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Geometric study on smoothing effects for dispersive evolution equations

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1. Introduction

Let M be a C^{∞} manifold with C^{∞} positive density μ, and put $\mathcal{H}=L^{2}(M,\mu)=L^{2}(M)$. Denote by $\Psi^{s}(M)$ the set of all pseudodifferential operators of type (1,0) of order s on M. For a function $f\in C^{1}(T^{*}M)$ (or $T^{*}M\setminus 0$), indicate by H_{f} the Hamilton vector field of f: in a canonical chart (x,ξ), $H_{f}=\sum_{j=1}^{d}\left(\frac{\partial f}{\partial\xi_{j}}\frac{\partial}{\partial x_{j}}-\frac{\partial f}{\partial x_{j}}\frac{\partial}{\partial\xi_{j}}\right)$.

Let $H\in\Psi^{m}(M)$ $(m>1)$ be a properly supported, formally self-adjoint operator with positively homogeneous principal symbol $\sigma_{prin}(H)=h>0$ on $T^{*}M\setminus 0$, whose Hamilton vector field H_{h} is complete on $T^{*}M\setminus 0$. Let Φ_{t} be the H_{q}-flow in $T^{*}M\setminus 0$, where $q=h^{1/m}$. Assume that

(H0) $\quad H|_{C_{0}^{\infty}(M)}$ is essentially self-adjoint.

Denote its self-adjoint extension by the same symbol H.

A typical example is the Laplace-Beltrami operator $-H=\Delta_{g}(\leq 0)$ on a C^{∞} complete Riemannian manifold (M,g) with the associated density $\mu=\mu_{g}$, where $m=2$. In this case, $H|_{C_{0}^{\infty}(M)}$ is essentially self-adjoint, and Φ_{t} is the geodesic flow.

This report discusses the relationship between smoothing effects of the quantum flow $\exp(-itH)$ and the global behavior of the classical flow

Φ_t. We shall explain: first, propagation of smoothing effects along the Hamilton flow Φ_t in the positive direction; second, absence of smoothing effects at every point $z_0 \in S^*M = \{z \in T^*M;\ h(z) = 1\}$ such that for every neighborhood U of it, $\sup_{z' \in S^*M} |\{t \in \mathbf{R}; \Phi_t(z') \in U\}| = \infty$, where $|\cdot|$ is the 1-dimensional Lebesgue measure; third, abstract theory of smoothing effects for a pair of self-adjoint operators in a Hilbert space. Combining all results, we shall conclude that the smoothing effects hold at every point nontrapped backwards, and fail at almost every point trapped backwards, by the Hamilton flow under certain global conditions. This approach is applicable to the Schrödinger equations associated with complete Riemannian metrics having strictly convex functions near infinity: (i) asymptotically Euclidean metric, (ii) conformally compact metric, (iii) generalized scattering metric, (iv) metric of separation of variables near infinity. The details are discussed in [Do4,5].

Now we explain some related works. For the Schrödinger evolution equation with non-flat principal symbol, there are works such as Craig-Kappeler-Strauss [CKS], Craig [Cr], Doi [Do1-3], Kapitanski-Safarov [KS]; Wunsch [Wu1,2]; Kajitani-Wakabayashi [KW], Robbiano-Zuily [RZ1,2] (analytic class); Kajitani [Ka] (Gevrey class). For the Schrödinger evolution equation related with the (quadratic) potential term, there are works such as Kapitanski-Rodianski [KR], Yajima [Ya1,2], Zelditch [Ze], Kapitanski-Rodianski-Yajima [KRY], Wunsch [Wu2]. For the nonlinear Schrödinger equation, see, for example, Chihara [Ch], and Kenig-Ponce-Vega [KPV].

