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Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions

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Abstract of the Talk

The talk will be concerned with the boundedness of the commutators of Calderon-Zygmund singular integral operators on Morrey spaces $L^{p,\Phi}(R^n)$ with growth functions $\Phi(x,r)$ satisfying the condition ; there exists a constant C, independent of $(x,r)\in R_+^{n+1}$, such that for any $(x,r)\in R_+^{n+1}$

(1) $$\int_r^\infty [\Phi(x,t)/t^{a+1}]dt \le C\Phi(x,r)/r^a, \quad \text{for some } a>0.$$

In this case, we write $\Phi\in G_a$ simply. We denote by $L^{p,\Phi}(R^n)$, $0<p<\infty$, the space of locally integrable functions f, defined on R^n, for which there exists a constant C, independent of balls $B=B(x,r)$, such that

(2) $$\int_{B(x,r)} |f(y)|^p dy \le C^p\Phi(x,r)$$

for all balls $B=B(x,r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (2). Then the space $L^{p,\Phi}(R^n)$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$. In particular, if $1\le p<\infty$, then the space $L^{p,\Phi}(R^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$.

Let $BMO(R^n)$ be the space of all functions of bounded mean oscillation and let $\Lambda_\alpha(R^n)$, $0<\alpha<n$, be the space of all Lipschitz continuous functions of order α. Let M be the Hardy-Littlewood maximal operator. We need two variants of M. For $0<q<\infty$ let $M_qf(x)=\{(M|f|^q)(x)\}^{1/q}$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x)=\sup_{x\in B}|B|^{-1}\int_B|f(y)-f_B|dy, \quad \text{where} \quad f_B=|B|^{-1}\int_B f(y)dy.$$

Let T be a Calderon-Zygmund singular integral operator defined by $Tf=k*f$ with the kernel k satisfying the conditions :

$$\|\hat{k}\|_\infty \le C, \quad |k(x)| \le C|x|^{-n} \quad \text{for } 0\ne x\in R^n,$$

$$|k(x)-k(x-y)|\le C|y|/|x|^{n+1} \quad \text{for } |y|\le|x|/2.$$

Let I_α, $0<\alpha<n$, be the Riesz potential of order α defined by

$$(I_\alpha f)(x)=\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Related to $I_\alpha f$, the fractional maximal function $M^*_{\alpha/n}f(x)$ is defined by

$$M^*_{\alpha/n}f(x) = f^*_{\alpha,1}(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy.$$

For a locally integrable function b and an operator S, we define the commutator $[b, S]$, between the operator S and the multiplication operator by b, by $[b, S] = bS - Sb$.

We have proved the following ([Miz$_2$]);

Theorem 1(Theorem 2.1.) *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C = C(p, \Phi) > 0$, independent of f, such that for all $f \in L^{p,\Phi}(R^n) \cap L^\infty_c(R^n)$*

$$\|Mf\|_{p,\Phi} \le C\|f^\sharp\|_{p,\Phi} \tag{3}$$

where $L^\infty_c(R^n)$ be the set of all essentially bounded functions on R^n with compact support.

We use the method due to Di Fazio and Ragusa. Our method is based on weighted maximal inequality due to Garcia-Cuerva and Rubio de Francia.

From this and the pointwise estimate due to Strőmberg ;

$$\{[b, T](f)\}^\sharp(x) \le C\|b\|_*\{M_q(Tf)(x) + (M_s f)(x)\}, \quad 1 < q, s < \infty,$$

for almost all $x \in R^n$, we obtain the boundedness of the commutators $[b, T]$ on Morrey spaces ([Miz$_2$]) ;

Theorem 2(Theorem 2.2.) *Let $1 < p < \infty$ and $b \in BMO(R^n)$. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists a constant $C = C(p, \Phi) > 0$, independent of b and f, such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L^\infty_c(R^n)$*

$$\|[b, T](f)\|_{p,\Phi} \le C\|b\|_*\|f\|_{p,\Phi}. \tag{4}$$

Also we can observe the following ([Miz$_2$]) ;

Theorem 3(Theorem 2.3.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f, such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L^\infty_c(R^n)$*

$$\|[b, T]f\|_{q,\Phi^{q/p}} \le C\|b\|_{\Lambda_\alpha(R^n)}\|f\|_{p,\Phi}. \tag{5}$$

