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TWISTED ENDOSCOPY IMPLIES THE GENERIC PACKET CONJECTURE

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CONTENTS

1. INTRODUCTION

Langlands correspondence is one of the most fundamental leading principle in the modern theory of automorphic forms. It predicts, for a global field k, the existence of a fine correspondence between the set of automorphic representations of $GL(n)_k$ and the set of isomorphism classes of n-dimensional representations of the hypothetical Langlands group $\mathcal{L}_k$ of k. Its generalizations to the automorphic representations of the general reductive groups is of particular importance, since they are also inevitable to establish the original (i.e. $GL(n)$) Langlands correspondence.

The multiplicity one theorem for $GL(n)_k$ reduced us to describe the correspondence only for the set of automorphic representations. But for the general reductive groups, we have to propose a recipe to deduce the *multiplicity* in the automorphic spectrum of each isomorphism classes of automorphic representations from its corresponding representation of $\mathcal{L}_k$ (or rather its Langlands parameter). By constructing the theory of endoscopy, Langlands obtained this type of conjectural recipe for the tempered automorphic representations [K, §12]. Then Arthur extended it to the case of non-tempered automorphic representations [A].

I feel that there is also a realistic program to establish these conjectures at least for some classical groups, by means of comparisons of Arthur-Selberg trace formulae in the setting of twisted endoscopy. In the course of this program, one needs to establish certain relationship between the representation theory of reductive groups over a local field and certain automorphic L and ε-factors. Let F be a local field and G a connected reductive quasisplit group over F. If an irreducible smooth representation π of $G(F)$ is *generic*, i.e. if it admits a Whittaker model, then Shahidi defined some of its corresponding automorphic L and ε-factors, and obtained the desired relationships [Sh]. Thus we might ask if we can extend his results to the representations which are not generic. The generic packet conjecture is a key part of this question.

Conjecturally the set $\Pi_{\mathrm{temp}}(G(F))$ of isomorphism classes of irreducible tempered representations of $G(F)$ should be partitioned into a disjoint union of finite subsets Π_φ, called *L-packets* parametrized by the so-called Langlands parameters φ for G_F. The elements of Π_φ should share the same L and ε-factors which can be directly defined from φ. Thus to extend Shahidi's definition of Euler factors, it is sufficient to find a generic element in each Π_φ. This is exactly the assertion of the generic packet conjecture.

Not so many is known about this conjecture. From now on, we assume that F is non-archimedean since the archimedean case was established by Vogan [V]. Then before discussing the generic packet conjecture, we must assume that the tempered L-packets are defined and satisfy some reasonable properties. At present, this is the case only when G is $SL(n)$ and $U(3)$. If a tempered L-packet Π_φ of G is a lift of an L-packet Π_{φ^H} of an endoscopic group H of G, *i.e.* Π_φ is and *endoscopic L-packet*, Shahidi reduced the generic packet conjecture for Π_φ to that for Π_{φ^H} [Sh, §9]. Gelbart-Rogawski-Soudry obtained a beautiful description of the endoscopic L-packets of $U(3)$ in terms of theta liftings, and deduced the conjecture for them [GRS]. More recently, Friedberg-Gelbart-Jacquet-Rogawski established the generic packet conjecture for any L-packets of $U(3)$ by means of the relative trace formulae [FGJR]. Their result also includes the global counter part of the conjecture. In fact, the relative trace formula proposed by Jacquet is a very promising tool which might be used to obtain some deep informations (e.g. properties of ε-functions) necessary in the Arthur-Langlands program.

In this report we shall examine an extended version of the Shahidi's approach. His method relies on the relationship between the genericity of a representation π and the asymptotic behavior of the distribution character Θ_π around the unit [Rd2], [MW]. Our result is a twisted analogue of this relationship (under some restriction on the residual characteristic). Then we look at the situation where many classical groups are realized as twisted endoscopic groups of $GL(n)$ [A2, §9]. Suppose that G is such a classical group. Then, assuming the existence of suitable twisted endoscopic lifting from G to $GL(n)$, a twisted version of Shahidi's argument allows us to deduce the generic packet conjecture.