Notation. $\mathbf{N} = \{1, 2, 3, \ldots\}$; $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$; $\mathbf{R}_+ = (0, \infty)$. For topological vector spaces X, Y, $L(X,Y)$ denotes the set of all continuous linear operators from X to Y, and $L(X) = L(X,X)$. For $x \in \mathbf{R}^d$, $\langle x \rangle = (1+|x|^2)^{1/2}$. For pseudodifferential operators, see [Hö, Chapter 18]. We quote only the definition of $S(m,g)$. For positive functions m and g_j $(j = 1,\ldots,n)$ on $\mathbf{R}^n$, the symbol class $S(m,g)$ consists of all functions $f \in C^\infty(\mathbf{R}^n)$ such that for every $k = 0, 1, \ldots$

$$|f|_{k,S(m,g)} = \sum_{|\alpha|\leq k} \sup_{z\in\mathbf{R}^n} (m(z)g(z)^\alpha)^{-1} |\partial_z^\alpha f(z)| < \infty,$$

where $g = \sum_{j=1}^n g_j(z)^2 dz_j^2$ and $g(z)^\alpha = g_1(z)^{\alpha_1}\cdots g_n(z)^{\alpha_n}$. Set $S^\lambda = S^\lambda(\mathbf{R}) = S(\langle t\rangle^\lambda, \langle t\rangle^{-2}dt^2)$ $(\lambda \in \mathbf{R})$, where $t \in \mathbf{R}$.

2. Propagation of smoothing effects

First, we fix the notation. An operator in $\Psi^s(M)$ is called compactly supported if its distribution kernel has a compact support; indicate by $\Psi^s_{cpt}(M)$ the set of all compactly supported operators in $\Psi^s(M)$. For $P \in \Psi^s(M)$, the essential support of P, denoted by ess-supp P or by $WF(P)$, is the smallest closed conic set of $T^*M \setminus 0$ such that P is of order $-\infty$ in the complement (see [Hö, Chapter 18.1]). For a subset U of S^*M, denote by $\Psi^s_*(U)$ the set of all $P \in \Psi^s_*(M)$ satisfying ess-supp $P \cap S^*M \subset U$, where $\Psi_* = \Psi$, Ψ_{cpt}.

Now we state our propagation theorem.

Theorem 2.1. *Let U be an open subset of S^*M, and put $\Gamma = \cup_{0\leq t\leq T}\Phi_t(U)$ $(T > 0)$. Let $s \in \mathbf{R}, r \geq 0$, $N >> 1$. For every $A_j = A_j^* \in \Psi^{s+(m-1)j}_{cpt}(\Gamma)$ $(j = 0, 1, r, r+1)$, there exist $P_0 = P_0^* \in \Psi^s_{cpt}(\Gamma)$, $B_{j+1} = B_{j+1}^* \in \Psi^{s+(m-1)(j+1)}_{cpt}(U)$ $(j = 0, r)$, and $C > 0$ such that the estimate below* *holds for every* $t \geq 0$, *and* $u_0 \in \cap_{n\in\mathbf{N}} D(H^n)$

$$
\begin{aligned}
&(\,(A_0+|t|^r A_r)u(t), u(t)\,) + \int_0^t (\,(A_1+|\tau|^r A_{r+1})\,u(\tau), u(\tau)\,)d\tau \\
&\leq (P_0 u(0), u(0)) + \int_0^t (\,(B_1+|\tau|^r B_{r+1})\,u(\tau), u(\tau)\,)d\tau \\
&\quad + C(1+t^{r+1})\|\,(1+|H|)^{-N/m} u_0\,\|^2.
\end{aligned}
$$

Here $u(t) = e^{-itH}u_0$.

Theorem 2.1 means that the smoothing effect associated with the time-dependent weight $(1+t\langle\xi\rangle^{m-1})^r\langle\xi\rangle^s$ propagetes along the Hamilton flow in the positive directon. The proof is reduced to the Euclidean case $(M,\mu) = (\mathbf{R}^d, |dx|)$, and is based on the construction of a time-dependent nonnegative observable $P(t)$ $(t \geq 0)$ satisfying

$$
\begin{aligned}
&-(\partial_t + i\mathrm{ad}_H)P(t) \geq Q_1(t) - Q_2(t) - R(t) \;\; (t \geq 0); \\
&P(t), Q_1(t), Q_2(t) \geq 0 \;\; (t \geq 0); \quad R(t):\ \text{an error term}
\end{aligned}
$$

in the framework of the Weyl-Hörmander calculus associated with the time-dependent symbol class $S((1+t\langle\xi\rangle^{m-1})^r\langle\xi\rangle^s, |dx|^2+\langle\xi\rangle^{-2}|d\xi|^2)$ $(t \geq 0)$ (see [Hö, Chapter 18]).

3. Lack of smoothing effects

Let $t_0 > 0$ be fixed, and set $I = [0, t_0]$. For a point $z_0 \in S^*M$, consider the assertions $(i)_r$ and $(ii)_r$ $(r \geq 0)$.