This follows from the result (due to Naki [N]) of the boundedness of Riesz potential on Morrey spaces and the pointwise estimate ;

$$|([b,T]f)(x)| \le \int_{R^n} |b(x)-b(y)||k(x-y)||f(y)|dy \le C\|b\|_{\Lambda_\alpha(R^n)} I_\alpha(|f|)(x).$$

Further we obtain the following result ([Miz$_2$]) from the boundedness of the fractional maximal operator $M^*_{\alpha/n}$ on Morrey spaces and the pointwise estimate due to Strőmberg ;

$$\{[b,I_\alpha](f)\}^\sharp(x) \le C\|b\|_*\{M_u(I_\alpha f)(x) + (M^*_{\alpha t/n}|f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$, where $1 < u, t < p < n/\alpha$.

Theorem 4(Theorem 3.1.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(R^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p,q,\Phi) > 0$, independent of b and f, such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L^\infty_c(R^n)$*

$$\|[b,I_\alpha]f\|_{q,\Phi^{q/p}} \le C\|b\|_*\|f\|_{p,\Phi}. \tag{6}$$

Similarly we can show the following ([Miz$_2$]) ;

Theorem 5(Theorem 3.2.) *Let $1 < p < q < \infty$, $0 < \alpha$, β, $0 < \alpha+\beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha+\beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p,q,\Phi) > 0$, independent of b and f, such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L^\infty_c(R^n)$*

$$\|[b,I_\beta]f\|_{q,\Phi^{q/p}} \le C\|b\|_{\Lambda_\alpha(R^n)}\|f\|_{p,\Phi}. \tag{7}$$

Our results (Theorems 1, 2 and 4) generalize partly the classical results due to Di Fazio and Ragusa [DiFRag]. Also we obtain the new results (Theorems 3 and 5).

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1. Introduction.

Let $\Phi = \Phi(x,r)$, be a growth function on $R_+^{n+1} = R^n \times R_+$, that is, a positive and non-decreasing function with repect to $r > 0$. We say that the growth function $\Phi(x,r)$ satisfies the Δ_2-condition (or doubling condition) for $r > 0$ if there exists constant $D = D(\Phi) \geq 1$, independent of (x,r), such that

$$\Phi(x,2r) \leq D\Phi(x,r), \quad (x,r) \in R_+^{n+1}, \tag{1.1}$$

or equivalently,

$$\Phi(x,2r)/D \leq \Phi(x,r) \leq \Phi(x,2r), \quad (x,r) \in R_+^{n+1}.$$

In this case, we write $\Phi \in \Delta_2$ simply. We consider the following functions in Δ_2 ;

$$\Phi(x,r) = \Psi(x)r^\lambda\{\log(1+r)\}^\mu, \quad \Psi(x) \in L^\infty(R^n), \quad 0 \leq \lambda < \infty, \quad -\infty < \mu < \infty.$$

Remark. Nakai [Nak] assumed a slightly weak condition on $\Phi(x,r)$ replacing (1.1) ; there exists a constant $C > 0$ such that, for all $(x,r) \in R_+^{n+1}$,

$$r \leq t \leq 2r \Longrightarrow C^{-1} \leq \Phi(x,t)/\Phi(x,r) \leq C. \tag{1.2}$$

However, for simplicity, we describe the results on the asumption of (1.1). Of course our results are also valid under the condition (1.2).

Function Spaces. Let R^n be the n-dimensional Euclidean space and let $B = B(x,r)$ be the ball centered at $x \in R^n$ and with radius $r > 0$. Let $Q = Q(x,r)$ be the cube centered at $x \in R^n$ and with sides of length $r > 0$, where the cube will always mean a compact cube with sides parallel to the axes and nonempty interior. $|B|$ and $|Q|$ stand for the Lebesgue measures of ball B and cube Q, respectively. Let $0 < p < \infty$.