Although our method works only in the case $G = U(3)$ at the moment, where the result is already known, it should apply more wider class of groups once the twisted endoscopic lift is established. On the other hand, since our approach uses the Lie algebra and the exponential map, the restriction on the residual characteristic is inevitable.

2. Degenerate Whittaker models and their twisted analogue

2.1. **Degenerate Whittaker models.** Let F be a non-archimedean local field of characteristic zero. We write $\mathcal{O}, \mathfrak{p}$, and $|\ |$ for the ring of integers of F, its unique maximal ideal and the normalized absolute value on F. Fix a non-trivial additive character ψ of F. Throughout, $\mathbb{C}^1$ denotes the group of complex numbers of absolute value 1.

We consider a connected reductive F-group G. Write $\mathfrak{g}$ for its Lie algebra and fix a non-degenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form $B(\ ,\ )$ on $\mathfrak{g}(F) \otimes_F \mathfrak{g}(F)$. To define the space of degenerate Whittaker vectors, we need the data (N, ϕ) where N is a nilpotent element in $\mathfrak{g}(F)$ and ϕ is a cocharacter of G defined over F such that

$$\mathrm{Ad}(\phi(t))N = t^{-2}N, \quad \forall t \in \mathbb{G}_m.$$

We write $\mathfrak{g}^N$ for the centralizer of N in $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{g}^N \oplus \mathfrak{m}$$

be a decomposition which is $\mathrm{Ad}(\phi(\mathbb{G}_m))$-stable. We have the graduation

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_i = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(\phi(t))X = t^i X,\ \forall t \in \mathbb{G}_m\}.$$

We write $\mathfrak{g}_i^N := \mathfrak{g}_i \cap \mathfrak{g}^N$, $\mathfrak{m}_i := \mathfrak{m} \cap \mathfrak{g}_i$, etc. We have the unipotent subgroup V whose Lie algebra is given by $\mathfrak{g}_1^N \oplus \sum_{i \geq 2} \mathfrak{g}_i$ and a character χ_N of $V(F)$ defined by

$$\chi_N : V(F) \ni \exp X \longmapsto \psi(B(N, X)) \in \mathbb{C}^\times.$$

The restriction to $\mathfrak{m}(F)$ of the alternating form $B_N(X, Y) := B(N, [X, Y])$ is non-degenerate. If $\mathfrak{m}_1 \neq \{0\}$, then we define $\mathcal{H}_N$ to be the Heisenberg group $\mathfrak{m}_1(F) \times \mathbb{C}^1$ attached to the symplectic space $(\mathfrak{m}_1(F), B_N)$. If this is not the case, set $\mathcal{H}_N = \mathbb{C}^1$. We write ρ_N for the unique irreducible representation of $\mathcal{H}_N$ on which the subgroup $\mathbb{C}^1$ acts by $\mathrm{id}_{\mathbb{C}^1}$. Let U be the unipotent subgroup of G with the Lie algebra $\sum_{i \geq 1} \mathfrak{g}_i$. Then we have the homomorphism

$$p_N : U(F) \ni \exp(\sum_{i \geq 1} X_i) \longmapsto (X_1^{\mathfrak{m}}, \psi(B(N, X_2))) \in \mathcal{H}_N.$$

Here $X_1^{\mathfrak{m}}$ is the $\mathfrak{m}$-component of X_1 under the decomposition $\mathfrak{g} = \mathfrak{g}^N \oplus \mathfrak{m}$.

Now let (π, E) be an irreducible admissible representation of $G(F)$. We put

$$E(V, \chi_N) := \mathrm{Span}\{\pi(v)\xi - \chi_N(v)\xi \,|\, v \in V(F),\ \xi \in E\}, \quad E_{V,\chi_N} := E/E(V, \chi_N).$$

Obviously, $U(F)$ acts on E_{V,χ_N} by some copy of $\rho_N \circ p_N$. The space of *degenerate Whittaker vectors* $\mathcal{W}_{N,\phi}(\pi)$ for π is defined to be

$$\mathcal{W}_{N,\phi}(\pi) := \mathrm{Hom}_{U(F)}(\rho_N \circ p_N, E_{V,\chi_N}).$$

If N is regular, this is the space of ordinary χ_N-Whittaker vectors.

2.2. **A twisted analogue of** [MW]. We begin with a review of the result of [MW].