$(i)_r$ There is an open neighborhood U of z_0 in S^*M such that for every $A \in \Psi_{cpt}^{(r+(1/2))(m-1)}(U)$ the mapping below is continuous:

$$
L^2_{cpt}(M) \ni u \mapsto |t|^r A e^{-itH} u \in L^2(I; L^2(M)).
$$

$(ii)_r$ There is an open neighborhood U of z_0 in S^*M such that for every $A \in \Psi_{cpt}^{r(m-1)}(U)$ the mapping below is continuous:

$$L^2_{cpt}(M) \ni u \mapsto |t|^r A e^{-itH} u \in C(I; L^2(M)).$$

The assertions are open in the sense that if they hold at z_0, then they hold at every point near z_0. By interpolation, if $(i)_0$ and $(i)_r$ hold, then $(i)_{r'}$ holds for every $0 \le r' \le r$; similarly, if $(ii)_r$ holds, then $(ii)_{r'}$ holds for every $0 \le r' \le r$, because $(ii)_0$ is always valid. Theorem 2.1 gives

Corollary 3.1. *If $(i)_0$ and $(i)_r$ are valid at z_0, then $(i)_{r'}$ and $(ii)_{r'}$ are valid at $\Phi_t(z_0)$ for every $t \ge 0$ and $0 \le r' \le r$.*

We prepare some notions related with the classical mecanics (S^*M, Φ_t).

Every 1-form θ satisfying $\theta \wedge dh = \frac{1}{d!}\sigma^d$ in $T^*M \setminus 0$ induces the unique Φ_t-invariant measure on S^*M, denoted by meas_h. Here σ is the canonical 2-form on T^*M, and $d = \dim M$.

Denote by $S_{cpt,\pm}$ the set of all $z \in S^*M$ such that $\{\Phi_t(z)\}_{\pm t \ge 0}$ is relatively compact.

Indicate by $S_{lim,\pm}$ the set of all $z \in S^*M$ such that there are $z' \in S^*M$ and a sequence of real numbers $\{t_j\}_{j\in\mathbf{N}}$ satisfying $\Phi_{t_j}(z') \to z$ and $t_j \to \pm\infty$ as $j \to \infty$ (i.e., z is a positive (resp. negative) limit point of z').

The set S_0 consists of all $z \in S^*M$ such that for every neighborhood U of z, $\sup_{z' \in S^*M} |\{t \in \mathbf{R}; \Phi_t(z') \in U\}| = \infty$, where $|\cdot|$ is the 1-dimensional Lebesgue measure. It is closed, and Φ_t-invariant; and $S_{lim,+} \cup S_{lim,-} \subset S_0 \subset \{\text{nonwandering points}\}$ (see [Do2, Proposition 1.2]). So, if $\mathrm{meas}_h(S^*M) < \infty$, then $\overline{S_{lim,\pm}} = S_0 = S^*M$.

Theorem 3.2. *All the assertions* $(i)_r$ $(r \geq 0)$ *and* $(ii)_r$ $(r > 0)$ *fail at every point of* S_0.

The proof is by contradictin as well as [Do2, Proof of Theorem 1.5]; assuming the smoothing estimate, we derive from it another estimate depending on a large parameter λ, and choose a λ-dependent initial data, which proves to break the estimate derived above as $\lambda \to \infty$ by virtue of an Egorov-type lemma containing λ.

4. Abstract theory of smoothing effects

Let $\mathcal{H}$ be a Hilbert space, and A and B a pair of self-adjoint operators on $\mathcal{H}$ satisfying $A \geq 1$, $B \geq 1$. We prepare first the weighted Sobolev spaces associated with A and B. Put $D^{(t,s)} = D(B^t A^s)$ $(t, s \geq 0)$, $\mathcal{S} = \cap_{t,s\geq 0} D^{(t,s)}$; $D^{(t,s)}$ has a natural Hilbert space structure with norm $\|u\|_{D^{(t,s)}} = \|B^t A^s u\|$. Assume (A1) and (A2) with $0 < \nu \leq 1$ being fixed.

(A1) For $z \notin \sigma(A)$, $(z - A)^{-1} \in L(D(B))$.

