^{p}(\mathbb{R}^{n})}. \]

In the last inequality, we used \( p \leq 2 \) and Theorem 2.

\( \square \)

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