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Kyoto University
A note on the Rankin-Selberg method
for Siegel cusp forms of genus 2

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1 Introduction and Notations

In [K-S] Kohnen and Skoruppa introduced and studied a new type of Dirichlet series, which
is associated with the Fourier-Jacobi expansion of a pair \( F, G \) of Siegel cusp forms of the same
weight and genus 2. The proof is based on the Rankin-Selberg method. In particular, it was
shown that this Dirichlet series is equal to the Spinor zeta function attached to \( F \) up to constant
on condition that \( F \) is a Hecke eigenform and \( G \) is in the "Maass space".

In the present note we extend a part of results in [K-S] to the case of any level. As an
application, we give a new proof of meromorphic continuation of the Spinor zeta function
attached to a Siegel cusp form \( F \) of any level (on a condition for Fourier coefficients of \( F \)), and
find certain functional equation satisfied by the Spinor zeta function of any level \( > 1 \). We also
prove the Spinor zeta function of \( F \) times a simple meromorphic function is entire if \( F \) is not
in a certain Maass space, which was proved in the level 1 case in [Ev 2], [K-S], [O].

We remark that it is relatively easy to study Kohnen-Skoruppa's Dirichlet series, even in the
case of higher level (or even in the case of half-integral weight), because of its simple integral
representation.

Notations. We use standard notations, found in [Ei-Z]. We let \( \Gamma^g := \text{Sp}_g(Z) \) be integral
symplectic \( 2g \times 2g \)-matrices and set

\[
\Gamma^g_0(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g | C \equiv O \pmod{N} \right\},
\]

where \( A, B, C, D \) are \( g \times g \)-matrices. We let \( \Gamma^1, J(N) \) be the semi-direct product of \( \Gamma^1_0(N) \) and
\( \mathbb{Z}^2 \) (see [Ei-Za, p.9]), which is called the Jacobi group of level \( N \).

\( \mathcal{H}_g \) denotes the Siegel upper half space of genus \( g \) consisting of complex \( g \times g \)-matrices with
positive definite imaginary part. We often write

\[
Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2, \quad X = \text{Re}(Z) = \begin{pmatrix} u & x \\ x & u' \end{pmatrix}, \quad Y = \text{Im}(Z) = \begin{pmatrix} v & y \\ y & v' \end{pmatrix}.
\]

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We usually set $|Y| = \det Y$.

Let $k$ be an even integer $> 2$. $\Gamma^2$ acts on $\mathcal{H}_2$ by

$$\gamma(Z) := (AZ + B)(CZ + D)^{-1} \quad \left( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^2, \ Z \in \mathcal{H}_2 \right),$$

and acts on any function $F(Z)$ on $\mathcal{H}_2$ by

$$F|_{k\gamma}(Z) := \det(CZ + D)^{-k}F(\gamma(Z)).$$

$\Gamma^{1,J}(N)$ acts on any function $\phi(\tau, z)$ on $\mathcal{H}_1 \times \mathbb{C}$ by

$$\phi|_{k,m\gamma}(\tau, Z) := \frac{1}{(c\tau+d)k}\exp\left(-\frac{4\pi my^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda(a\tau+b)}{c\tau+d}+\mu\right)$$

$$(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu) \in \Gamma^{1,J}(N), \ (\tau, z) \in \mathcal{H}_1 \times \mathbb{C},$$

where $m$ denotes an integer $\geq 0$.

We write simply $\exp(*)$ for $\exp(2\pi i*)$.

**Definition.** Let $\chi$ be a Dirichlet character modulo $N$. A Siegel modular form of integral weight $k$, level $N$ and character $\chi$ is a holomorphic function on $\mathcal{H}_2$ satisfying

(i) $F|_{k\gamma} = \chi(\det D)F$ \quad $$(\forall\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^2(N))$$

and the vector space of all such functions $F$ is denoted by $M_k(N, \chi)$. If $F \in M_k(N, \chi)$ satisfies

(ii) $\Phi(F|_{k\gamma}) = 0$ \quad $$(\forall\gamma \in \Gamma^2, \ \Phi \text{ is the Siegel operator, cf. [A, p.75]})$$

$F$ is called a Siegel cusp form and the vector space of all such functions $F$ is denoted by $S_k(N, \chi)$. A Jacobi cusp form $\phi$ of weight $k$, level $N$, character $\chi$ and index $m$ is a holomorphic function on $\mathcal{H}_1 \times \mathbb{C}$ satisfying

(i) $\phi|_{k,m\gamma} = \chi(\det D)\phi$ \quad $$(\forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu) \in \Gamma^{1,J}(N))$$

(ii) $\phi|_{k,0}\gamma = \sum_{n, r \in \mathbb{N}} \sum_{D=c^2-4mn<0} c(D, r)q^n\zeta^r$ \quad $$(\forall\gamma \in \Gamma^1, \ N_\gamma \text{ is a natural number depending on } \gamma)$$

and the vector space of all such functions $\phi$ is denoted by $J_{k,m}^{cusp}(N, \chi)$.