Definition 1.1 (Morrey spaces). (Confer Mizuhara [Miz$_1$]). *We denote by $L^{p,\Phi} = L^{p,\Phi}(R^n)$ the space of locally integrable functions f, defined on R^n, for which there exists a constant C, independent of balls $B = B(x,r)$, such that*

$$\int_{B(x,r)} |f(y)|^p dy \leq C^p\Phi(x,r) \tag{1.3}$$

for all balls $B = B(x,r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (1.3). *Then the space $L^{p,\Phi}$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$ in the sense of* Triebel [Tri]. *In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$. The balls $B = B(x,r)$ in* (1.3) *can be replaced by cubes $Q = Q(x,r)$.*

When $\Phi(x,r) = r^\lambda$, $\lambda \geq 0$, then L^{p,r^λ} is the classical Morrey space denoted by $L^{p,\lambda}$ simply. The classical Morrey spaces $L^{p,\lambda}$, $0 < \lambda < n$, were originally introduced by Morrey [Mor] in 1938 and used by himself and the others in the problems related to the calculus of variations and the theory of elliptic PDE's. We refer to Campanato [Cam], Giaguinta [Gia], Kufner-John-Fučik [KufJohFuc] and Peetre [P_2].

The some properties of $L^{p,\lambda}$ are known ; *If* $1 \leq p < \infty$, *then* $L^{p,0} = L^p(R^n)$ *and* $L^{p,n} = L^\infty(R^n)$ *isometrically. If* $n < \lambda$, *then* $L^{p,\lambda} = \{0\}$. *If* $1 \leq p < \infty$ *and* $0 < \lambda < n$, *then* $L^{p,\lambda}$ *does not include nonzero constants.* Hence, in the classical Morrey spaces, $L^{p,\lambda}$ for $0 < \lambda < n$ is interesting. Also Hölder's inequality implies the imbedding theorem ; if $(n-\lambda)/q = (n-\mu)/p$, $p \leq q$, then $L^{q,\lambda} \subset L^{p,\mu}$.

Let $BMO(R^n)$ be the John-Nirenberg space of all functions of bounded mean oscillation (see John-Nirenberg [JoN]), that is, $BMO(R^n)$ is a Banach space, modulo constants, with norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_B |B|^{-1} \int_B |b(y) - b_B| dy, \quad \text{where} \quad b_B = |B|^{-1} \int_B b(y) dy.$$

The space $BMO(R^n)$ is identified with the dual space of the Hardy space $H^1(R^n)$ in the sense of Fefferman-Stein ([FeS_2]).

Let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order α on R^n. The space $\Lambda_\alpha(R^n)$ is homogeneous in the sense of dilations. The dual space of $H^p(R^n)$ can be identified with the Lipschitz space $\Lambda_\alpha(R^n)$, $\alpha = n(1/p - 1)$.

Classical operators. Let f be a locally integrable function on R^n. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy$$

where the supremum is taken over all balls B containing x and $|B|$ is the volume of the ball B. We introduce two variants of M. Let $0 < q < \infty$ and

$$M_q f(x) = \{(M|f|^q)(x)\}^{1/q}.$$

Then Hölder's inequality shows that $Mf = M_1 f \leq M_q f$ if $1 \leq q < \infty$ and $M_q f \leq M_1 f = Mf$ if $0 < q \leq 1$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where} \quad f_B = |B|^{-1} \int_B f(y) dy.$$

Let T be a Calderon-Zygmund singular integral operator $Tf = k * f$ defined by the kernel k satisfying the conditions;

$$\|\hat{k}\|_\infty \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in R^n,$$

$$|k(x) - k(x-y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2.$$

For $\epsilon > 0$, put

$$T_\epsilon f(x) = \int_{|y|>\epsilon} k(y) f(x-y) dy \quad \text{and} \quad T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

Let I_α, $0 < \alpha < n$, be the fractional integral operator (or Riesz potential operator) of order α defined by

$$(I_\alpha f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for a suitable function f. Related to $I_\alpha f$, the fractional maximal function $M^*_{\alpha/n} f(x)$, which appeared in [MucWhe] as $f^*_{\alpha,1}(x)$, is defined by

$$M^*_{\alpha/n} f(x) = f^*_{\alpha,1}(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

We define the commutator $[b, S]$ between an operator S and the multiplication operator by a locally integrable function b, by $[b, S] = bS - Sb$.

