Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions

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Abstract of the Talk

The talk will be concerned with the boundedness of the commutators of Calderon-Zygmund singular integral operators on Morrey spaces $L^{p,\Phi}(R^n)$ with growth functions $\Phi(x,r)$ satisfying the condition ; there exists a constant $C$, independent of $(x,r) \in R^{n+1}$, such that for any $(x,r) \in R^{n+1}$

$$\int_r^\infty \frac{\Phi(x,t)}{t^{a+1}} dt \leq C \Phi(x,r)/r^a, \text{ for some } a > 0.$$ 

In this case, we write $\Phi \in G_a$ simply. We denote by $L^{p,\Phi}(R^n)$, $0 < p < \infty$, the space of locally integrable functions $f$, defined on $R^n$, for which there exists a constant $C$, independent of balls $B = B(x,r)$, such that

$$\int_{B(x,r)} |f(y)|^p dy \leq C \Phi(x,r) \text{ for all balls } B = B(x,r).$$

Let $BMO(R^n)$ be the space of all functions of bounded mean oscillation and let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order $\alpha$. Let $M$ be the Hardy-Littlewood maximal operator. We need two variants of $M$. For $0 < q < \infty$ let $M_q f(x) = \{ (M|f|^q)(x) \}^{1/q}$. The sharp maximal function $f^\#(x)$ is defined by

$$f^\#(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \text{ where } f_B = |B|^{-1} \int_B f(y) dy.$$

Let $T$ be a Calderon-Zygmund singular integral operator defined by $Tf = k*f$ with the kernel $k$ satisfying the conditions:

$$\|k\|_{\infty} \leq C, \text{ } |k(x)| \leq C|x|^{-n} \text{ for } 0 \neq x \in R^n,$$

$$|k(x) - k(x-y)| \leq C|y|/|x|^{n+1} \text{ for } |y| \leq |x|/2.$$ 

Let $I_\alpha$, $0 < \alpha < n$, be the Riesz potential of order $\alpha$ defined by

$$(I_\alpha f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$
Related to $I_{\alpha}f$, the fractional maximal function $M_{\alpha/n}^{*}f(x)$ is defined by

$$M_{\alpha/n}^{*}f(x) = f_{\alpha,1}^{*}(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha}/n} \int_{Q} |f(y)|dy.$$ 

For a locally integrable function $b$ and an operator $S$, we define the commutator $[b, S]$, between the operator $S$ and the multiplication operator by $b$, by $[b, S] = bS - Sb$.

We have proved the following ([Miz2]);

**Theorem 1** (Theorem 2.1.) Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C = C(p, \Phi) > 0$, independent of $f$, such that for all $f \in L^{p, \Phi}(\mathbb{R}^n) \cap L_{c}^\infty(\mathbb{R}^n)$

$$||Mf||_{p, \Phi} \leq C||f\#||_{p, \Phi}$$

where $L_{c}^\infty(\mathbb{R}^n)$ be the set of all essentially bounded functions on $\mathbb{R}^n$ with compact support.

We use the method due to Di Fazio and Ragusa. Our method is based on weighted maximal inequality due to Garcia-Cuerva and Rubio de Francia.

From this and the pointwise estimate due to Strömberg;

$$\{[b, T](f)\}^s(x) \leq C\|b\|_{*}\{M_q(Tf)(x) + (M_{\epsilon f})(x)\}, \quad 1 < q, s < \infty,$$

for almost all $x \in \mathbb{R}^n$, we obtain the boundedness of the commutators $[b, T]$ on Morrey spaces ([Miz2]);

**Theorem 2** (Theorem 2.2.) Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p, \Phi}$. More precisely, there exists a constant $C = C(p, \Phi) > 0$, independent of $b$ and $f$, such that for all $b \in BMO(\mathbb{R}^n)$ and $f \in L^{p, \Phi}(\mathbb{R}^n) \cap L_{c}^\infty(\mathbb{R}^n)$

$$\|[b, T](f)\|_{p, \Phi} \leq C\|b\|_{*}\|f\|_{p, \Phi}.$$ 

Also we can observe the following ([Miz2]);