The Petersson inner product on these spaces are normalized by

$$\langle F,G \rangle_N := \int_{\mathcal{H}_2} F(Z) \overline{G}(Z) |Y|^{k-3} dX dY$$

$$(F, G \in M_k(N, \chi), \ Z = X + iY \in \mathcal{H}_2, \ \text{One of } F, G \text{ is in } S_k(N, \chi)), \ \langle \phi, \psi \rangle_N := \int_{\mathcal{H}_1 \times \mathbb{C}} \phi(\tau, z) \overline{\psi}(\tau, z) \mu^{k-3} \exp\left(-\frac{4\pi my^2}{v}\right) du dv dx dy$$

$$(\phi, \psi \in J_{k,m}^{cusp}(N, \chi), \ \tau = u + iv \in \mathcal{H}_1, \ z = x + iy \in \mathbb{C}).$$
2 Statement of Result

Definition. Take $F \in S_k(N, \chi)$, $G \in M_k(N, \chi)$ and a natural number $M$ which divides $N$. For $\gamma \in \Gamma^2 = \mathrm{Sp}_2(\mathbb{Z})$, we write the Fourier-Jacobi expansions of $F|_{k\gamma}$ and $G|_{k\gamma}$ by

$$F|_{k\gamma} = \sum_{n \geq 1} \phi_{n,\gamma}(\tau, z) e\left(\frac{n\tau'}{N}\right) \text{ and } G|_{k\gamma} = \sum_{n \geq 1} \psi_{n,\gamma}(\tau, z) e\left(\frac{n\tau'}{N}\right).$$

Then we define a Dirichlet series $D_{F,G,M}(s)$ as $\zeta(2s - 2k + 4)$ times

$$\sum_{n \geq 1} \left\{ \int_{\gamma \in \Gamma^2(N) \setminus \Gamma^2_0(M)} \phi_{n,\gamma}(\tau, z) \overline{\psi}_{n,\gamma}(\tau, z) \exp\left(-\frac{4\pi n y^2}{vN}\right) v^{k-3} dv dx dy \right\} n^{-s},$$

(1)
on the assumption that $D_{F,G,M}(s)$ converges for sufficiently large $\Re(s)$, where $\mathcal{F}$ is a fundamental domain $\Gamma^1,J(M) \setminus \mathcal{H} \times \mathbb{C}$. We define its gamma factor by

$$D_{F,G,M}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,G,M}(s).$$

In a special case of $M = N$, the Dirichlet series above is an obvious generalization of Rankin’s Dirichlet series in the case of genus 1 (cf. [R]). In fact, if we write the Fourier-Jacobi expansions of $F$ and $G$ by

$$F(Z) = \sum_{n \geq 1} \phi_n(\tau, z) e(n\tau') \text{ and } G(Z) = \sum_{n \geq 1} \psi_n(\tau, z) e(n\tau'),$$

then

$$D_{F,G,N}(s) = \frac{1}{N^s} \zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\langle \phi_n, \psi_n \rangle_N}{n^s}. \zeta(2s - 2k + 4)$$

On the other hand, if $F(Z) \in S_k(N, \chi)$ is a Hecke eigenform with

$$T(n)F = \lambda_F(n)F$$

for all the Hecke operators $T(n)$ with $(n, N) = 1$, we can associate with $F$ the Spinor zeta function $Z_F(s)$ which has an Euler product of the form

$$Z_F(s) := \prod_{p \text{ prime}} Q_{F,p}(\chi(p)p^{-s}) \text{ (Re}(s) \gg 0),$$

$$Q_{F,p}(t) := \left\{ 1 - \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - \chi(p^2)p^{2k-4})t^2 - \chi(p^2)p^{2k-3} + \lambda_F(p^4)p^{4k-6}t^4 \right\}^{-1},$$

(2)

see [A, (4.3.35), Proposition 3.3.35, Exercise 3.3.38 and (4.4.21)]. We define its gamma factor by

$$Z_F^*(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) Z_F(s).$$

Note that the gamma factor of $D_{F,G,M}(s)$ concides with that of $Z_F(s)$. 
The modular forms which play an important role in relating (1) to (2) are Poincaré series. First, for a negative discriminant $D = r^2 - 4n$, we define the $D$-th Jacobi Poincaré series $P_{D,N}^{\tau}(\tau, z)$ of level $N$ and index $1$ by

$$\lambda_{k,D}P_{D,N}^{\tau}(\tau, z) := \sum_{\gamma \in \Gamma^1(J)(\infty) \backslash \Gamma^1(J(N))} \bar{\chi}(\gamma)\mathbf{e}(n\tau + rz)|_{k,1}\gamma \in J_{k,1}^{\text{cusp}}(N, \chi),$$

where we write $\lambda_{k,D} := \frac{1}{2}\Gamma(k-\frac{3}{2})(\pi|D|)^{-k+3/2}$, $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, $\lambda, \mu \in \Gamma^1(J(N))$ and $\Gamma^1(J(\infty)) := \left( \begin{array}{cc} \pm 1 & b \\ 0 & \pm 1 \end{array} \right), 0, \mu \in \Gamma^1(J(N))$. Next, we define a Siegel modular form $P_{D,N}^{\tau}(Z) \in M_k(N, \chi)$ as the image of $P_{D,N}^{\tau}(\tau, z)$ under the Maass lifting (for the definition, see (6) in the section 3).

