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Time local well-posedness for the KP II equation

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Introduction

We consider the well-posedness for the Cauchy problem of the Kadomtsev-Petviashvili II (KP II) equation:

$$\partial_x(\partial_t u+\partial_x^3 u+\partial_x(u^2))+\partial_y^2 u=0,\ (t,x,y)\in[-T,T]\times\mathbb{R}^2, \tag{1}$$

$$u(0,x,y)=u_0(x,y),\ (x,y)\in\mathbb{R}^2, \tag{2}$$

where the unknown function u is a real valued and T gives a time interval to be determined later.

While the equation (1) is known as the KP II equation, the following equation corresponds to the KP I:

$$\partial_x(\partial_t u+\partial_x^3 u+\partial_x(u^2))-\partial_y^2 u=0.$$

These KP equations describe a propagation of a weakly nonlinear dispersive long wave which is essentially one dimensional with a weak transverse effect [3].

Our purpose of this note is to consider the well-posedness for the Cauchy problem of the KP II equation in a weak class. As usual, we rewrite the Cauchy problem (1)-(2) as the integral equation and use the contraction argument. The usual contraction argument seems meet with the difficulty of the derivative loss which stems from the derivative nonlinearity of the equation (1).

Known results

There are a large amount of work for the Cauchy problem (1)-(2) (see e.g., [2,5,6,7]). Recently J. Bourgain [2] showed the time local well-posedness in $L^2(\mathbb{R}^2)$. By the conservation law of the L^2 norm, the time global well-posedness was also obtained in the same space.

In this note we consider the equation (1) in the integral form. On the other hand, N. Tzvetkov [6] considered the following equation in the distribution sense instead of the equation (1):

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) + \partial_x^{-1}\partial_y^2 u = 0. \tag{3}$$

For the Cauchy problem (3)-(2), the time local well-posedness was shown in $\widetilde{H}_{x,y}^{s_1,s_2}$ for $s_1 > -1/4$ and $s_2 \geq 0$ [6], where $\widetilde{H}_{x,y}^{s_1,s_2}$ is the anisotropic Sobolev space with the following norm:

$$\|f\|_{\widetilde{H}_{x,y}^{s_1,s_2}} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2} f\|_{L^2_{x,y}} + \|(-\partial_x^2)^{-1/2}(1-\partial_y^2)^{s_2/2} f\|_{L^2_{x,y}}.$$

In [6], the condition $(-\partial_x^2)^{-1/2}(1-\partial_y^2)^{s_2/2}u_0 \in L^2(\mathbb{R}^2)$ is necessary. This homogeneous Sobolev space of index -1 in x variable seems natural for the equation (3). But even if the equation is considered in the integral form (1) instead of the differential form (3), the proof of [6] seems to require the use of the homogeneous Sobolev space of negative index. We remark [6] does not cover, nor be covered by [2].

Results of this note

To state our theorems, we give some notations.

Definition 1. Let $\widetilde{g}(\tau,\xi,\eta)$ denote the Fourier transformation of $g(t,x,y)$ in the time and the space variables, i.e.,

$$\widetilde{g}(\tau,\xi,\eta) = \iiint_{\mathbb{R}^3} \exp(-it\tau - ix\xi - iy\eta) g(t,x,y)\,dt\,dx\,dy.$$

Let $\psi \in C_0^\infty(\mathbb{R})$ denote a smooth cut off function such that $\psi = 1$ on $[-1,1]$ and support$\psi \subseteq (-2,2)$. For $\delta > 0$, we put $\psi_\delta(t) = \psi(t/\delta)$. Let

$$k(\tau,\xi,\eta) = \tau - \xi^3 + \frac{\eta^2}{\xi}.$$

For $s_1, s_2 \in \mathbb{R}$, we define the anisotropic Sobolev space $H_{x,y}^{s_1,s_2}$ with the following norm:

$$\|f\|_{H_{x,y}^{s_1,s_2}} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2} f\|_{L^2_{x,y}}.$$

For $s_1, s_2, b \in \mathbb{R}$, we define the space $X_{s_1,s_2,b}$ to be the completion of the Schwartz function space on $\mathbb{R}^3$ with the following norm:

$$\|g\|_{X_{s_1,s_2,b}} = \|g\|_{s_1,s_2,b} + \|g\|_{-3/4,s_2,3/4},$$

where

$$\|g\|_{s_1,s_2,b} = \|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle k(\tau,\xi,\eta)\rangle^{b}\widetilde{g}(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}}.$$