**Theorem 3** (Theorem 2.3.) Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-\alpha}$. If $b \in \Lambda_{\alpha}(\mathbb{R}^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p, \Phi}(\mathbb{R}^n)$ into $L^{q, \Phi q/p}(\mathbb{R}^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of $b$ and $f$, such that for all $b \in \Lambda_{\alpha}(\mathbb{R}^n)$ and $f \in L^{p, \Phi}(\mathbb{R}^n) \cap L_{c}^\infty(\mathbb{R}^n)$

$$\|[b, T]f\|_{q, \Phi q/p} \leq C\|b\|_{\Lambda_{\alpha}(\mathbb{R}^n)}\|f\|_{p, \Phi}.$$
This follows from the result (due to Naki [N]) of the boundedness of Riesz potential on Morrey spaces and the pointwise estimate:

$$|([b, T]f)(x)| \leq \int_{\mathbb{R}^n} |b(x) - b(y)||k(x-y)||f(y)|dy \leq C\|b\|_{A_{\alpha}(\mathbb{R}^n)}I_\alpha(|f|)(x).$$

Further we obtain the following result ([Miz2]) from the boundedness of the fractional maximal operator $M_{\alpha/n}^*$ on Morrey spaces and the pointwise estimate due to Strömberg:

$$\{[b, I_\alpha](f)^u\}(x) \leq C\|b\|_{*}\{M_u(I_{\alpha}f)(x) + (M_{\alpha/n}^*|f|^t)^{1/t}(x)\}$$

for almost all $x \in \mathbb{R}^n$, where $1 < u, t < p < n/\alpha$.

**Theorem 4 (Theorem 3.1.)** Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^p,\Phi(R^n)$ into $L^{q,\Phi^{q/p}}(\mathbb{R}^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of $b$ and $f$, such that for all $b \in BMO(\mathbb{R}^n)$ and $f \in L^p,\Phi(R^n) \cap L^{\infty}_c(R^n)$

$$\|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C\|b\|_{*}\|f\|_{p,\Phi}.$$  

Similarly we can show the following ([Miz2])

**Theorem 5 (Theorem 3.2.)** Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-\alpha-\beta}$ and $\Phi^{q/p} \in G_n$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^p,\Phi(R^n)$ into $L^{q,\Phi^{q/p}}(\mathbb{R}^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of $b$ and $f$, such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^p,\Phi(R^n) \cap L^{\infty}_c(R^n)$

$$\|[b, I_\beta]f\|_{q,\Phi^{q/p}} \leq C\|b\|_{\Lambda_\alpha(R^n)}\|f\|_{p,\Phi}.$$  

Our results (Theorems 1, 2 and 4) generalize partly the classical results due to Di Fazio and Ragusa [DiFRag]. Also we obtain the new results (Theorems 3 and 5).

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1. Introduction.

Let $\Phi = \Phi(x,r)$, be a growth function on $R_+^{n+1} = R^n \times R_+$, that is, a positive and non-decreasing function with respect to $r > 0$. We say that the growth function $\Phi(x,r)$ satisfies the $\Delta_2$-condition (or doubling condition) for $r > 0$ if there exists constant $D = D(\Phi) \geq 1$, independent of $(x,r)$, such that

\begin{equation}
\Phi(x, 2r) \leq D \Phi(x,r), \quad (x,r) \in R_+^{n+1},
\end{equation}

or equivalently,

$$
\frac{\Phi(x, 2r)}{D} \leq \Phi(x,r) \leq D \Phi(x,2r), \quad (x,r) \in R_+^{n+1}.
$$

In this case, we write $\Phi \in \Delta_2$ simply. We consider the following functions in $\Delta_2$;

$$
\Phi(x,r) = \Psi(x)r^\lambda \{\log(1+r)\}^\mu, \quad \Psi(x) \in L^\infty(R^n), \quad 0 \leq \lambda < \infty, \quad -\infty < \mu < \infty.
$$

Remark. Nakai [Nak] assumed a slightly weak condition on $\Phi(x,r)$ replacing (1.1); there exists a constant $C > 0$ such that, for all $(x,r) \in R_+^{n+1},$

\begin{equation}
(1.2)
\quad r \leq t \leq 2r \implies C^{-1} \leq \frac{\Phi(x,t)}{\Phi(x,r)} \leq C.
\end{equation}

However, for simplicity, we describe the results on the assumption of (1.1). Of course our results are also valid under the condition (1.2).