Let (π, E) be an irreducible admissible representation of $G(F)$. The distribution $\Theta_\pi : C_c^\infty(G(F)) \ni f \mapsto \mathrm{tr}\pi(f) \in \mathbb{C}$, where

$$\pi(f)\xi := \int_{G(F)} f(x)\pi(x)\xi\, dx, \quad \xi \in E$$

is a well-defined distribution called the *character* of π. On a suitable neighborhood of 0 in $\mathfrak{g}(F)$, the exponential map exp is well-defined and is an injection into $G(F)$. Then a theorem of Harish-Chandra asserts that, on some neighborhood U_π of 0 in $\mathfrak{g}(F)$, the distribution $\Theta_\pi \circ \exp$ is well-defined and admits an expansion:

$$\Theta_\pi(f) = \sum_{\mathfrak{O}\in\mathcal{N}(\mathfrak{g}(F))} c_{\mathfrak{O}}(\pi)\mu_{\mathfrak{O}}(\widehat{f\circ\exp}), \quad \forall f \in C_c^\infty(\exp(U_\pi)).$$

Here $\mathcal{N}(\mathfrak{g}(F))$ denotes the set of nilpotent $\mathrm{Ad}(G(F))$-orbits in $\mathfrak{g}(F)$, and $\mu_{\mathfrak{O}}$ is an invariant measure on $\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(F))$. Also $\widehat{f\circ\exp}$ is the Fourier transform

$$\widehat{f\circ\exp}(X) := \int_{\mathfrak{g}(F)} f(\exp Y)\psi(B(X,Y))\, dY.$$

Of course the coefficients $c_{\mathfrak{O}}(\pi)$ depends on the choice of measures. We shall adopt the following convention on this choice. The measure dX on $\mathfrak{g}(F)$ is chosen to be self-dual with respect to $\psi \circ B$. By means of B_N, we identify the tangent space $T_N\mathfrak{O}$ of $\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(F))$ at $N \in \mathfrak{O}$ with $\mathfrak{g}(F)/\mathfrak{g}^N(F) \simeq \mathfrak{m}(F)$. We fix the measure self-dual with respect to $\psi \circ B_N$ on this. Then it determines the invariant measure $\mu_{\mathfrak{O}}$ on $\mathfrak{O}$. Also fix an invariant measure dx on $G(F)$ such that the Jacobian of exp around 0 with respect to dX and dx has absolute value 1.

Finally we write $\mathfrak{O} \geq \mathfrak{O}'$ ($\mathfrak{O}$, $\mathfrak{O}' \in \mathcal{N}(\mathfrak{g}(F))$) if $\mathfrak{O}'$ is contained in the closure (with respect to the p-adic topology) of $\mathfrak{O}$. This is a partial order on $\mathcal{N}(\mathfrak{g}(F))$. Set

$$\mathcal{N}_B(\pi) := \{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(F)) \,|\, c_{\mathfrak{O}}(\pi) \neq 0\},$$
$$\mathcal{N}_{\mathrm{Wh}}(\pi) := \{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(F)) \,|\, \mathcal{W}_{N,\phi}(\pi) \neq 0,\, \exists\phi\}.$$

We write $\mathcal{N}_\bullet(\pi)^{\max}$ for the set of maximal elements in $\mathcal{N}_\bullet(\pi)$ with respect to the partial order defined above ($\bullet$ assigns B or Wh).

Theorem 2.1 ([MW] I.16, 17). *Suppose that the residual characteristic of F is odd. Then*
(i) $\mathcal{N}_B(\pi)^{\max} = \mathcal{N}_{\mathrm{Wh}}(\pi)^{\max}$.
(ii) For $\mathfrak{O} \in \mathcal{N}_{\mathrm{Wh}}(\pi)^{\max}$ and ϕ as in the definition of $\mathcal{W}_{N,\phi}(\pi)$, we have $\dim \mathcal{W}_{N,\phi}(\pi) = c_{\mathfrak{O}}(\pi)$.

If G is split and $\mathcal{N}_B(\pi)$ contains an regular nilpotent orbit, this is due to Rodier [Rd2]. The contribution of Mœglin-Waldspurger is the construction of $\mathcal{W}_{N,\phi}(\pi)$ when N is not even unipotent. The same construction in the case of finite F was found independently by Kawanaka [Ka].