(A2) $D(A) \cap D(B)$ is dense in $D(B^{1-\nu})$; the multiple commutator $\mathrm{ad}_A^N B$, firstly defined as a quadratic form on $D(A) \cap D(B)$, is extended to an operator in $L(D(B^{1-\nu}), D(B^0))$ inductively on $N \in \mathbf{N}$; further, $\mathrm{ad}_A^N B \in L(D(B^{t+1-\nu}), D(B^t))$ for every $t \geq 0$, and $N \in \mathbf{N}$.

Here $\mathrm{ad}_A^0 B = B$, $\mathrm{ad}_A B = [A, B] = AB - BA$. Then $\mathcal{S}$ has a natural Fréchet space structure, and is dense in $D^{(t,s)}$ $(t, s \geq 0)$; and $A^s, B^t \in L(\mathcal{S})$ $(t, s \in \mathbf{R})$ so that $D^{(t,s)} = \{u \in \mathcal{S}'; B^t A^s u \in \mathcal{H}\}$ is well-defined for every $t, s \in \mathbf{R}$, where $\mathcal{S}'$ is the set of continuous anti-linear functionals on $\mathcal{S}$. Set $H^{(t,s)} = D^{(\nu t, s)}$, $m = 1/\nu \geq 1$, $\Lambda = B^{1/m}$.

Next we introduce a new operator class $Q^{(b,a)}$ and its subclass $R^{(b,a)}$ in the spirit of Gérard, Isozaki and Skibsted [GIS], which corresponds roughly to the class of pseudodifferential operators associated with the symbol class $S(\langle\xi\rangle^b\langle x\rangle^a, \langle x\rangle^{-2}|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$ (cf. [Hö, Chapter 18]).

Definition 4.1. $P^{(b,a)}$ is the set of all $P \in L(\mathcal{S}) \cap L(\mathcal{S}')$ such that $P \in L(H^{(t+b,s+a)}, H^{(t,s)})$ for every $t, s \in \mathbf{R}$.

Definition 4.2. $Q^{(b,a)}$ is the set of all $P \in P^{(b,a)}$ such that for every $N \in \{1, 2, \ldots\}, L_1, \cdots, L_N \in \{A, B\}$

$$\mathrm{ad}_{L_1} \cdots \mathrm{ad}_{L_N} P \in P^{(b+\beta m-N, a+\alpha-N)}.$$

Here $\alpha = \#\{1 \leq j \leq N; L_j = A\}$, $\beta = \#\{1 \leq j \leq N; L_j = B\}$.

By definition, it follows easily that

$$Q^{(b,a)} \cdot Q^{(b',a')} \subset Q^{(b+b',a+a')}; \quad (Q^{(b,a)})^* \subset Q^{(b,a)}.$$

However, we can not expect that $[\,Q^{(b,a)}, Q^{(b',a')}\,] \subset Q^{(b+b'-1,a+a'-1)}$, because $Q^{(b,a)}$ is, in some sense, a dual object of the algebra generated by A and B, which could be too small in general. So let us consider the biggest subspace $R^{(b,a)}$ of $Q^{(b,a)}$ such that $[R^{(b,a)}, Q^{(b',a')}] \subset Q^{(b+b'-1,a+a'-1)}$.

Definition 4.3. $R^{(b,a)}$ is the set of all $P \in Q^{(b,a)}$ such that for every $b', a' \in \mathbf{R}$, $Q \in Q^{(b',a')}$

$$\mathrm{ad}_P Q \in Q^{(b+b'-1,a+a'-1)}.$$

Then we have

$$R^{(b,a)} \cdot R^{(b',a')} \subset R^{(b+b',a+a')}; \quad (R^{(b,a)})^* \subset R^{(b,a)};$$
$$[R^{(b,a)}, R^{(b',a')}] \subset R^{(b+b'-1,a+a'-1)}.$$

We assume (A3) as a compatibility condition:

(A3) $A \in Q^{(0,1)}$, $B \in Q^{(m,0)}$; that is, for every $N \in \{0,1,\ldots\}$, $L_0,\cdots,L_N \in \{A,B\}$

$$\mathrm{ad}_{L_N}\cdots\mathrm{ad}_{L_1}L_0 \in P^{(\beta m-N,\alpha-N)}.$$

Here $\alpha = \#\{0 \le j \le N; L_j = A\}$, $\beta = \#\{0 \le j \le N; L_j = B\}$.