Function Spaces. Let $R^n$ be the $n$-dimensional Euclidean space and let $B = B(x,r)$ be the ball centered at $x \in R^n$ and with radius $r > 0$. Let $Q = Q(x,r)$ be the cube centered at $x \in R^n$ and with sides of length $r > 0$, where the cube will always mean a compact cube with sides parallel to the axes and nonempty interior. $|B|$ and $|Q|$ stand for the Lebesgue measures of ball $B$ and cube $Q$, respectively. Let $0 < p < \infty$.

Definition 1.1 (Morrey spaces). (Confer Mizuhara [Miz1]). We denote by $L^{p,\Phi} = L^{p,\Phi}(R^n)$ the space of locally integrable functions $f$, defined on $R^n$, for which there exists a constant $C$, independent of balls $B = B(x,r)$, such that

\begin{equation}
\int_{B(x,r)} |f(y)|^p dy \leq C^p \Phi(x,r)
\end{equation}

for all balls $B = B(x,r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant $C$ satisfying (1.3). Then the space $L^{p,\Phi}$ becomes a quasi-Banach space with quasi-norm $\| \cdot \|_{p,\Phi}$ in the sense of Triebel [Tri]. In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}$ becomes a Banach space with norm $\| \cdot \|_{p,\Phi}$. The balls $B = B(x,r)$ in (1.3) can be replaced by cubes $Q = Q(x,r)$. 

When $\Phi(x, r) = r^\lambda$, $\lambda \geq 0$, then $L^{p, \lambda}$ is the classical Morrey space denoted by $L^{p, \lambda}$ simply. The classical Morrey spaces $L^{p, \lambda}$, $0 < \lambda < n$, were originally introduced by Morrey [Mor] in 1938 and used by himself and the others in the problems related to the calculus of variations and the theory of elliptic PDE's. We refer to Campanato [Cam], Giaguinta [Gia], Kufner-John-Fučik [KufJohFuc] and Peetre [P2].

The same properties of $L^{p, \lambda}$ are known: If $1 \leq p < \infty$, then $L^{p, 0} = L^p(R^n)$ and $L^{p, n} = L^\infty(R^n)$ isometrically. If $n < \lambda$, then $L^{p, \lambda} = \{0\}$. If $1 \leq p < \infty$ and $0 < \lambda < n$, then $L^{p, \lambda}$ does not include nonzero constants. Hence, in the classical Morrey spaces, $L^{p, \lambda}$ for $0 < \lambda < n$ is interesting. Also Hölder's inequality implies the imbedding theorem; if $(n-\lambda)/q = (n-\mu)/p$, $p \leq q$, then $L^{q, \lambda} \subset L^{p, \mu}$.

Let $BMO(R^n)$ be the John-Nirenberg space of all functions of bounded mean oscillation (see John-Nirenberg [JoN]), that is, $BMO(R^n)$ is a Banach space, modulo constants, with norm $\| \cdot \|_*$ defined by

$$\|b\|_* = \sup_B |B|^{-1} \int_B |b(y) - b_B| dy, \quad \text{where} \quad b_B = |B|^{-1} \int_B b(y) dy.$$ 

The space $BMO(R^n)$ is identified with the dual space of the Hardy space $H^1(R^n)$ in the sense of Fefferman-Stein ([FeS2]).

Let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order $\alpha$ on $R^n$. The space $\Lambda_\alpha(R^n)$ is homogeneous in the sense of dilations. The dual space of $H^p(R^n)$ can be identified with the Lipschitz space $\Lambda_\alpha(R^n)$, $\alpha = n(1/p - 1)$.