In this note we show the boundednesss of the commutator $[b, T]$, for $b \in BMO(R^n)$ or $b \in \Lambda_\alpha(R^n)$, on Morrey spaces $L^{p,\Phi}(R^n)$ with some growth function Φ. Our results (Theorems 2.1, 2.2, 3.1) generalize partly the recent results due to Di Fazio and Ragusa [DiFRag] on the classical Morrey spaces $L^{p,\lambda}(R^n), 0 < \lambda < n, 1 < p < \infty$. Further we obtain the new results (Theorems 2.3 and 3.2). The letters C's will denote positive constants, which may have different values in each line.

2. Commutators between Calderon-Zygmund singular integral operators and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_\alpha(R^n)$.

G_a-condition. We consider the following condition on growth function $\Phi(x, r)$;

$$\frac{\Phi(x,t)}{t^a} \in L^1([r, \infty), dt/t)$$

for all $r > 0$ and any $x \in R^n$, and, in addition, there exists a constant C, independent of $(x, r) \in R^{n+1}_+$, such that

$$\int_r^\infty [\Phi(x,t)/t^{a+1}] dt \leq C\Phi(x,r)/r^a, \quad (x,r) \in R^{n+1}_+, \tag{2.1}$$

for some $a > 0$. In this case, we write $\Phi \in G_a$ simply.

We can observe the following property of G_a, $a > 0$;

Lemma 2.1. (i) *If* $0 < a < a' < n$, *then* $G_a \subset G'_a \subset G_n \subset \Delta_2$.

(ii) *If* $\Phi \in \Delta_2$ *with doubling constant* D, $1 \leq D < 2^n$, *then* $\Phi \in G_n$.

(iii) *If* $a > 0$, *then* $G_a \subset G_{a\gamma}$ *for some* γ, $0 < \gamma < 1$. *More precisely, if* $\Phi \in G_a$, $a > 0$, *there exist constants* $\gamma = \gamma(C, a)$, $0 < \gamma < 1$, *and* $C' = C'(C, a, \gamma) > 0$ *such that for any* $(x, r) \in R^{n+1}_+$

$$\int_r^\infty [\Phi(x,t)/t^{a\gamma+1}]dt \leq C'\Phi(x,r)/r^{a\gamma}. \tag{2.2}$$

Proof. (i), (ii) These are easy to see.

(iii) Let

$$\Phi_a(x,r) = \int_r^\infty [\Phi(x,t)/t^{a+1}]dt.$$

Then (2.1) implies

$$\Phi_a(x,r) \leq C\Phi(x,r)/r^a.$$

For $0 < r < R$, we have, integrating by parts and using (2.1),

$$\int_r^R [\Phi(x,t)/t^{a\gamma+1}]dt = \int_r^R [\Phi(x,t)/t^{a+1}t^{a(\gamma-1)}]dt$$

$$= \left[-\Phi_a(x,t)t^{a(1-\gamma)}\right]_r^R - \int_r^R [-\Phi_a(x,t)a(1-\gamma)t^{a(1-\gamma)-1}]dt$$

$$= -\Phi_a(x,R)R^{a(1-\gamma)} + \Phi_a(x,r)r^{a(1-\gamma)} + a(1-\gamma)\int_r^R [\Phi_a(x,t)t^{a(1-\gamma)-1}]dt$$

$$\leq C\Phi(x,r)/r^{a\gamma} + a(1-\gamma)C\int_r^R [\Phi(x,t)/t^{a\gamma+1}]dt.$$

Hence we obtain

$$\int_r^R [\Phi(x,t)/t^{a\gamma+1}]dt \leq \frac{C}{1-a(1-\gamma)C}\Phi(x,r)/r^{a\gamma},$$

and we have (2.2) with

$$C' = \frac{C}{1-a(1-\gamma)C} > 0.$$

Thus we have (2.2) for some γ such that $1 - (1/aC) < \gamma < 1$.

Q.E.D.

First using this Lemma, we show the following ;

Theorem 2.1. *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C > 0$, independent of f, such that*

$$\|Mf\|_{p,\Phi} \le C\|f^\sharp\|_{p,\Phi} \tag{2.3}$$

for all $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$, where $L_c^\infty(R^n)$ is the set of all essentially bounded functions on R^n with compact support.