Technically, we need to develop an analogy of Weyl-Hörmander calculus associated with the symbol class

$$S((\langle x\rangle + t\langle\xi\rangle^{m-1})^r\langle\xi\rangle^b\langle x\rangle^a, \langle x\rangle^{-2}|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$$

depending uniformly on the time-parameter $t \ge 0$, which we do not explain here.

Let $X \ge 1$ and $H \ge 0$ be a pair of self-adjoint operators on $\mathcal{H}$, satisfying (A1)-(A3) with $A = X$ and $B = 1 + H$, and a Mourre-type condition near infinity with respect to X.

(A4) There exist $R > 0, \delta > 0, K > 0$ such that as a quadratic form on $\mathcal{S}$ the following estimate holds for every real-valued function $\alpha \in S^0(\mathbf{R})$ with $\operatorname{supp}\alpha \subset (R,\infty)$

$$\alpha(X)[\,iH,\,[\,iH,\,X^2\,]\,]\alpha(X) \ge 2\delta^2\alpha(X)\Lambda^{2(m-1)}\alpha(X) - 2K\alpha(X)\Lambda^{2m-3}\alpha(X).$$

Here $\Lambda = (1+H)^{1/m}$. Introduce

$$E = \Lambda^{(1-m)/2} i[H,X]\Lambda^{(1-m)/2} \in R^{(0,0)}.$$

Let $f, f_1, g, g_1 \in C^\infty(\mathbf{R};\mathbf{R})$ such that $f(t) = 1$ for $t >> 1$, $f_1 = 1$ in a neighborhood of $\operatorname{supp} f$, $\operatorname{supp} f_1 \subset (R,\infty)$, $g = 1$ in a neighborhood of

$(-\infty, -\delta]$, $g_1 = 1$ in a neighborhood of $\operatorname{supp} g$, $\operatorname{supp} g_1 \subset (-\infty, 0)$. Then one of our main results is:

Theorem 4.4. *For $a \geq 0$, $b \in \mathbf{R}$, $N >> 1$, $\varepsilon > 0$, there exists $C > 0$ such that the following estimate holds: for every $t \geq 0$ and $u \in \mathcal{S}$*

$$\begin{aligned}
&t^a \| \Lambda^{(b+(m-1)a)/2} f(X) g(E) e^{-itH} u \|^2 \\
&+ \int_0^t \tau^a \| \Lambda^{(b+(m-1)(a+1))/2} X^{-(1+\varepsilon)/2} f(X) g(E) e^{-i\tau H} u \|^2 d\tau \\
&\leq C \| \Lambda^{b/2} X^{a/2} f_1(X) g_1(E) u \|^2 + C(1 + t^{a+1}) \| \Lambda^{(b-N)/2} u \|^2.
\end{aligned}$$

The proof is based on the construction of a time-dependent nonnegative observable $P(t)$ $(t \geq 0)$ with nonpositive Heisenberg derivative with respect to H in the framework of commutator calculus above:

$$\begin{aligned}
&-(\partial_t + i\mathrm{ad}_H) P(t) \geq Q(t) - R(t) \ (t \geq 0); \\
&P(t),\ Q(t) \geq 0 \ (t \geq 0); \quad R(t) : \text{ an error term.}
\end{aligned}$$

5. Global picture of smoothing effects

We return to the manifold setting in Sections 2 and 3. Let X be a multiplication operator by a function $r \in C^\infty(M)$ such that $r \geq 1$, and that $\{x \in M; r(x) \leq L\}$ is compact for every $L > 0$. Assume (H1) and (H2) in addition to (H0).

(H1) For every $N \in \{0, 1, \ldots\}, L_0, \cdots, L_N \in \{X, H\}$, $\alpha' \in \mathbf{R}$,

$$\Lambda^{N-\beta m} X^{N-\alpha-\alpha'} (\mathrm{ad}_{L_N} \cdots \mathrm{ad}_{L_1} L_0) X^{\alpha'}|_{C_0^\infty(M)}$$

extends to an operator in $L(\mathcal{H})$. Here $\alpha = \#\{0 \leq j \leq N; L_j = X\}$, $\beta = \#\{0 \leq j \leq N; L_j = H\}$.