Now let us state our main result.

**Theorem.** Let $F$ be a Siegel cusp form in $S_k(N, \chi)$ ($k$: even integer $> 2$). For a natural number $M$ dividing $N$ such that $\chi$ is defined modulo $M$, we define a trace of $F$ by

$$\text{Tr}^M_{T}(F) := \sum_{\gamma \in \Gamma_0^M(M) \backslash \Gamma_2^M(M)} F|_{k}\gamma(Z) \in S_k(M, \chi).$$

Suppose that $\text{Tr}^M_{T}(F)$ is a non-zero Hecke eigenform. Then for any negative fundamental discriminant $D$ and a Siegel modular form $P_{D,M}^{\tau}(Z) \in M_k(M, \chi)$ defined above, we have a relation

$$D_{F,P_{D,M}^{\tau}}(s) = d_{\text{Tr}^M_{T}(F),D}(s)Z_{F}(s).$$

Here for $\text{Tr}^M_{T}(F)(Z) = \sum_{Q>0} \tilde{A}(Q)e(\text{tr}QZ)$, by writing the indices of Fourier coefficients by integral ideals of some order in quadratic fields, we define a Dirichlet series

$$d_{\text{Tr}^M_{T}(F),D}(s) := \frac{1}{N^s} \sum_{\mathfrak{A}(\mathfrak{Z})|N^{\infty}} \tilde{A}(\mathfrak{Z})N^{-s(k+2)}(\text{Re}(s) \gg 0),$$

where $\mathfrak{Z}$ runs through all integral ideals of the maximal order in $\mathbb{Q}(\sqrt{D})$ such that each of the prime ideals which divides $\mathfrak{Z}$ also divides $M$ and $N\mathfrak{Z}$ denotes the norm of $\mathfrak{Z}$. This Dirichlet series is also defined by a following meromorphic function on the whole $s$-plane:

$$d_{\text{Tr}^M_{T}(F),D}(s) := \frac{1}{N^s h(D)} \sum_{\xi} \prod_{p|M} \left( 1 - \frac{\xi(p)}{N\mathfrak{Z}_{p}^{-k+2}} \right)^{-1} \xi(\mathfrak{Z})\tilde{A}(\mathfrak{Z}),$$

where $h(D)$ denotes the class number of $\mathbb{Q}(\sqrt{D})$, $\mathfrak{Z}$ runs through all prime ideals dividing $M$ of the maximal order in $\mathbb{Q}(\sqrt{D})$, $\{\mathfrak{Z}_i\}_{i=1,\ldots,h(D)}$ denotes a set of representatives of ideal class group and $\xi$ runs through all ideal class characters.

We shall write down our relation (4) in the special case of $M = N$. Let

$$F(Z) = \sum_{T>0} A(T)e(trTZ) = \sum_{m>0} \phi_m(\tau, z)e(m\tau') \in S_k(N, \chi)$$

be a non-zero Hecke eigenform for all the Hecke operators $T(n)$ with $(n, N) = 1$, then for any negative fundamental discriminant $D$ we have an explicit relation

$$\zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\langle \phi_n, P_{D,N}(V_n)|_N \rangle}{n^s} = \sum_{\mathfrak{Z}|N^\infty} \frac{A(\mathfrak{Z})}{N\mathfrak{Z}^{s-k+2}} \times Z_F(s),$$

where $V_n$ denotes the $n$-th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N, \chi)$ to $J_{k,n}^{\text{cusp}}(N, \chi)$ (see below).
3 Proof

The proof proceeds along the lines of the second proof of [K-S], which uses the “Maass lifting” of Jacobi Poincaré series and “Andrianov’s formula”.

We generalize Maass lifting as follows:

**Theorem-Definition** ([Saito-Kurokawa]-Maass lifting). (cf. [Ei-Za] and [M-Ra-V]) Let $\phi(\tau, z)$ be a Jacobi cusp form of index 1 in $J_{k,1}^{\text{cusp}}(N, \chi)$. Then we have a lifting map from $J_{k,1}^{\text{cusp}}(N, \chi)$ to $M_{k}(N, \chi)$ via

$$\phi(\tau, z) \mapsto \operatorname{Lift}(\phi) := \sum_{m \geq 1} \phi|V_{m}(\tau, z)e(m\tau'),$$

where $V_{m}$ is the $m$-th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N)$ to $J_{k,m}^{\text{cusp}}(N)$ and defined by

$$(\phi|V_{m})(\tau, z) := m^{k-1} \sum \chi(a)(cr + d)^{-k} e\left(\frac{-mcz^{2}}{cr + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{mz}{c\tau + d}\right).$$

We call this map the Maass lifting. We call the image $\operatorname{Lift}(J_{k,1}^{\text{cusp}}(N, \chi))$ the Maass space of level $N$ and character $\chi$.