**Classical operators.** Let $f$ be a locally integrable function on $R^n$. The Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y)| dy$$

where the supremum is taken over all balls $B$ containing $x$ and $|B|$ is the volume of the ball $B$. We introduce two variants of $M$. Let $0 < q < \infty$ and

$$M_qf(x) = \{(M|f|^q)(x)\}^{1/q}.$$ 

Then Hölder's inequality shows that $Mf = M_1f \leq M_qf$ if $1 \leq q < \infty$ and $M_qf \leq M_1f = Mf$ if $0 < q \leq 1$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where} \quad f_B = |B|^{-1} \int_B f(y) dy.$$
Let $T$ be a Calderon-Zygmund singular integral operator $Tf = k \ast f$ defined by the kernel $k$ satisfying the conditions:

\[ \|k\|_\infty \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in \mathbb{R}^n, \]

\[ |k(x) - k(x - y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2. \]

For $\epsilon > 0$, put

\[ T_\epsilon f(x) = \int_{|y|>\epsilon} k(y)f(x-y)dy \quad \text{and} \quad T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|. \]

Let $I_\alpha$, $0 < \alpha < n$, be the fractional integral operator (or Riesz potential operator) of order $\alpha$ defined by

\[ (I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy \]

for a suitable function $f$. Related to $I_\alpha f$, the fractional maximal function $M_{\alpha/n}^* f(x)$, which appeared in [MucWhe] as $f_{\alpha,1}^*(x)$, is defined by

\[ M_{\alpha/n}^* f(x) = f_{\alpha,1}^*(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy. \]

We define the commutator $[b, S]$ between an operator $S$ and the multiplication operator by a locally integrable function $b$, by $[b, S] = bS - Sb$.

In this note we show the boundedness of the commutator $[b, T]$ , for $b \in BMO(\mathbb{R}^n)$ or $b \in \Lambda_\alpha(\mathbb{R}^n)$, on Morrey spaces $L^{p,\Phi}(\mathbb{R}^n)$ with some growth function $\Phi$. Our results (Theorems 2.1, 2.2, 3.1) generalize partly the recent results due to Di Fazio and Ragusa [DiFRag] on the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n, 1 < p < \infty$. Further we obtain the new results (Theorems 2.3 and 3.2). The letters $C$'s will denote positive constants, which may have different values in each line.

### 2. Commutators between Calderon-Zygmund singular integral operators and multiplication operator by a function $b \in BMO(\mathbb{R}^n) \cup \Lambda_\alpha(\mathbb{R}^n)$.

$G_\alpha$-condition. We consider the following condition on growth function $\Phi(x, r)$;

\[ \frac{\Phi(x, t)}{t^a} \in L^1([r, \infty), dt/t) \]

for all $r > 0$ and any $x \in \mathbb{R}^n$, and, in addition, there exists a constant $C$, independent of $(x, r) \in \mathbb{R}^{n+1}_+$, such that

\[ (2.1) \int_r^\infty \frac{\Phi(x, t)}{t^{a+1}} dt \leq C\Phi(x, r)/r^a, \quad (x, r) \in \mathbb{R}^{n+1}_+. \]
for some $a > 0$. In this case, we write $\Phi \in G_a$ simply.

We can observe the following property of $G_a$, $a > 0$:

**Lemma 2.1.** (i) If $0 < a' < n$, then $G_a \subset G_{a'} \subset G_n \subset \Delta_2$.

(ii) If $\Phi \in \Delta_2$ with doubling constant $D$, $1 \leq D < 2^n$, then $\Phi \in G_n$.

(iii) If $a > 0$, then $G_a \subset G_{a\gamma}$ for some $\gamma$, $0 < \gamma < 1$. More precisely, if $\Phi \in G_a$, $a > 0$, there exist constants $\gamma = \gamma(C, a)$, $0 < \gamma < 1$, and $C' = C'(C, a, \gamma) > 0$ such that for any $(x, r) \in R_n^{+1}$

$$(2.2) \quad \int_r^\infty [\Phi(x, t)/t^{a\gamma+1}] dt \leq C' \Phi(x, r)/r^{a\gamma}.$$