Proof. We use the method due to Di Fazio-Ragusa [DifRag]. We recall the weighted version of the maximal inequality due to Fefferman-Stein [FS$_2$] ; there exists a constant C such that

$$\int_{R^n} \{Mf(x)\}^p w(x)dx \le C \int_{R^n} \{f^\sharp(x)\}^p w(x)dx \tag{2.4}$$

for all $w \in A_\infty$ and all $f \in L_w^p(R^n)$, for $0 < p < \infty$ (see Garcia-Cuerva-Rubio de Francia [GarRub; p.410]) where A_q, $1 \le q \le \infty$, is the Muckenhoupt class of weight functions.

Let $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$ and B a ball. We take $w(x)$ as $(M\chi)^\gamma \in A_1$, $0 < \gamma < 1$, where $\chi = \chi_B(x)$ is the characteristic function of the ball $B = B(x_0, r)$.

Then we get by (2.4),

$$\int_B \{Mf(x)\}^p dx = \int_{R^n} \{Mf(x)\}^p \chi_B(x)dx$$

$$\le \int_{R^n} \{Mf(x)\}^p \{M\chi_B(x)\}^\gamma dx \le C \int_{R^n} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx$$

$$= C \int_{B(x_0,r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx$$

$$+ C \sum_{k=1}^{\infty} \int_{B(x_0,2^k r)-B(x_0,2^{k-1}r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx$$

$$\le C\left\{\int_{B(x_0,r)} \{f^\sharp(x)\}^p dx + \sum_{k=1}^{\infty} (2^{-kn\gamma}) \int_{B(x_0,2^k r)} \{f^\sharp(x)\}^p dx\right\}$$

$$\le C\|f^\sharp\|_{p,\Phi}^p \left\{\Phi(x_0, r) + \sum_{k=1}^{\infty} (2^{-k})^{n\gamma} \Phi(x_0, 2^k r)\right\}$$

$$\le C\|f^\sharp\|_{p,\Phi}^p \sum_{k=0}^{\infty} \frac{\Phi(x_0, 2^k r)}{2^{kn\gamma}} \sim C\|f^\sharp\|_{p,\Phi}^p r^{n\gamma} \int_r^\infty \frac{\Phi(x_0, t)}{t^{n\gamma+1}} dt.$$

Since, by Lemma 2.1, the last term is bounded by

$$C\|f^\sharp\|_{p,\Phi}^p \Phi(x_0, r),$$

we have

$$\|Mf\|_{p,\Phi} \le C\|f^\sharp\|_{p,\Phi}.$$

Thus we have (2.3) for some $C > 0$, independent of $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

Our second aim is to show the following ;

Theorem 2.2. *Let $1 < p < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists constant C, independent of b and f, such that*

$$\|[b,T](f)\|_{p,\Phi} \le C\|b\|_*\|f\|_{p,\Phi} \tag{2.5}$$

for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need Theorem 2.1 and the following three lemmas ;

Lemma 2.2. *Let $1 < q,\ s < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. Then there exists constant C independent of b and f such that*

$$\{[b,T](f)\}^\sharp(x) \le C\|b\|_*\{M_q(Tf)(x) + (M_sf)(x)\}$$

for almost all $x \in R^n$ and all $f \in L_c^\infty(R^n)$

Proof. This is the pointwise estimate due to Strőmberg (see [Tor, p.418.] and Janson [Jan; pp.268-269.]).

Q.E.D.

Lemma 2.3. *Let $0 < q < p < \infty$. We assume that $\Phi \in G_n$. Then the maximal operator M_q is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$\|M_qf\|_{p,\Phi} \le C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. The proof depends on the weighted maximal inequality due to Fefferman-Stein [FefSte$_1$]. In the restricted case $1 \le q < p < \infty$, the corresponding result is proved by Nakai [Nak; Theorem 1]. It is not difficult to extend the result to the case $0 < q < p < \infty$. Confer also Chiarenza-Frasca [ChiFra] and Mizuhara [Miz$_1$].

Q.E.D.

Lemma 2.4. *Let $1 < p < \infty$. We assume that $\Phi \in G_n$. Then the Calderon-Zygmund singular integral operator T is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$\|Tf\|_{p,\Phi} \le C\|f\|_{p,\Phi} \tag{2.6}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. This is the result due to Nakai [Nak; Theorem 2] in the setting of more general growth functions. Confer also Peetre [Pee_1], Chiarenza-Frasca [ChiFra] and Mizuhara [Miz_1].