Before the proof, we give a definition.

**Definition.** We define the Jacobi subgroup of level $N$ of $\Gamma_{0}^{2}(N)$ by

$$C_{2,1}(N) := \left\{ \left( \begin{array}{ccc} a & b & \mu \\
\lambda' & 1 & \mu' \kappa \\
c & d & -\chi \\
0 & 0 & 1 \end{array} \right) \in \Gamma_{0}^{2}(N) \right\}, \quad (\lambda', \mu') = (\lambda, \mu) \left( \begin{array}{cc} a & b \\
c & d \end{array} \right)$$

which is a central extension of $\Gamma^{1,J}(N)$ by $\mathbb{Z}$.

**Proof.** The proof is a direct generalization of [Ei-Z, Theorem 6.2 and Theorem 4.2]. By straightforward calculations, we see $\phi|V_{m}$ transforms like a Jacobi form of index $m$. Therefore

$$\phi|V_{m}(\tau, z)e(m\tau')$$

transforms like a Siegel modular form under the action of $C_{2,1}(N)$, hence a sum $\operatorname{Lift}(\phi)$ also does.

On the other hand, if we write the Fourier expansion of $\phi$ by

$$\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}} c(r^{2} - 4n, r)q^{n}\zeta^{r} \quad (q := e(\tau), \, \zeta := e(z)),$$

then a standard calculation shows

$$\phi|V_{m}(\tau, z) = \sum_{n, r \in \mathbb{Z}} \left( \sum_{a | (n, r, m)} \chi(a)a^{k-1}c\left(\frac{r^{2} - 4mn}{a^{2}}, \frac{r}{a}\right) \right)q^{n}\zeta^{r},$$
hence we have
\[
\text{Lift}(\phi)(\frac{\tau}{z}) = \sum_{(n, r, m) > 0} \left( \sum_{a(n, r, m)} \chi(a) a^{-1} \zeta^{r} p^{m} \right) q^{n} \zeta^{r} p^{m} \quad (p := e(\tau')).
\]

Also we can easily see \text{Lift}(\phi) is symmetric in \( n \) and \( m \), so we deduce that \text{Lift}(\phi) transforms like a Siegel modular form with respect to the matrix
\[
V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Therefore \text{Lift}(\phi) satisfies the transformation law of Siegel modular forms by using Lemma 1 below on generators for \( \Gamma_{0}^{2}(N) \).

\[\square\]

Remark. We have not succeeded in proving \text{Lift}(\phi) is a cusp form \( \in S_{k}(N, \chi) \) in general.

Lemma 1. \( \Gamma_{0}^{2}(N) \) is generated by \( C_{2,1}(N) \) (the Jacobi subgroup of level \( N \)) and the element
\[
V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Proof. Any integral primitive vector \( X = (x_{1}, x_{2}, x_{3}, x_{4}) \) could be reduced by the left multiplication by the element of type
\[
M(x, y, z) = \begin{pmatrix} 1 & 0 & 0 & x \\ y & 1 & x & z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
to a vector with \( \text{g.c.d.}(x_{2}, x_{4}) = 1 \). Next using the element of type
\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad c \equiv 0 \pmod{N},
\]
we may reduce the primitive vector \( X \) with \( N|x_{3}, x_{4} \) to \( X = (x_{1}, x_{2}, x_{3}, 0) \). Moreover \( X \) reduces to \( (x_{1}, 1, x_{3}, 0) \) by using a matrix of type \( M(x, y, z) \), and then by the left multiplication by the element of type
\[
\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c \equiv 0 \pmod{N},
\]
X could be reduced to $X = (x_1, 1, 0, 0)$ (note that g.c.d.$(x_1, x_3) = 1$ and $N|x_3$).

For any element $\gamma = (X_1, X_2, X_3, X_4) \in \Gamma_0^2(N)$, we reduce the 2-th column vector $X_2$ to the form $^t(x_1, 1, 0, 0)$ and multiplying an element $VM(x, y, z)V$ finally to $(0, 1, 0, 0)$. It is easily shown that this type matrix belongs to the parabolic subgroup $C_{2,1}(N)$, so Lemma 1 is proved.