(H2) There exist $R > 0, \delta > 0, K > 0$ such that as a quadratic form on $C_0^\infty(M)$ the following estimate holds for every real-valued function $\alpha \in S^0(\mathbf{R})$ with $\operatorname{supp} \alpha \subset (R, \infty)$

$$\alpha(X)[iH, [iH, X^2]]\alpha(X) \geq 2\delta^2\alpha(X)\Lambda^{2(m-1)}\alpha(X) - 2K\alpha(X)\Lambda^{2m-3}\alpha(X).$$

The conditions (H1) and (H2) imply (A1)-(A4), and hence Theorem 4.4 holds in this setting. Moreover, the Mourre-type condition (H2) implies the classical correspondence:

(H2)' $H_h^2(r^2) \geq 2\delta^2$ in $\{z = (x, \xi) \in S^*M; r(x) \geq R\}$.

Here R, δ are the same as in (H2). For $R' \geq R$ and $0 < \delta' < \delta$, define $S_-(R', \delta') = \{z = (x, \xi) \in S^*M; r(x) > R', \ -H_h r(z) > \delta'\}$. Then we have

Lemma 5.1. *(1) $\Phi_t S_-(R', \delta') \subset S_-(R', \delta')$ $(t \leq 0)$;*

(2) For every $z_0 \notin S_{cpt,-}$ there is $T > 0$ such that $\Phi_t(z_0) \in S_-(R', \delta')$ if $t \leq -T$.

So it is reasonable to call $S_-(R', \delta')$ incoming region.

Now we translate the abstract results in Section 4. Recall that $E = \Lambda^{(1-m)/2} i[H, X]\Lambda^{(1-m)/2}$. The operator $f(X)g(E)$ in Theorem 4.4 belongs to $\Psi^0(M)$, and its principal symbol is represented by $f(r)g(r^{1-m}H_h r)$ in $\{z \in T^*M; h(z) > 1/2\}$. Hence it is elliptic in a suitable incoming region $S_-(R', \delta')$. So Theorem 4.4 implies that $(i)_r$ and $(ii)_r$ hold at every point $z_0 \in S_-(R', \delta')$. Combining this with Theorem 2.1 and Lemma 5.1, we have that $(i)_r$ and $(ii)_r$ are valid at every point $z_0 \in S^*M \setminus S_{cpt,-}$. On the other hand, $S_{cpt,-}$ is equal to S_0 modullo a null set under the condition (H2)'. Our conclusion is

Theorem 5.2. *The assertions $(i)_r (r \geq 0)$ and $(ii)_r (r > 0)$ hold at every point $z_0 \notin S_{cpt,-}$, and fail at almost every point $z_0 \in S_{cpt,-}$.*

6. Application

6.1. Asymptotically Euclidean metric on $\mathbf{R}^d$

Let $g = \sum_{j,k=1}^{d} g_{jk}(x) dx^j \otimes dx^k$ be a C^∞ Riemannian metric on $M = \mathbf{R}^d$. Assume

(i) with $C \geq 1$: $C^{-1}|dx|^2 \leq g \leq C|dx|^2$ in $\mathbf{R}^d$;

(ii) $|\partial^\alpha g_{jk}(x)| \leq C_\alpha (1 + |x|)^{-|\alpha|}$, $x \in \mathbf{R}^d$ for all $\alpha \in \mathbf{Z}_+^d, 1 \leq j, k \leq d$;

(iii) there is $f \in S(\langle x \rangle^2, \langle x \rangle^{-2}|dx|^2)$, $f \geq 1$, such that $\mathrm{Hess}_g f \geq g$ outside a compact set.

Then $H = -\Delta_g$, $X = \sqrt{f}$ satisfy (A1)-(A4) with $B = 1 + H$, $A = X$, $m = 1/\nu = 2$. Remark that (iii)' implies (iii) with $f(x) = 1 + |x|^2$:

(iii)' $|\partial_i g_{jk}(x)| = o(|x|^{-1})$ as $|x| \to \infty$ for all $1 \leq i, j, k \leq d$.