Theorem 1. *Let $s_1 > -1/4$ and $s_2 \geq 0$. Then there exists $b > 1/2$ with the following properties. For any $u_0 \in H^{s_1,s_2}_{x,y}$, there exist $T > 0$ and a unique solution u of the Cauchy problem (1)-(2) in the time interval $[-T,T]$ such that*

$$u \in C([-T,T] : H^{s_1,s_2}_{x,y}), \ \psi_T(t)u \in X_{s_1,s_2,b}.$$

For any $T' \in (0,T)$, there exists $\epsilon > 0$ such that the map $v_0 \mapsto v$ is Lipschitz from $\{v_0 : \|v_0 - u_0\|_{H^{s_1,s_2}_{x,y}} < \epsilon\}$ to $\|v-u\|_{L^\infty_{T'}(H^{s_1,s_2}_{x,y})} + \|\psi_{T'}(v-u)\|_{X_{s_1,s_2,b}}$. In addition, if $u_0 \in H^{s_1',s_2'}_{x,y}$ with $s_1' \geq s_1$ and $s_2' \geq s_2$, then the above result holds with s_1 and s_2 replaced by s_1' and s_2', respectively, in the same time interval $[-T,T]$.

Remark 1. The admissible values of (s_1, s_2) achieved by Theorem 1 are the same as in [6]. In Theorem 1, we do not require the use of the homogeneous space of negative order (see [6]). Therefore, Theorem 1 recovers the L^2 time global well-posedness [2].

Remark 2. Scaling argument. If u solves the Cauchy problem (1)-(2), so does

$$u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y), \tag{4}$$

for any $\lambda \in \mathbb{R}$ with the data $u_\lambda(0) = u_\lambda(0,x,y)$. Then the order difference on λ between λx and $\lambda^2 y$ in (4) suggests that the anisotropic Sobolev space $H^{s_1,s_2}_{x,y}$ may be natural for the Cauchy problem (1)-(2), rather than the usual Sobolev space $H^s(\mathbb{R}^2)$. Taking the homogeneous norm, we have

$$\|u_\lambda(0)\|_{\dot{H}^{s_1,s_2}_{x,y}} = \lambda^{s_1+2s_2+1/2}\|u_0\|_{\dot{H}^{s_1,s_2}_{x,y}}.$$

Then the norm of $\dot{H}^{s_1,s_2}_{x,y}$ is invariant for $s_1 + 2s_2 + 1/2 = 0$ under the scaling transformation (4). This argument suggests that the critical values for the Cauchy problem (1)-(2) in $H^{s_1,s_2}_{x,y}$ may be $s_1 + 2s_2 + 1/2 = 0$. Note that the admissible values (s_1, s_2) achieved in Theorem 1 are far from the critical values.

To state our second theorem of this note, we introduce some notations.

Definition 2. For $s_1 > -1/2$ and $s_2 \in \mathbb{R}$, we define the anisotropic homogeneous Sobolev space $\bar{H}^{s_1,s_2}_{x,y}$ as follows:

$$\bar{H}^{s_1,s_2}_{x,y} = \{f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{\bar{H}^{s_1,s_2}_{x,y}} < \infty\},$$

where

$$\|f\|_{\bar{H}^{s_1,s_2}_{x,y}} = \|(-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2} f\|_{L^2_{x,y}}.$$

For $s_1, s_2, b \in \mathbb{R}$ and $a > -1/2$, we define the space $Y_{s_1,s_2,a,b}$ as follows:

$$Y_{s_1,s_2,a,b} = \{g \in \mathcal{S}'(\mathbb{R}^3) : \|g\|_{Y_{s_1,s_2,a,b}} = \|g\|_{s_1,s_2,b} + |||g|||_{a,s_2,a+1} < \infty\},$$

where

$$|||g|||_{a,s_2,a+1} = \||\xi|^a \langle\eta\rangle^{s_2} \langle k(\tau,\xi,\eta)\rangle^{a+1} \tilde{g}(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}}.$$

Theorem 2. *Let $s_1 > 3a+1 > -1/2$, $-1/2 < a < -1/4$ and $s_2 \geq 0$. Then there exists $b > 1/2$ with the following properties. For any $u_0 \in H^{s_1,s_2}_{x,y} \cap \bar{H}^{a,s_2}_{x,y}$, there exist $T > 0$ and a unique solution u of the Cauchy problem (1)-(2) in the time interval $[-T,T]$ such that*

$$u \in C([-T,T] : H^{s_1,s_2}_{x,y} \cap \bar{H}^{a,s_2}_{x,y}), \ \psi_T(t)u \in Y_{s_1,s_2,a,b}.$$