We define a Siegel modular form as the Maass lifting of Jacobi Poicaré series defined in (3), i.e.

$$P_{D,M}(Z) := \operatorname{Lift}(P_{D,M}) = \sum_{m \geq 1} (P_{D,M}|V_m)(\tau, z)e(m\tau') \in M_k(M, \chi).$$

(6)

Now, we recall an important property of Jacobi Poicaré series:

**Lemma 2.** $P_{D,N}(\tau, z)$ (the $D$-th Jacobi Poicaré series in $J_{k,1}^{\text{cusp}}(N, \chi)$ defined in (3)) is characterized by

$$\langle \phi, P_{D,N} \rangle_N = c(D, r) \quad (\forall \phi \in J_{k,1}^{\text{cusp}}(N)),$$

where $c(D, r)$ denotes the $(D, r)$-th Fourier coefficient of $\phi$, i.e.

$$\phi(\tau, z) = \sum_{\substack{\nu, \tau' \in \mathbb{Z} \quad \nu^2 - 4\nu < 0 \quad n \in \mathbb{Z}}} c(D, r)q^n \zeta^{\nu/2} \quad (q := e(\tau), \ z := e(z)).$$

(Note that $c(D, r)$ depend only on $D = r^2 - 4n$ and $r \pmod 2$).

**Proof.** This is proved using the unfolding trick, in the same way of [G-K-Z, p.520].

For a half integral symmetric matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $D := b^2 - 4ac$, we can associate with $T$ a binary quadratic form

$$Q(x, y) = [a, b, c](x, y) = ax^2 + bxy + cy^2$$

of discriminant $D$, and a proper $o$-ideal of some order $o$ of the quadratic field $\mathbb{Q}(\sqrt{D})$:

$$\mathfrak{O} = a\mathbb{Z} + \frac{-b + \sqrt{D}}{2}\mathbb{Z}.$$

We occasionally write $A(Q)$, $A(a, b, c)$ or $A(\mathfrak{O})$ instead of $A(T)$ for Fourier coefficients of Siegel modular forms.

**Proof of Theorem.** We put the assumption that $D_{F, P_{D,M}}(s)$ converges sufficiently large $\operatorname{Re}(s)$ and put forward calculations, and later will remove the assumption by the convergence of Spinor zeta functions. Write the Fourier and the Fourier-Jacobi expansion of $\operatorname{Tr}^{N}_{M}(F)$ by

$$\operatorname{Tr}^{N}_{M}(F)(Z) = \sum_{T > 0} \tilde{A}(T) e(\text{tr}TZ) = \sum_{m > 0} \tilde{\phi}_m(\tau, z) e(m\tau')$$

respectively, where $T$ runs over all positive definite half integral matrices.
We recall the definition (6) of the Siegel modular form $\mathcal{P}_{D,N}(Z) \in M_{k}^{*}(N, \chi)$. We note that for any $\gamma \in \Gamma_{0}^{2}(M)$

$$\mathcal{P}_{D,M}|_{k\gamma}(Z) = \mathcal{P}_{D,M}(Z) = \sum_{m>0} P_{D,M}|V_{m}(\tau, z)e(m\tau'),$$

so in the notations of (2) in Definition

$$\psi_{n,\gamma} = \begin{cases} 0 & \text{if } n \text{ is not divisible by } N \\ P_{D,M}|V_{n}(\tau, z) & \text{if } n = Nm \end{cases}$$

Therefore the $Nm$-th coefficient of $\zeta(2s - 2k + 4)^{-1}D_{F,\mathcal{P}}M(D,M,s)$ is equal to

$$\int_{\mathcal{F}} \sum_{\gamma \in \Gamma_{0}^{2}(M)\setminus \Gamma_{0}^{2}(N)} \phi_{Nm,\gamma}(\tau, z) \overline{\psi}_{Nm,\gamma}(\tau, z) \exp\left(\frac{-4\pi my^{2}}{v}\right) v^{k-3} du dv dx dy = \left\langle \sum_{\gamma} \phi_{Nm,\gamma}, P_{D,M}|V_{m}\right\rangle_{M}.$$ 

We remark that $\sum_{\gamma} \phi_{Nm,\gamma}(\tau, z) = \tilde{\phi}_{m}(\tau, z)$ is nothing but the $m$-th Fourier-Jacobi coefficient of $\text{Tr}_{M}^{N}(F)$ and it is a Jacobi form of index $m$ and level $M$. Hence we can rewrite the above as

$$\langle \tilde{\phi}_{m}, P_{D,M}|V_{m}\rangle_{M} = \langle \tilde{\phi}_{m}|V_{m}^{*}, P_{D,M}\rangle_{M},$$

where $V_{m}^{*} : J_{k,m}^{\text{cusp}}(M, \chi) \rightarrow J_{k}^{\text{cusp}}(N, \chi)$ denotes the adjoint operator of $V_{m} : J_{k,1}^{\text{cusp}}(M, \chi) \rightarrow J_{k,m}^{\text{cusp}}(N, \chi)$. Now we must calculate the action of $V_{m}^{*}$ on Fourier coefficients explicitly.