We note that we can give a short proof following the method of the author [Miz_1] which depends on the weighted maximal inequality due to Cordoba-Fefferman [CorFef] (see also [GarRub]) ;

there exists constant C, depending only on T, p and $0 < \gamma < 1$, such that

$$\int_{R^n} |Tf(x)|^p \phi(x)dx \le C \int_{R^n} |f(x)|^p (M\phi)^\gamma(x)dx \tag{2.7}$$

for all f anf $\phi(x) \ge 0$. A standard proof using (2.7) implies (2.6).

Q.E.D.

Proof of Theorem 2.2. We apply the method of Di Fazio-Ragusa [DifRag] to our case. We suppose that $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 2.2 imply that, for $1 < q,\ s < p < \infty$,

$$\|[b,T](f)\|_{p,\Phi} \le \|M\{[b,T](f)\}\|_{p,\Phi}$$

$$\le C\|\{([b,T](f)\}^\sharp\|_{p,\Phi} \le C\|b\|_*\{\|M_q(Tf)\|_{p,\Phi} + \|M_s f\|_{p,\Phi}\}.$$

Since, Lemma 2.3 and Lemma 2.4 imply

$$\|M_q(Tf)\|_{p,\Phi} \le C\|Tf\|_{p,\Phi} \le C\|f\|_{p,\Phi} \quad \text{and} \quad \|M_s f\|_{p,\Phi} \le C\|f\|_{p,\Phi},$$

we obtain

$$\|[b,T](f)\|_{p,\Phi} \le C\|b\|_*\|f\|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$. Thus we have (2.5).

Q.E.D.

When $b \in \Lambda_\alpha(R^n),\ 0 < \alpha < n$, we obtain the following :

Theorem 2.3. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(2.8) \qquad \|[b,T]f\|_{q,\Phi^{q/p}} \leq C\|b\|_{\Lambda_\alpha(R^n)}\|f\|_{p,\Phi}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need the following lemma ;

Lemma 2.5. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional integral operator I_α is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(2.9) \qquad \|I_\alpha f\|_{q,\Phi^{q/p}} \leq C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}$.

Proof. This is the result due to Nakai [Nak; Theorem 3].

Q.E.D.

Proof of Theorem 2.3. Let $b \in \Lambda_\alpha(R^n)$. Then

$$|([b,T]f)(x)| \leq \int_{R^n} |b(x) - b(y)||k(x-y)||f(y)|dy$$

$$\leq C\|b\|_{\Lambda_\alpha} \int_{R^n} |x-y|^\alpha |x-y|^{-n}|f(y)|dy = C\|b\|_{\Lambda_\alpha} I_\alpha(|f|)(x).$$

Hence we have, by Lemma 2.5,

$$\|[b,T]f\|_{q,\Phi^{p/q}} \leq C\|b\|_{\Lambda_\alpha}\|I_\alpha(|f|)\|_{q,\Phi^{p/q}} \leq C\|b\|_{\Lambda_\alpha}\|f\|_{p,\Phi}.$$

Thus we have (2.8) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

3. Commutators between the fractional integral operator and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_\alpha(R^n)$.

In this section first we show the following when $b \in BMO(R^n)$;

Theorem 3.1. *Let* $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. *We assume that* $\Phi \in G_{n-p\alpha}$ *and* $\Phi^{q/p'} \in G_n$. *If* $b \in BMO(R^n)$, *then the commutator* $[b, I_\alpha]$ *is a bounded operator from* $L^{p,\Phi}(R^n)$ *into* $L^{q,\Phi^{q/p}}(R^n)$ *and*

$$\|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C\|b\|_*\|f\|_{p,\Phi} \tag{3.1}$$

for some constant C *independent of* $b \in BMO(R^n)$ *and* $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$.

To prove the theorem, we need the following pointwise estimate and the bounbedness of the fractional maximal operator ;

Lemma 3.1. *Let* $0 < \alpha < n$, $1 < u$, $t < n/\alpha$ *and* $b \in BMO(R^n)$. *Then there exists constant* C *independent of* b *and* f *such that*

$$\{[b, I_\alpha](f)\}^\sharp(x) \leq C\|b\|_*\{M_u(I_\alpha f)(x) + (M^*_{\alpha t/n}|f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$ *and all* $f \in L^\infty_c(R^n)$

Proof. This is the pointwise estimate due to Strőmberg (see [Tor, p.419.] and Di Fazio-Ragusa [DifRag; p.326, Lemma 2.]).