Now we turn to the twisted analogue of this result. Let θ be an F-automorphism of finite order ℓ of G. We suppose that (π, E) is θ-stable, that is, $\theta(\pi) := \pi\circ\theta^{-1}$ is isomorphic to π. By choosing an intertwiner $\pi(\theta) : \theta(\pi) \xrightarrow{\sim} \pi$ satisfying $\pi(\theta)^\ell = \mathrm{id}_E$, we extend π to an irreducible representation of $G(F) \rtimes \langle\theta\rangle$. If we take $B(\ ,\ )$, N and ϕ above to be θ-invariant, then the resulting space $\mathcal{W}_{N,\phi}(\pi)$ is $\pi(\theta)$-stable.

We need Clozel's result on the asymptotic behavior of twisted character [C]. Recall Harish-Chandra's descent for twisted invariant distributions, *i.e.* distributions invariant under the θ-conjugacy:

$$\mathrm{Ad}_\theta(g)x := gx\theta(g)^{-1}, \quad g,\, x \in G(F).$$

Writing $\mathfrak{g}(\theta) := (1-\theta)\mathfrak{g} \subset \mathfrak{g}$, set $\Omega_\theta := \mathrm{Ad}_\theta(G(F))G^{\theta\prime}(F)$ where

$$G^{\theta\prime}(F) := \{g \in G^\theta(F) \mid \det(\mathrm{Ad}(g)\circ\theta - 1|\mathfrak{g}(\theta)) \neq 0\}.$$

There exists a surjective linear map

$$C_c^\infty(G(F)\times G^{\theta\prime}(F)) \ni \alpha(g,m) \longmapsto \varphi(x) \in C_c^\infty(\Omega_\theta)$$

such that

$$\int_{G(F)}\int_{G^\theta(F)} \alpha(g,m)\Phi(\mathrm{Ad}_\theta(g)m)\,dm\,dg = \int_{G(F)} \varphi(x)\Phi(x)\,dx, \quad \forall\Phi \in C_c^\infty(G(F)).$$

Dual to this is a map of distributions

$$\mathcal{D}(\Omega_\theta) \ni T \longmapsto \tau \in \mathcal{D}(G(F)\times G^{\theta\prime}(F))$$

given by $\langle\tau,\alpha\rangle := \langle T,\varphi\rangle$. Though the map $\alpha \mapsto \varphi$ is not injective,

$$\varphi^\theta(m) := \int_{G(F)} \alpha(g,m)\,dg \in C_c^\infty(G^{\theta\prime}(F))$$

is well-defined.

Lemma 2.2. *For any* $\mathrm{Ad}_\theta(G(F))$*-invariant* $T \in \mathcal{D}(\Omega_\theta)$*, there exists an* $\mathrm{Ad}(G^\theta(F))$*-invariant distribution* $\sigma_T \in \mathcal{D}(G^{\theta\prime}(F))$ *such that*

$$\langle T,\varphi\rangle = \langle\sigma_T,\varphi^\theta\rangle, \quad \forall\alpha \in C_c^\infty(G(F)\times G^{\theta\prime}(F)).$$

The twisted character of π, a twisted invariant distribution, is defined by

$$\Theta_{\pi,\theta}(f) := \mathrm{tr}(\pi(f)\circ\pi(\theta)), \quad f \in C_c^\infty(G(F)).$$

We apply Lem.2.2 to have an invariant distribution ϑ_π on $G^{\theta\prime}(F)$ such that

$$\Theta_{\pi,\theta}(\varphi) = \vartheta_\pi(\varphi^\theta), \quad \forall\varphi \in C_c^\infty(\Omega_\theta).$$