**Proposition 1.** Let $V_{m}^{*} : J_{k,m}^{\text{cusp}}(N, \chi) \rightarrow J_{k,1}^{\text{cusp}}(N, \chi)$ be the adjoint operator of $V_{m} : J_{k,1}^{\text{cusp}}(N, \chi) \rightarrow J_{k,m}^{\text{cusp}}(N, \chi)$ with respect to the Petersson inner products. Then we have

$$\left\langle \tilde{\phi}_{m}|V_{m}^{*}, P_{D,M}\right\rangle_{M} = \sum_{D<0, r \in \mathbb{Z}} c(D, r)e\left(\frac{r^{2} - D}{4m}\tau + rz\right) |V_{m}^{*}|.$$

(Here, $c(D, r)$ denotes the Fourier coefficient of a Jacobi form of index $m$ and note that $c(D, r)$ depends only on $D$ and $r \pmod{2m}$.)

**Proof.** In our general case (i.e. level $N \geq 1$ and with character $\chi$), we can proceed along the same calculation on [K-S, p.554-557].

Using Proposition 1 and the characterization of $P_{D,M}$ in Lemma 2, we have

$$\langle \tilde{\phi}_{m}|V_{m}^{*}, P_{D,M}\rangle_{M} = \sum_{d|m, (m/d, M)=1} \tilde{\chi}(m/d)d^{k-2} \sum_{s \pmod{2d}, s^{2} \equiv D \pmod{4d}} \tilde{A}\left(\frac{m}{d} \left(\frac{s^{2} - D}{4d}, s, d\right)\right).$$
where \( \tilde{A}(*) \) denotes the Fourier coefficients of \( \text{Tr}^N_M(F) \). Let \( \{Q_i\}_{i=1,\ldots,h} \) be a set of representatives of binary quadratic forms of discriminant \( r^2 - 4n \) and let

\[
n(Q_i; d) := \# \left\{ s \pmod{2d} \mid s^2 \equiv D \pmod{4d}, \frac{s^2 - D}{4d}, s, d \sim Q_i \right\}
\]

be the number of \( s \pmod{2d} \) such that \( s^2 \equiv D \pmod{4d} \) and the quadratic form \( Q(x,y) = \frac{s^2 - D}{4d}x^2 + sxy + dy^2 \) is equivalent to \( Q_i \). Then we have

\[
\langle \tilde{\phi} | V_m^*, P_{D,M} \rangle_M = \sum_{i=1}^h \sum_{md|} \overline{x}(m/d) dk - 2n(Q_i; d) \tilde{A}(\frac{m}{d} Qi).
\]

By [Z, Proposition 3 (i)] we can see

\[
\sum_{n \geq 1} n(Q_i; n)n^{-s} = \zeta_{Q_i}(s) \zeta(2s)^{-1},
\]

where \( \zeta_{Q_i}(s) \) is the (partial) zeta function of the class of \( Q_i \) (= the zeta function of the ideal class of \( \mathbf{Q}(\sqrt{D}) \) corresponding in the usual way to the class of \( Q_i \)), so we obtain

\[
D_{F,P_{D,M},M}(s) = N^{-s} \sum_{i=1}^h \zeta_{Q_i}(s - k + 2) R_{Q_i,\text{Tr}^N_M(F),M}(s),
\]

with

\[
R_{Q_i,\text{Tr}^N_M(F),M}(s) := \sum_{n \geq 1, (n,M)=1} \overline{\chi}(n) \tilde{A}(nQ_i)n^{-s}.
\]

We now recall Andrinov's formula, which is mentioned in [A, Theorem 4.3.16] in a most general form. Take any negative fundamental discriminant \( D \) and any Hecke eigenform \( F(Z) = \sum Q A(Q) \mathbf{e}(\text{tr}QZ) \in S_k(M, \chi) \). Then for any class character \( \xi \) of the class group \( H(D) \) and any completely multiplicative function \( \omega \) on \( N_{(M)} := \{ n \in \mathbf{N} | (n,M) = 1 \} \), it holds that

\[
A_{\xi}(s) \prod_{\text{p prime ideal (p,M)=1}} \left( 1 - \zeta \frac{(N\mathbf{p})\omega(N\mathbf{p})\xi(p)}{(N\mathbf{p})^{s-k+2}} \right) \prod_{\text{p prime (p,M)=1}} Q_{F,p}(\omega(p)p^{-s})
\]

\[
= \sum_{i=1}^{h(D)} \xi(Q_i) \sum_{n \in N_{(M)}} \frac{\omega(n) A(nQ_i)}{n^s},
\]

with

\[
A_{\xi}(s) := \sum_{i=1}^{h(D)} \xi(Q_i) A(Q_i),
\]

where \( h = h(D) = \# H(D) \) is the class number of discriminant \( D \). Inverting this,

\[
\sum_{n \in N_{(M)}} \frac{\omega(n) A(nQ_i)}{n^s}
\]

\[
= \frac{1}{h} \prod_{(p,M)=1} Q_{F,p}(\omega(p)p^{-s}) \sum_{i=1}^{h(D)} \xi(Q_i) A_{\xi}(s) \prod_{\text{p prime ideal (p,M)=1}} \left( 1 - \zeta \frac{(N\mathbf{p})\omega(N\mathbf{p})\xi(p)}{(N\mathbf{p})^{s-k+2}} \right).
\]
Instituting this formula for $F = \operatorname{Tr}^N_M(F)$, $\omega = \chi$ in (7), we have

\[
D_{F,\mathcal{P}_{D},M}(s) = \frac{Z_{\operatorname{Tr}^N_M(F)}(s)}{N^s h} \sum_{i=1}^{h} \zeta_i Q_i(s - k + 2) \sum_{\xi} \overline{\xi}(Q_i) \tilde{A}_\xi(s) \prod_{\text{prime ideal } (\wp,M) = 1} \left(1 - \frac{\xi(\wp)}{(N\wp)^{s-k+2}}\right),
\]

since, by writing the above Euler product by $L(s, \xi)$, it holds $L(s, \overline{\xi}) = L(s, \xi)$.