**Proof.** (i), (ii) These are easy to see.

(iii) Let

$$\Phi_a(x, r) = \int_r^\infty [\Phi(x, t)/t^{a+1}] dt.$$

Then (2.1) implies

$$\Phi_a(x, r) \leq C \Phi(x, r)/r^a.$$

For $0 < r < R$, we have, integrating by parts and using (2.1),

$$\int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt = \int_r^R [\Phi(x, t)/t^{a+1}t^{a(\gamma-1)}] dt$$

$$= [-\Phi_a(x, t)t^{a(1-\gamma)}]_r^R - \int_r^R [-\Phi_a(x, t)a(1-\gamma)t^{a(\gamma-1)-1}] dt$$

$$= -\Phi_a(x, R)t^{a(1-\gamma)} + \Phi_a(x, r)t^{a(1-\gamma)} + a(1-\gamma) \int_r^R \Phi_a(x, t)t^{a(\gamma-1)-1}] dt$$

$$\leq C\Phi(x, r)/r^{a\gamma} + a(1-\gamma) C \int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt.$$

Hence we obtain

$$\int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt \leq \frac{C}{1 - a(1-\gamma)C} \Phi(x, r)/r^{a\gamma},$$

and we have (2.2) with

$$C' = \frac{C}{1 - a(1-\gamma)C} > 0.$$

Thus we have (2.2) for some $\gamma$ such that $1 - (1/aC) < \gamma < 1$.

Q.E.D.

First using this Lemma, we show the following;
Theorem 2.1. Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C > 0$, independent of $f$, such that

$$(2.3) \quad \|Mf\|_{p,\Phi} \leq C\|f^\#\|_{p,\Phi}$$

for all $f \in L^{p,\Phi} \cap L^\infty_c(\mathbb{R}^n)$, where $L^\infty_c(\mathbb{R}^n)$ is the set of all essentially bounded functions on $\mathbb{R}^n$ with compact support.

Proof. We use the method due to Di Fazio-Ragusa [DiRag]. We recall the weighted version of the maximal inequality due to Fefferman-Stein [FS2]; there exists a constant $C$ such that

$$(2.4) \quad \int_{\mathbb{R}^n} \{Mf(x)\}^p w(x) dx \leq C \int_{\mathbb{R}^n} \{f^\#(x)\}^p w(x) dx$$

for all $w \in A_\infty$ and all $f \in L^p_w(\mathbb{R}^n)$, for $0 < p < \infty$ (see Garcia-Cuerva-Rubio de Francia [GarRub; p.410]) where $A_q$, $1 \leq q \leq \infty$, is the Muckenhoupt class of weight functions.

Let $f \in L^{p,\Phi} \cap L^\infty_c(\mathbb{R}^n)$ and $B$ a ball. We take $w(x)$ as $(M\chi)^\gamma \in A_1$, $0 < \gamma < 1$, where $\chi = \chi_B(x)$ is the characteristic function of the ball $B = B(x_0, r)$.

Then we get by (2.4),

$$\int_B \{Mf(x)\}^p dx = \int_{\mathbb{R}^n} \{Mf(x)\}^p \chi_B(x) dx$$

$$\leq \int_{\mathbb{R}^n} \{Mf(x)\}^p \{M\chi_B(x)\}^\gamma dx \leq C \int_{\mathbb{R}^n} \{f^\#(x)\}^p \{M\chi_B(x_0, r)(x)\}^\gamma dx$$

$$= C \int_{B(x_0, r)} \{f^\#(x)\}^p \{M\chi_B(x_0, r)(x)\}^\gamma dx$$

$$+ C \sum_{k=1}^{\infty} \int_{B(x_0, 2^k r) \setminus B(x_0, 2^{k-1} r)} \{f^\#(x)\}^p \{M\chi_B(x_0, r)(x)\}^\gamma dx$$

$$\leq C \left\{ \int_{B(x_0, r)} \{f^\#(x)\}^p dx + \sum_{k=1}^{\infty} (2^{-k\gamma}) \int_{B(x_0, 2^k r)} \{f^\#(x)\}^p dx \right\}$$

$$\leq C \|f^\#\|_{p,\Phi}^p \left\{ \Phi(x_0, r) + \sum_{k=1}^{\infty} (2^{-k\gamma}) \Phi(x_0, 2^k r) \right\}$$

$$\leq C \|f^\#\|_{p,\Phi}^p \sum_{k=0}^{\infty} \frac{\Phi(x_0, 2^k r)}{2^{km\gamma}} \sim c \|f^\#\|_{p,\Phi}^p \int_r^{\infty} \frac{\Phi(x_0, t)}{t^{n\gamma+1}} dt.$$
we have
\[ \|Mf\|_{p,\Phi} \leq C\|f\|_{p,\Phi}. \]
Thus we have (2.3) for some \( C > 0 \), independent of \( f \in L^{p,\Phi} \cap L^\infty_c(R^n) \).