For any $T' \in (0,T)$, there exists $\epsilon > 0$ such that the map $v_0 \mapsto v$ is Lipschitz from $\{v_0 : \|v_0 - u_0\|_{H^{s_1,s_2}_{x,y}} + \|v_0 - u_0\|_{\bar{H}^{a,s_2}_{x,y}} < \epsilon\}$ to $\|v-u\|_{L^\infty_{T'}(H^{s_1,s_2}_{x,y})} + \|v - u\|_{L^\infty_{T'}(\bar{H}^{a,s_2}_{x,y})} + \|\psi_{T'}(v-u)\|_{Y_{s_1,s_2,a,b}}$.
In addition, if $u_0 \in H^{s_1',s_2'}_{x,y} \cap \bar{H}^{a',s_2'}_{x,y}$ with $s_1' \geq s_1$, $-1/4 > a' \geq a$, $s_1' > 3a'+1$ and $s_2' \geq s_2$, then the above result holds with s_1, a and s_2 replaced by s_1', a' and s_2', respectively, in the same time interval $[-T,T]$.

Remark 3. When $s_2 = 0$, the scaling argument of Remark 2 suggests that Theorem 2 may be the best possible result, except for the limiting case $s_1 = -1/2$ which remains an open problem. We assume $s_2 \geq 0$ in Theorems 1 and 2. In the case of $s_2 < 0$, the well-posedness result remains to be an open problem.

Remark 4. The algebraic relation (9) below plays an important role to overcome the difficulty of a loss of the derivative. In the case of the KP I equation, our method seems useless to avoid this difficulty (see [2, §10]).

The proofs of Theorems 1 and 2 are based on the Fourier restriction norm method [2,4]. This method has been used for the dispersive equation with the quadratic nonlinearity, such as the KdV and the KP II equations.

There are several differences between the Cauchy problem for the KdV equation and that for the KP II equation. In the case of the KdV equation, the following smoothing effect is valid to overcome a loss of the derivative:

$$\|\partial_x \exp(-t\partial_x^3)u_0\|_{L_x^\infty L_t^2} \le c\|u_0\|_{L_x^2}.$$

On the other hand, it seems difficult to see such a kind of smoothing effect for the KP II equation case. Then using an argument similar to that of the KdV equation case, we need to consider the integrability in η variable, where η is the Fourier variable in y. To get such the integrability, we define the spaces $X_{s_1,s_2,b}$ and $Y_{s_1,s_2,a,b}$ for the proofs of Theorems 1 and 2, respectively. More precisely, the second terms that $\|\cdot\|_{-3/4,s_2,3/4}$ and $|||\cdot|||_{a,s_2,a+1}$ play a role to get such an integrability for the proofs of Theorems 1 and 2, respectively. These Fourier restriction norm spaces $X_{s_1,s_2,b}$ and $Y_{s_1,s_2,a,b}$ are the essential difference from [2,6].

Proofs of Theorems 1 and 2

We regard the Cauchy problem (1)-(2) as the following integral equation:

$$u(t) = W(t)u_0 - \int_0^t W(t-s)\partial_x(u(s)^2)ds,$$

where $W(t) = \exp(-t(\partial_x^3 + \partial_x^{-1}\partial_y^2))$.

First we state the lemma concerning the estimates of the linear and the nonlinear part of the KP II equation in the function spaces we consider.

Lemma 1. *Let* $1/2 < b < b' < 1$. *For* $\delta \in (0,1)$ *and* $s_1, s_2 \in \mathbb{R}$, *we have*

$$\|\psi_\delta(\cdot)W(\cdot)u_0\|_{s_1,s_2,b} \le c\delta^{1/2-b}\|u_0\|_{H_{x,y}^{s_1,s_2}}, \tag{5}$$

$$\|\psi_\delta(\cdot)\int_0^{\cdot} W(\cdot - t')F(t')dt'\|_{s_1,s_2,b} \le c\delta^{1/2-b}\|F\|_{s_1,s_2,b-1}, \tag{6}$$

$$\|\psi_\delta(\cdot)F\|_{s_1,s_2,b-1} \le c\delta^{b'-b}\|F\|_{s_1,s_2,b'-1}. \tag{7}$$

See [4], for the proof of Lemma 1.

Remark 5. Lemma 1 holds with $\|\cdot\|_{s_1,s_2,b}$ and $\|\cdot\|_{H_{x,y}^{s_1,s_2}}$ replaced by $|||\cdot|||_{s_1,s_2,b}$ and $\|\cdot\|_{\bar{H}_{x,y}^{s_1,s_2}}$, respectively.