We note that

\[
d_{F,D}(s) = \frac{1}{N^s h} \sum_{\xi} \prod_{\wp|M} \left(1 - \frac{\overline{\xi}(\wp)}{N\wp^{s-k+2}}\right)^{-1} \tilde{A}_\xi(s)
\]

is a meromorphic function on the whole $s$-plane. Expanding the right hand side we get

\[
D_{F,\mathcal{P}_{D},M}(s) = \frac{Z_{\operatorname{Tr}^N_M(F)}(s)}{N^s h} \sum_{i=1}^{h} \tilde{A}(Q_i) \sum_{\xi} \sum_{\wp|M} \frac{\xi(\wp)^{-1}}{N\wp^{s-k+2}},
\]

and summing up for $\xi$'s, we have the relation (4) and the expression (5).

Now we can remove the assumption on convergence of $D_{F,\mathcal{P}_{D},M}(s)$ for sufficiently large $\operatorname{Re}(s)$ by using convergence of $Z_F(s)$. This completes the proof of Theorem.

\[\square\]

### 4 Applications

We summarize the known facts about the analytic properties for $D_{F,G,M}(s)$'s. We define Eisenstein series of Klingen-Siegel type of weight 0 and level $N$ by

\[
E_{s,N}(Z) := \sum_{\gamma \in C_2(N) \backslash \Gamma_0(N)} \left(\frac{\det \operatorname{Im} \gamma(Z)}{\operatorname{Im} \gamma(Z)}\right)^s,
\]

where $C_2(N)$ stands for the Jacobi subgroup of level $N$ (see Definition in the section 3) and $Z_1$ denotes the left upper entry of $Z \in \mathcal{H}_2$. We define its gamma factor by

\[
E_{s,N}^\gamma(Z) := \pi^{-s} \Gamma(s) \zeta(2s) \prod_{p \mid N} \left(1 - \frac{1}{p^{2s}}\right) E_{s,N}(Z).
\]

In this last section, for Siegel modular forms $F \in S_k(N,\chi), G \in M_k(N,\chi)$ and a natural number $M$ dividing $N$, we put

\[
D_{F,G,M}(s) := \prod_{p \mid M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D_{F,G,M}(s), \quad D_{F,G,M}^\gamma(s) := \prod_{p \mid M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D_{F,G,M}^\gamma(s),
\]

\[
Z_{F,N}(s) := \prod_{p \mid N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z_F(s), \quad Z_{F,N}^\gamma(s) := \prod_{p \mid N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z_F^\gamma(s).
\]

Then $D_{F,G,M}(s)$ has a following integral representation:
Lemma 3 ([H 1, Lemma 2]). We have
\[ N^s D_{F,G;M}^s(s) = \pi^{-k+2} \langle FE_{s-k+2;M}^s, G \rangle_N. \]

Also we can prove functional equations of Eisenstein series \( E_{s,N}(Z) \) for arbitrary level:

**Lemma 4.** Let \( N \) be a natural number. Then, the function \( E_{s,N}^*(Z) \) has a meromorphic continuation to \( \mathbb{C} \) with possible simple poles at \( s=0,2 \) and satisfies a functional equation
\[
\frac{1}{N^{2-s}} \sum_{d|N} d^{2(2-s)} E_{s-N,d}^*(Z) = \frac{1}{N^s} \sum_{d|N} d^{2s} E_{s,d}^*(Z),
\]
or equivalently
\[
E_{2-s,N}^*(Z) = \frac{1}{N^{2}} \sum_{d|N} e^{2s} \prod_{p|d} (1 - p^{2s-2}) E_{s,d}^*(Z).
\]

**Proof.** (For details, see [H 3].) We will prove for any natural numbers \( m \) and \( N \) the formula
\[
N^s E_{s,m}(NZ) = - \sum_{1 \neq M \mid N \atop (m,M)=1} \mu(M) \sum_{d \mid M} \mu(d) (N/M)^s E_{s,l,c.m.(m,d)}(\mathbb{C}(N/M)Z) + N^2 E_{s,m,N}(Z),
\]
where \( \mu(*) \) denotes the Möbius function. We note that for a square-free number \( M \) with \( (m,M) > 1 \)
\[
\sum_{d \mid M} \mu(d) E_{s,l,c.m.(m,d)}((N/M)Z) = \sum_{d_1 \mid M/(m,M)} \mu(d_1) \sum_{d_2 \mid (m,M)} \mu(d_2) E_{s,m,d_1}((N/M)Z) = 0,
\]
then we have
\[
N^s E_{s,m}(NZ) = - \sum_{1 \neq M \mid N \atop (m,M)=1} \mu(M) \sum_{d \mid M} \mu(d) (N/M)^s E_{s,m,d}(\mathbb{C}(N/M)Z) + N^2 E_{s,m,N}(Z).
\]