Q.E.D.

Our second aim is to show the following ;

**Theorem 2.2.** Let \( 1 < p < \infty, b \in BMO(R^n) \) and \( T \) be a Calderon-Zygmund singular integral operator. We assume that \( \Phi \in G_n \). Then the commutator \( [b, T] \) is bounded in \( L^{p,\Phi} \). More precisely, there exists constant \( C \), independent of \( b \) and \( f \), such that

\[ \|[b, T](f)\|_{p,\Phi} \leq C\|b\|_\ast\|f\|_{p,\Phi} \]

for all \( b \in BMO(R^n) \) and \( f \in L^{p,\Phi} \cap L^\infty_c(R^n) \).

To prove the theorem we need Theorem 2.1 and the following three lemmas ;

**Lemma 2.2.** Let \( 1 < q, s < \infty, b \in BMO(R^n) \) and \( T \) be a Calderon-Zygmund singular integral operator. Then there exists constant \( C \) independent of \( b \) and \( f \) such that

\[ \{[b, T](f)\}^\sharp(x) \leq C\|b\|_\ast\{M_q(Tf)(x) + (Ms_f)(x)\} \]

for almost all \( x \in R^n \) and all \( f \in L^\infty_c(R^n) \)

**Proof.** This is the pointwise estimate due to Strömberg (see [Tor, p.418.] and Janson [Jan; pp.268-269.]).

Q.E.D.

**Lemma 2.3.** Let \( 0 < q < p < \infty \). We assume that \( \Phi \in G_n \). Then the maximal operator \( M_q \) is a bounded operator in \( L^{p,\Phi}(R^n) \) and

\[ \|M_qf\|_{p,\Phi} \leq C\|f\|_{p,\Phi} \]

for some constant \( C \) independent of \( f \in L^{p,\Phi}(R^n) \).

**Proof.** The proof depends on the weighted maximal inequality due to Fefferman-Stein [FefSte1]. In the restricted case \( 1 \leq q < p < \infty \), the corresponding result is proved by Nakai [Nak; Theorem 1]. It is not difficult to extend the result to the case \( 0 < q < p < \infty \). Confer also Chiarenza-Frasca [ChiFra] and Mizuhara [Miz1].
Lemma 2.4. Let $1 < p < \infty$. We assume that $\Phi \in G_\alpha$. Then the Calderon-Zygmund singular integral operator $T$ is a bounded operator in $L^{p,\Phi}(R^n)$ and

\begin{equation}
\|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi}
\end{equation}

for some constant $C$ independent of $f \in L^{p,\Phi}(R^n)$.

Proof. This is the result due to Nakai [Nak; Theorem 2] in the setting of more general growth functions. Confer also Peetre [Pee1], Chiarenza-Frasca [ChiFra] and Mizuhara [Miz1].

We note that we can give a short proof following the method of the author [Miz1] which depends on the weighted maximal inequality due to Cordoba-Fefferman [CorFef] (see also [GarRub])

there exists constant $C$, depending only on $T, p$ and $0 < \gamma < 1$, such that

\begin{equation}
\int_{R^n} |Tf(x)|^p \phi(x) dx \leq C \int_{R^n} |f(x)|^p (M\phi)\gamma(x) dx
\end{equation}

for all $f$ and $\phi(x) \geq 0$. A standard proof using (2.7) implies (2.6).