The following two propositions play an important role in the proofs of Theorems 1 and 2, respectively.

Proposition 2. *Let $s_1 > -1/4$ and $s_2 \geq 0$. Then there exist $b > 1/2$ and $\theta > 0$ with the following properties. For $T \in (0,1)$, we have*

$$\|\psi_T \partial_x(u^2)\|_{s_1,s_2,b-1} \leq cT^\theta (T^{b-1/2}\|u\|_{s_1,s_2,b} + T^{1/4}\|u\|_{-3/4,s_2,3/4}) \times T^{b-1/2}\|u\|_{s_1,s_2,b}, \tag{8}$$

$$\|\psi_T \partial_x(u^2)\|_{-3/4,s_2,3/4-1} \leq cT^\theta (T^{b-1/2}\|u\|_{s_1,s_2,b})^2.$$

Proposition 3. *Let $s_1 > 3a+1 > -1/2$, $-1/2 < a < -1/4$ and $s_2 \geq 0$. Then there exist $b > 1/2$ and $\theta > 0$ with the following properties. For $T \in (0,1)$, we have*

$$\begin{aligned}&\|\psi_T \partial_x(u^2)\|_{s_1,s_2,b-1} + |||\psi_T \partial_x(u^2)|||_{a,s_2,a}\\ \leq& cT^\theta (T^{b-1/2}\|u\|_{s_1,s_2,b} + T^{a+1/2}|||u|||_{a,s_2,a+1})T^{b-1/2}\|u\|_{s_1,s_2,b}.\end{aligned}$$

For the proofs of Propositions 2 and 3, we use the following identity [2]:

$$\begin{aligned}&k(\tau,\xi,\eta) - k(\tau_1,\xi_1,\eta_1) - k(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)\\ =& -\frac{3|\xi(\xi-\xi_1)\xi_1|^2 + |\xi\eta_1 - \xi_1\eta|^2}{\xi(\xi-\xi_1)\xi_1}.\end{aligned}$$

This identity implies

$$\begin{aligned}&\max\{|k(\tau,\xi,\eta)|, |k(\tau_1,\xi_1,\eta_1)|, |k(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)|\}\\ \geq& |\xi(\xi-\xi_1)\xi_1| + \frac{|\xi\eta_1 - \xi_1\eta|^2}{3|\xi(\xi-\xi_1)\xi_1|}.\end{aligned} \tag{9}$$

Using this inequality, we overcome the difficulty of the derivative loss and get the integrability in η variable. In the following, we only give a sketch of the proof of (8) in Proposition 2.

Proof. For simplicity we consider (8) for the case of $s_2 = 0$. We first consider the following:

(10)

$$\left(\iiint_{\mathbb{R}^3} \frac{|\xi|^2\langle\xi\rangle^{2s_1}}{\langle k(\tau,\xi,\eta)\rangle^{2(1-b')}} \left(\iiint_{\mathbb{R}^3} |u(\tau_1,\xi_1,\eta_1)||u(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)| d\tau_1 d\xi_1 d\eta_1\right)^2 d\tau d\xi d\eta\right)^{1/2},$$

where $2b-1/4<b'<1$. For briefly we only consider (10) in the following case:

(11) $|\xi_1|\le 1$ and $|k(\tau_1,\xi_1,\eta_1)|\ge \max\{|k(\tau,\xi,\eta)|,|k(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)|\}$.

Using the Schwarz inequality, we have that (10) is bounded by
(12)

$$
\begin{aligned}
&c\sup_{\tau,\xi,\eta}\frac{|\xi|\langle\xi\rangle^{s_1}}{\langle k(\tau,\xi,\eta)\rangle^{1-b'}}\\
&\times\left(\iiint_{\mathbb{R}^3}\frac{\langle\xi_1\rangle^{3/2}\langle\xi-\xi_1\rangle^{-2s_1}}{\langle k(\tau_1,\xi_1,\eta_1)\rangle^{3/2}\langle k(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)\rangle^{2b}}d\tau_1 d\xi_1 d\eta_1\right)^{1/2}\\
&\times\|u\|_{-3/4,0,3/4}\|u\|_{s_1,0,b}.
\end{aligned}
$$