6.2. Conformally compact metric

Let $\overline{M}$ be a C^∞ compact manifold with boundary ∂M, and let $x \in C^\infty(\overline{M}, \mathbf{R})$ be a defining function of ∂M; that is, $M := \overline{M} \setminus \partial M = \{x > 0\}$, $\partial M = \{x = 0\}, dx \neq 0$ on ∂M. Let g_0 be a C^∞ Riemannian metric on $\overline{M}$, and define the Riemannian metric on M by $g = a(x)^{-2} g_0$, where $a \in C^\infty(\mathbf{R}_+, \mathbf{R}_+)$. Then g is complete if and only if $\int_0^1 a(s)^{-1} ds = \infty$. Put $b(t) = \int_t^{t_0} a(s)^{-1} ds + 1$, where $t_0 > \sup_{p \in M} x(p)$. Assume

(i) $b(+0) = \infty$ (i.e., g is complete);

(ii) $|a^{(k)}(t)| \leq C_k' a(t)(a(t)b(t))^{-k}$, $0 < t < t_0$, for $k = 1, 2, \ldots$;

(iii) $\liminf_{t \to +0} a'(t) b(t) > 0$.

Then $H = -\Delta_g$, $X = b \circ x$ satisfy (A1)-(A4) with $B = 1 + H$, $A = X$, $m = 1/\nu = 2$.

Remark. Clearly, $a(t) = t^r$ $(r > 1)$ satisfies (i)-(iii).

6.3. Generalized scattering metric

Let $\overline{M}$ be a C^∞ compact manifold with boundary ∂M, and let $x \in C^\infty(\overline{M}, \mathrm{R})$ be a defining function of ∂M; that is, $M := \overline{M} \setminus \partial M = \{x > 0\}$, $\partial M = \{x = 0\}, dx \neq 0$ on ∂M. Choose an open neighborhood U of ∂M in $\overline{M}$, and $y \in C^\infty(U; \partial M)$ so that $U \ni p \to (x(p), y(p)) \in [0, \varepsilon) \times \partial M$ is diffeomorphic $(0 < \varepsilon << 1)$, by which we identify U and $[0, \varepsilon) \times \partial M$. Let g be a C^∞ Riemannian metric on $M = \overline{M} \setminus \partial M$ such that on $(0, \varepsilon) \times \partial M$

$$g_{(x,y)} = h(x, y, dx/x^2, dy/x)$$

where $h(x, y, dx, dy)$ is a C^∞ Riemannian metric on $[0, \varepsilon) \times \partial M$. Assume further there is $\delta > 0$ such that

$$\mathrm{Hess}_g(1/x^2) \geq \delta\, g \quad \text{near infinity.}$$

Then $X = 1/x$ (near infinity), $H = -\Delta_g$ satisfy (A1)-(A4) with $A = X, B = H + 1, m = 1/\nu = 2$.

The metric g on M is called a scattering metric if g takes the following form near infinity: $(x, y) \in (0, \varepsilon) \times \partial M$

$$g_{(x,y)} = \frac{|dx|^2}{x^4} + \frac{g'(x, y, dx, dy)}{x^2},$$

where g' is a C^∞ symmetric tensor field of type (0,2) on $[0, \varepsilon) \times \partial M$ satisfying that $g'(0, y, 0, dy)$ is a C^∞ Riemannian metric on ∂M (cf. [Wu1]).

In our notation, $h(x,y,dx,dy) = |dx|^2 + g'(x,y,xdx,dy)$. In this case, the convexity of $1/x^2$ is satisfied. See [Wu1] for sharper results concerning the scattering metric.

6.4. Metric of separation of variables near infinity

Let (M,g) be a C^∞ Riemannian manifold. Assume that there exist a C^∞ compact Riemannian manifold (N,ω), and a C^∞ diffeomorphism χ from $(0,\infty)\times N$ to an open subset U of M satisfying

$$\chi^* g = dt\otimes dt + f(t)^2\omega;\quad M\setminus\chi((1,\infty)\times N) \text{ is compact},$$

where $f\in C^\infty((0,\infty);\mathbf{R})$ satisfies

(i) $|f^{(k)}(t)/f(t)| \le C_k t^{-k}$, $t > 1/8\,(k=0,1,\ldots)$;

(ii) with $\delta > 0$, $tf'(t)/f(t) \ge \delta\,(t >> 1)$.

Then $H = -\Delta_g$, $X = r$ satisfy (A1)-(A4) with $A = X, B = 1+H, m = 1/\nu = 2$. Here $r\in C^\infty(M,\mathbf{R})$ satisfies $r\ge 1$ and $\chi^* r = t\,(t>2)$.

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