Proof of Theorem 2.2. We apply the method of Di Fazio-Ragusa [DiFaz] to our case. We suppose that $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 2.2 imply that, for $1 < q, s < p < \infty$,

$$
\|([b, T](f))\|_{p,\Phi} \leq \|M\{[b, T](f)\}\|_{p,\Phi} \leq C\|([b, T](f))\|_{p,\Phi} \leq C\|b\|_\star \{\|M_q(Tf)\|_{p,\Phi} + \|M_s f\|_{p,\Phi}\}.
$$

Since, Lemma 2.3 and Lemma 2.4 imply

$$
\|M_q(Tf)\|_{p,\Phi} \leq C\|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi} \quad \text{and} \quad \|M_s f\|_{p,\Phi} \leq C\|f\|_{p,\Phi},
$$

we obtain

$$
\|([b, T](f))\|_{p,\Phi} \leq C\|b\|_\star \|f\|_{p,\Phi}
$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$. Thus we have (2.5).

Q.E.D.

When $b \in \Lambda_\alpha(R^n)$, $0 < \alpha < n$, we obtain the following:
Theorem 2.3. Let $1 < p < q < \infty, \ 0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b,T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

\[(2.8) \quad \| [b,T] f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha(R^n)} \| f \|_{p,\Phi} \]

for some constant $C$ independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$.

To prove the theorem we need the following lemma:

Lemma 2.5. Let $1 < p < q < \infty, \ 0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional integral operator $I_\alpha$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

\[(2.9) \quad \| I_\alpha f \|_{q,\Phi^{q/p}} \leq C \| f \|_{p,\Phi} \]

for some constant $C$ independent of $f \in L^{p,\Phi}$.

Proof. This is the result due to Nakai [Nak; Theorem 3]. Q.E.D.

Proof of Theorem 2.3. Let $b \in \Lambda_\alpha(R^n)$. Then

\[
|([b,T] f)(x)| \leq \int_{R^n} |b(x) - b(y)||k(x-y)||f(y)| dy \\
\leq C \| b \|_{\Lambda_\alpha} \int_{R^n} |x-y|^\alpha|x-y|^{-n}|f(y)| dy = C \| b \|_{\Lambda_\alpha} I_\alpha(|f|)(x).
\]

Hence we have, by Lemma 2.5,

\[
\| [b,T] f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| I_\alpha(|f|) \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| f \|_{p,\Phi}.
\]

Thus we have (2.8) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$. Q.E.D.

3. Commutators between the fractional integral operator and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_\alpha(R^n)$.

In this section first we show the following when $b \in BMO(R^n)$;
Theorem 3.1. Let \( 1 < p < q < \infty \), \( 0 < \alpha = n(1/p - 1/q) < n \). We assume that \( \Phi \in G_{n-p\alpha} \) and \( \Phi^{q/p} \in G_n \). If \( b \in BMO(R^n) \), then the commutator \([b, I_{\alpha}]\) is a bounded operator from \( L^{p,\Phi}(R^n) \) into \( L^{q,\Phi^{q/p}}(R^n) \) and

\[
\| [b, I_{\alpha}]f \|_{q,\Phi^{q/p}} \leq C \| b \|_* \| f \|_{p,\Phi}
\]

for some constant \( C \) independent of \( b \in BMO(R^n) \) and \( f \in L^{p,\Phi} \cap L^{\infty}(R^n) \).

To prove the theorem, we need the following pointwise estimate and the boundedness of the fractional maximal operator:

**Lemma 3.1.** Let \( 0 < \alpha < n \), \( 1 < u \), \( t < n/\alpha \) and \( b \in BMO(R^n) \). Then there exists constant \( C \) independent of \( b \) and \( f \) such that

\[
\{[b, I_{\alpha}](f)(x)\} \leq C \| b \|_* \{M_u(I_{\alpha}f)(x) + (M_{\alpha/n}^*|f|^t)^{1/t}(x)\}
\]

for almost all \( x \in R^n \) and all \( f \in L^\infty(R^n) \)

**Proof.** This is the pointwise estimate due to Strömberg (see [Tor, p.419.] and Di Fazio-Ragusa [DifRag, p.326, Lemma 2.]).

Q.E.D.

**Lemma 3.2.** Let \( 1 < p < q < \infty \), \( 0 < \alpha = n(1/p - 1/q) < n \). We assume that \( \Phi \in G_{n-p\alpha} \). Then the fractional maximal operator \( M_{\alpha/n}^* \) is a bounded operator from \( L^{p,\Phi}(R^n) \) into \( L^{q,\Phi^{q/p}}(R^n) \) and

\[
\| M_{\alpha/n}^*f \|_{q,\Phi^{q/p}} \leq C \| f \|_{p,\Phi}
\]

for some constant \( C \) independent of \( f \in L^{p,\Phi} \).