We use (9) first, integrate in τ_1 variable and use the change of the variable η_1 as $\mu=(\xi\eta_1-\xi_1\eta)^2/\xi(\xi-\xi_1)\xi_1$. Noting that for the points in (11), $\langle\xi\rangle\sim\langle\xi-\xi_1\rangle$, we have that the contribution of the region (11) to (12) is bounded by

$$
\begin{aligned}
&c\frac{|\xi|}{\langle k(\tau,\xi,\eta)\rangle^{1-b'}}\left(\int_{|\xi_1|\le1}\int_{-\infty}^{\infty}\frac{|\xi_1|^{1/2}}{\langle|\xi(\xi-\xi_1)\xi_1|+|\mu|\rangle^{3/2}|\mu|^{1/2}}d\mu d\xi_1\right)^{1/2}\\
&\times\|u\|_{-3/4,0,3/4}\|u\|_{s_1,0,b}\\
\le& c\frac{|\xi|}{\langle k(\tau,\xi,\eta)\rangle^{1-b'}}\left(\int_{|\xi_1|\le1}\frac{d\xi_1}{\langle\xi\rangle^2|\xi_1|^{1/2}}\right)^{1/2}\|u\|_{-3/4,0,3/4}\|u\|_{s_1,0,b}\\
\le& c\|u\|_{-3/4,0,3/4}\|u\|_{s_1,0,b}.
\end{aligned}
$$

Combining the above argument with (7) of Lemma 1, we have that the left hand side of (8) restricted to the region (11) is bounded by

$$
\begin{aligned}
&cT^{b'-b}\|u\|_{-3/4,0,3/4}\|u\|_{s_1,0,b}\\
\le& cT^{b'-2b+1/4}(T^{1/4}\|u\|_{-3/4,0,3/4})(T^{b-1/2}\|u\|_{s_1,0,b}).
\end{aligned}
$$

□

Proof of Theorem 1.
We put $r=\|u_0\|_{H^{s_1,s_2}_{x,y}}$ for $s_1>-1/4$ and $s_2\ge 0$. For $T\in(0,1)$ and $b\in\mathbb{R}$, we put $\|w\|_{X^T}=T^{b-1/2}\|w\|_{s_1,s_2,b}+T^{1/4}\|w\|_{-3/4,s_2,3/4}$. Now for $T\in(0,1)$, we define

$$\mathfrak{B}(r)=\{w\in X_{s_1,s_2,b}:\|w\|_{X^T}\le 2cr\},$$

$$\Phi(u)(t) = \psi_T(t)W(t)u_0 - \psi_T(t)\int_0^t W(t-s)\psi_T(s)\partial_x(u(s)^2)ds.$$

We choose b same as in Proposition 2. Thus by (5)-(6) of Lemma 1 and Proposition 2, we have

$$\begin{aligned}
&\|\Phi(u)\|_{X^T}\\
\le & c\|u_0\|_{H^{s_1,s_2}_{x,y}} + c\|\psi_T\partial_x(u^2)\|_{s_1,s_2,b-1} + c\|\psi_T\partial_x(u^2)\|_{-3/4,s_2,3/4-1}\\
\le & cr + cT^\theta\|u\|^2_{X^T}\\
\le & cr + cT^\theta r^2,
\end{aligned}$$

for $u \in \mathfrak{B}(r)$, where $\theta > 0$ is the same as in Proposition 2. Similarly, if $u, v \in \mathfrak{B}(r)$, we obtain

$$\|\Phi(u) - \Phi(v)\|_{X^T} \le cT^\theta r\|u - v\|_{X^T}.$$

Then we conclude that if we choose $T > 0$ sufficiently small, then Φ is a contraction map. Then we obtain the unique local existence result in $X_{s_1,s_2,b}$ of the Cauchy problem (1)-(2) by the contraction argument. □

Replacing Proposition 2 by Proposition 3 in the proof of Theorem 1, we obtain Theorem 2.

References

[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schorödinger equation, part II: The KdV equation*, GAFA. **3** (1993), 107–156, 209–262.

[2] ———, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA. **3** (1993), 315–341.

[3] B. B. Kadomtsev and V. I. Petviashivili, *On the stability of solitary waves in weakly dispersive media*, Soviet Phys. Dokl. **15** (1970), 539–541.

[4] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[5] J. C. Saut and N. Tzvetkov, *The Cauchy problem for higher order KP equations*, preprint (1998).

[6] N. Tzvetkov, *On the Cauchy problem for Kadomtsev-Petviashvili equation*, preprint (1998).

[7] S. Ukai, *Local solutions of the Kadomtsev-Petviashvili equation*, J. Fac. Soc. Univ. Tokyo Sect. IA Math. **36** (1989), 193–209.