Now by using the assumption of induction on \( N \), we have
\[
N^s E_{s,m}(NZ) = - \sum_{1 \neq M \mid N \atop (m,M)=1} \mu(M) \sum_{d \mid M} \mu(d) \prod_{p \mid e} (p^{2s} - 1) \prod_{p \mid m} p^{2s} \prod_{f \geq 1} p^{2fs} E_{s,m,d}(\mathbb{C}(N/M)Z) + N^2 E_{s,m,N}(Z).
\]
Now we can see the sum of the first and third lines on the RHS is equal to 0 by using the following Claim and get the formula (8).

Claim. We fix natural numbers $d$, $e$, $m$ and $N$ such that $de|N$, $(d,m) = 1$ and $d$ is square-free, then we have

$$
\sum_{M \in \mathbb{N}, \, d|N, \, (e,M) = 1} \mu(M) = \begin{cases} 
\mu(d) & \text{if } de = N \\
0 & \text{if } de < N
\end{cases}.
$$

Then the assertions for meromorphic continuation and poles are obvious by (8) and induction on $N$, and the symmetric functional equation follows by specializing (8) to the case $m = 1$ and using the functional equation $E_{s-1}(Z) = E_{s,1}(Z)$ (cf. [K-S, Main Lemma]). We can easily prove the other functional equation from the symmetric one.

By Lemma 3 and 4, we can deduce

**Proposition 2** ([H 1, Proposition 1 and the section 4] and [H 3]). All $D_{F,G;M}(s)$'s with $M|N$ have a meromorphic continuation to $\mathbb{C}$, are entire if $(F, G)_N = 0$ and otherwise has a simple pole at $s = k$ as its only singularity with the residue

$$
\text{Res}_{s=k}D_{F,G;M}(s) = \frac{4^k \pi^{k+2}}{(k-1)!N^k M^2} \prod_{p|N} (1 - \frac{1}{p^2}) (F, G)_N.
$$

Furthermore there exists a functional equation

$$
N^{2(k-s)}D^*_{F,G;N}(2k - 2 - s) = \sum_{M|N} M^{2(s-k+2)} \prod_{p|N/M} (1 - p^{-2(s-k+1)}) D^*_{F,G;M}(s).
$$

Using Proposition 2 and Theorem in the case of $M = N$ we have

**Corollary 1.** Let $F \in \mathcal{S}_k(N, \chi)$ be a non-zero Hecke eigenform of level $N$. Suppose that $d_{F,D}(s)$ defined by (5) is not identically zero for some fundamental discriminant $D$. Then $Z_{F,N}(s)$ has a meromorphic continuation to the whole $s$-plane, the possible poles of $d_{F,D}(s)Z_{F,N}(s)$ are $s = k$. If $d_{F,D}(k)(F, P_{N,D})_N \neq 0$, then we have

$$
\frac{1}{\pi^{k+2}(F, P_{N,D})_N} \text{Res}_{s=k}Z_{F,N}(s) = \frac{4^k}{(k-1)!N^{k+2} d_{F,D}(k)} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \in \mathbb{Q}(F, \mathbb{e}(1/h(D))),
$$

where $\mathbb{Q}(F, \mathbb{e}(1/h(D)))$ is the field generated by the Fourier coefficients of $F$ and a primitive $h(D)$-th root of unity over $\mathbb{Q}$.

Furthermore there exists a functional equation satisfied by the Spinor zeta function $Z_{F,N}(s)$ and the Dirichlet series $D_{F,P_{M,D},M}(s)$'s with $M|N$. Explicitly, it holds

$$
N^{2(k-s)}d_{F,D}(2k - 2 - s)Z^*_{F,N}(2k - 2 - s) = \sum_{M|N} M^{2(s-k+2)} \prod_{p|N/M} \left(1 - p^{2(s-k+1)}\right) D^*_{F,G;M}(s).
$$
Remark. Similar results of Corollary 1 are given in [Ma] by the different method. For principal congruence subgroups. Similar results pf Corollary 1 are reported in [Ev 1, English transl. p.457] (without proof).

Cororally 2. (cf. [Ev 2], [K-S], [O].) Let $F \in S_k(N, \chi)$ be a non-zero Hecke eigenform. Suppose $F$ is in the orthogonal compliment of $\text{Lift}(J_{k,1}^\text{cusp}(N, \chi))$ (the Maass space, see the section 3), then $d_{F,D}(s)Z_{F,N}(s)$ is holomorphic for all $s$.

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