**Proof.** Let \( B(z, r) \) be any ball centered at \( z \) and with radius \( r > 0 \) such that \( x \in B(z, r) \). Since

\[
I_{\alpha}(|f|)(x) := \int_{R^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy \geq \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy
\]

\[
\geq \frac{C'}{r^{n-\alpha}} \int_{B(x,r)} |f(y)|dy \simeq \frac{C''}{|B|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|dy,
\]

we have the pointwise estimate;

\[
(M_{\alpha/n}^*f)(x) \leq I_{\alpha}(|f|)(x)
\]
for almost all $x \in \mathbb{R}^n$ and all $f \in L^{p,\Phi}$. Hence Lemma 2.5 implies the result.

Q.E.D.

**Proof of Theorem 3.1.** Let $b \in BMO(\mathbb{R}^n)$. Then Theorem 2.1 and Lemma 3.1 imply that, for $1 < u$, $t < q < p < \infty$,

$$
\| [b, I_\alpha](f) \|_{q, \Phi q/p} \leq \| M [b, I_\alpha](f) \|_{q, \Phi q/p} \leq \| (M \{ [b, I_\alpha](f) \}^q)^{1/q} \|_{q, \Phi q/p} \leq C \| b \| \| I_\alpha f \|_{q, \Phi q/p} + \| (M \{ [b, I_\alpha](f) \}^q)^{1/q} \|_{q, \Phi q/p}.
$$

Also, under the assumption on $\Phi$, Lemma 2.3 and Lemma 2.5 imply

$$
\| M_u (I_\alpha f) \|_{q, \Phi q/p} \leq C \| I_\alpha f \|_{q, \Phi q/p} \leq C \| f \|_{p, \Phi}
$$

and Lemma 3.2 imply

$$
\| (M_{\alpha+t/n} | f |^t)^{1/t} \|_{q, \Phi q/p} = \| M_{\alpha+t/n} | f |^t \|_{q, \Phi q/p} \leq C \| | f |^t \|_{p, \Phi} = C \| f \|_{p, \Phi}.
$$

Hence we obtain

$$
\| [b, I_\alpha](f) \|_{q, \Phi q/p} \leq C \| b \| \| I_\alpha f \|_{q, \Phi q/p} \leq C \| f \|_{p, \Phi}.
$$

for $b \in BMO(\mathbb{R}^n)$ and $f \in L^{p,\Phi} \cap L^\infty(\mathbb{R}^n)$. Thus we have (3.1).

Q.E.D.

We close this section showing the following;

**Theorem 3.2.** Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(\mathbb{R}^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(\mathbb{R}^n)$ into $L^{q,\Phi q/p}(\mathbb{R}^n)$ and

$$
\| [b, I_\beta] f \|_{q, \Phi q/p} \leq C \| b \| \| I_\alpha \| f \|_{p, \Phi}
$$

for some constant $C$ independent of $b \in \Lambda_\alpha(\mathbb{R}^n)$ and $f \in L^{p,\Phi} \cap L^\infty(\mathbb{R}^n)$.

**Proof of Theorem 3.2.** Let $b \in \Lambda_\alpha(\mathbb{R}^n)$. Then

$$
|([b, I_\beta] f)(x)| \leq C \| b \| \| I_\alpha \| (|f|)(x).
$$

for almost all $x \in \mathbb{R}^n$. Hence we have, by Lemma 2.5,

$$
\| [b, I_\beta] f \|_{q, \Phi q/p} \leq C \| b \| \| I_\alpha \| \| f \|_{q, \Phi q/p} \leq C \| b \| \| I_\alpha \| f \|_{p, \Phi}.
$$

Thus we have (3.3) for some $C > 0$, independent of $b \in \Lambda_\alpha(\mathbb{R}^n)$ and $f \in L^{p,\Phi} \cap L^\infty(\mathbb{R}^n)$.

Q.E.D.
References.


[Ste] Stein, E.S. Singular integrals and differentiability properties of functions, Princeton


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