Theorem 2.3 ([C] Th.3). *Write* $\mathcal{N}(\mathfrak{g}^\theta(F))$ *for the set of nilpotent* $\mathrm{Ad}(G^\theta(F))$*-orbits in* $\mathfrak{g}^\theta(F)$*. Then there exist a neighborhood* $U_{\pi,\theta}$ *of* 0 *in* $\mathfrak{g}^\theta(F)$ *and complex numbers* $c_{\mathfrak{O},\theta}(\pi)$*,* $(\mathfrak{O} \in \mathcal{N}(\mathfrak{g}^\theta(F)))$ *such that*

$$\Theta_{\pi,\theta}(\varphi) = \sum_{\mathfrak{O}\in\mathcal{N}(\mathfrak{g}^\theta(F))} c_{\mathfrak{O},\theta}(\pi)\widehat{\mu}_{\mathfrak{O}}(\varphi^\theta\circ\exp)$$

holds for any $\varphi \in C_c^\infty(\Omega_\theta)$ *with* $\mathrm{supp}(\varphi^\theta \circ \exp) \subset U_{\pi,\theta}$.

Similarly as in the ordinary case, we put

$$\mathcal{N}_{B,\theta}(\pi) := \{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}^\theta(F)) \,|\, c_{\mathfrak{O},\theta}(\pi) \neq 0\},$$
$$\mathcal{N}_{\mathrm{Wh},\theta}(\pi) := \{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}^\theta(F)) \,|\, \mathrm{tr}(\pi(\theta)|\mathcal{W}_{N,\phi}(\pi)) \neq 0,\ \exists\phi\}.$$

Now we can state:

Theorem 2.4. *Suppose that* ℓ *is prime to the residual characteristic of* F*, which we assumed to be odd.*
(1) $\mathcal{N}_{B,\theta}(\pi)^{\max}$ *coincides with* $\mathcal{N}_{\mathrm{Wh},\theta}(\pi)^{\max}$.
(2) Let $\mathfrak{O} \in \mathcal{N}_{B,\theta}(\pi)^{\max}$*. Then for any choice of* $N \in \mathfrak{O}$ *and* ϕ*, we have* $\mathrm{tr}(\pi(\theta)|\mathcal{W}_{N,\phi}(\pi)) = c_{\mathfrak{O},\theta}(\pi)$.

3. Twisted endoscopy implies the generic packet conjecture

3.1. **Twisted endoscopy problems to be considered.** We first review some basic facts on twisted endoscopy [KS].

Let k be a number field. We write ${}^LG = \widehat{G} \rtimes W_k$ for the L-group of a connected reductive k-group G. Langlands conjecture says that the automorphic representations of $G(\mathbb{A}_k)$ should be parametrized by the $\widehat{G}$-conjugacy classes of suitable homomorphisms $\varphi : \mathcal{L}_k \to {}^LG$, called the Langlands parameters. Endoscopy is aimed at the study of those automorphic representations whose Langlands parameters have the image contained in the fixed part ${}^LG^{{}^L\theta}$ of ${}^LG$ under certain semisimple automorphism ${}^L\theta$. Ordinary endoscopy deals with the case ${}^L\theta \in \mathrm{Int}(\widehat{G})$ and twisted endoscopy handles the general case.

Since the Langlands parameters are considered up to $\widehat{G}$-conjugation, we may assume that ${}^L\theta$ is of the form

$${}^L\theta(g \rtimes w) = \widehat{\theta}(g)a(w) \rtimes w, \quad g \in \widehat{G},\ w \in W_k,$$

where $\widehat{\theta}$ is a semisimple automorphism of $\widehat{G}$ and $a(w)$ is a $Z(\widehat{G})$-valued 1-cocycle on W_k. In this case, we say that the twisted endoscopy problem is associated to $(G, \theta, \mathbf{a})$, where θ is an k-automorphism of G whose outer class is dual to $\widehat{\theta}$ and $\mathbf{a}$ is the class of a in $\mathrm{H}^1(W_k, Z(\widehat{G}))$. The class of $\mathbf{a}$ in $\mathrm{H}^1(W_k, Z(\widehat{G}))/\ker^1(W_k, Z(\widehat{G}))$ corresponds to an automorphic character ω of $G(k)\backslash G(\mathbb{A}_k)$ by Langlands' correspondence for tori. Here

$$\ker^1(W_k, Z(\widehat{G})) := \ker[\mathrm{H}^1(W_k, Z(\widehat{G})) \to \prod_v \mathrm{H}^1(W_{k_v}, Z(\widehat{G}))].$$

Then according to the Langlands conjecture, the twisted endoscopy for $(G, \theta, \mathbf{a})$ should describe the automorphic representations or an L- or Arthur packets Π of $G(\mathbb{A}_k)$ which satisfies $\omega \otimes \theta(\Pi) = \Pi$. There is also an evident local counter part to this.