Q.E.D.

Lemma 3.2. *Let* $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. *We assume that* $\Phi \in G_{n-p\alpha}$. *Then the fractional maximal operator* $M^*_{\alpha/n}$ *is a bounded operator from* $L^{p,\Phi}(R^n)$ *into* $L^{q,\Phi^{q/p}}(R^n)$ *and*

$$\|M^*_{\alpha/n}f\|_{q,\Phi^{q/p}} \leq C\|f\|_{p,\Phi} \tag{3.2}$$

for some constant C *independent of* $f \in L^{p,\Phi}$.

Proof. Let $B(z, r)$ be any ball centered at z and with radius $r > 0$ such that $x \in B(z, r)$. Since

$$I_\alpha(|f|)(x) := \int_{R^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy \geq \int_{B(z,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy$$

$$\geq \frac{C'}{r^{n-\alpha}}\int_{B(z,r)} |f(y)|dy \simeq \frac{C''}{|B|^{1-\alpha/n}}\int_{B(z,r)} |f(y)|dy,$$

we have the pointwise estimate ;

$$(M^*_{\alpha/n}f)(x) \leq I_\alpha(|f|)(x)$$

for almost all $x \in R^n$ and all $f \in L^{p,\Phi}$. Hence Lemma 2.5 implies the result.

Q.E.D.

Proof of Theorem 3.1. Let $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 3.1 imply that, for $1 < u,\ t < q < p < \infty$,

$$\|[b, I_\alpha](f)\|_{q,\Phi^{q/p}} \le \|M\{[b, I_\alpha](f)\}\|_{q,\Phi^{q/p}} \le \|\{([b, I_\alpha](f)\}^\sharp\|_{q,\Phi^{q/p}}$$
$$\le C\|b\|_*\{\|M_u(I_\alpha f)\|_{q,\Phi^{q/p}} + \|(M^*_{\alpha t/n}|f|^t)^{1/t}\|_{q,\Phi^{q/p}}\}.$$

Also, under the assumption on Φ, Lemma 2.3 and Lemma 2.5 imply

$$\|M_u(I_\alpha f)\|_{q,\Phi^{q/p}} \le C\|I_\alpha f\|_{q,\Phi^{q/p}} \le C\|f\|_{p,\Phi}$$

and Lemma 3.2 imply

$$\|(M^*_{\alpha t/n}|f|^t)^{1/t}\|_{q,\Phi^{q/p}} = \|M^*_{\alpha t/n}|f|^t\|^{1/t}_{q/t,\Phi^{q/p}}$$
$$\le C\|(|f|^t)\|^{1/t}_{p/t,\Phi} = C\|f\|_{p,\Phi}.$$

Hence we obtain

$$\|[b, I_\alpha](f)\|_{q,\Phi^{q/p}} \le C\|b\|_*\|f\|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$. Thus we have (3.1).

Q.E.D.

We close this section showing the following ;

Theorem 3.2. *Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha+\beta = n(1/p-1/q) < n$, $1 < p < n/(\alpha+\beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$\|[b, I_\beta]f\|_{q,\Phi^{q/p}} \le C\|b\|_{\Lambda_\alpha}\|f\|_{p,\Phi} \tag{3.3}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$.

Proof of Theorem 3.2. Let $b \in \Lambda_\alpha(R^n)$. Then

$$|([b, I_\beta]f)(x)| \le C\|b\|_{\Lambda_\alpha} I_{\alpha+\beta}(|f|)(x).$$

for almost all $x \in R^n$. Hence we have, by Lemma 2.5,

$$\|[b, I_\beta]f\|_{q,\Phi^{q/p}} \le C\|b\|_{\Lambda_\alpha}\|I_{\alpha+\beta}(|f|)\|_{q,\Phi^{q/p}} \le C\|b\|_{\Lambda_\alpha}\|f\|_{p,\Phi}.$$

Thus we have (3.3) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L^\infty_c(R^n)$.

Q.E.D.

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