Now we return to the case of local F and let E be either a quadratic extension of F or F itself. We consider the twisted endoscopy problem associated to $(L, \theta, \mathbf{1})$, where

$L := \mathrm{Res}_{E/F} GL(n)$, θ is the automorphism of L defined by

$$\theta(g) := \mathrm{Ad}\left(\begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n-1} & & & \end{pmatrix}\right)(\tilde{\sigma}({}^t g^{-1})).$$

Here $\tilde{\sigma}$ is the F-automorphism of L attached to the generator σ of $\mathrm{Gal}(E/F)$. The group G defined by

$$G := \begin{cases} \text{quasisplit } U(n) & \text{if } E \neq F, \\ Sp(m) & \text{if } E = F \text{ and } n = 2m+1, \\ SO(m,m) & \text{if } E = F \text{ and } n = 2m \end{cases}$$

is the "principal" endoscopic group of $(L, \theta, \mathbf{1})$.

For any $\delta \in L$, we write $L_{\delta,\theta}$ for the group of fixed points in L under $\mathrm{Ad}(\delta) \circ \theta$. $\delta \in L$ is said to be *θ-semisimple* if $\mathrm{Ad}(\delta) \circ \theta$ induces a semisimple automorphism of L_{der}. A θ-semisimple $\delta \in L$ is *θ-regular* if the identity component $L^0_{\delta,\theta}$ of $L_{\delta,\theta}$ is a torus, and *strongly θ-regular* if $L_{\delta,\theta}$ is abelian. We write $L_{\theta,\mathrm{sr}}(F)$ for the set of strongly θ-regular elements in $L(F)$. At each $\delta \in L_{\theta,\mathrm{sr}}(F)$ we define the *θ-orbital integral* by

$$O_{\delta,\theta}(f) := \int_{L_{\delta,\theta}(F)\backslash L(F)} f(g^{-1}\delta\theta(g))\,\frac{dg}{dt}.$$

Two strongly θ-regular δ, $\delta' \in L(F)$ is *stably θ-conjugate* if they are θ-conjugate in $L(\overline{F})$. We define the *stable θ-orbital integral* at $\delta \in L_{\theta,\mathrm{sr}}(F)$ by

$$SO_{\delta,\theta}(f) := \sum_{\substack{\delta' \text{ stably } \theta\text{-conj. to } \delta \\ \text{mod. } \theta\text{-conj.}}} O_{\delta,\theta}(f).$$

In [KS] I, Kottwitz and Shelstad constructed the norm map, which we denote by $\mathrm{N}_{L/G}$, from the set of stable θ-conjugacy classes in $L_{\theta,\mathrm{sr}}(F)$ to that of strongly regular stable conjugacy classes in $G(F)$. Also they defined a function $\Delta_{L/G}(\gamma,\delta)$ on $G_{\mathrm{sr}}(F) \times L_{\theta,\mathrm{sr}}(F)$ called the transfer factor. Of course their construction applies to the most general setting. In our case, we know that

$$\Delta_{L/G}(\gamma,\delta) = \begin{cases} 1 & \text{if } \gamma \in \mathrm{N}_{L/G}(\delta), \\ 0 & \text{otherwise.} \end{cases}$$

To define the endoscopic lifting, we need the following conjecture.

Conjecture 3.1 (Transfer conjecture). *For $f \in C_c^\infty(L(F))$, there exists $f^G \in C_c^\infty(G(F))$ such that*

$$SO_\gamma(f^G) = \sum_\delta \Delta_{L/G}(\gamma,\delta) O_{\delta,\theta}(f).$$

Here δ runs over the θ-conjugacy classes whose norm contains γ.

As opposed to the ordinary (i.e. $\theta = \mathrm{id}$) case, we do not have the precise notion of stable distributions in the twisted case. But we assume this in the following. We also have to postulate the existence of discrete L-packets. An irreducible admissible representation π of $G(F)$ is *square integrable* if it appears discretely in Harish-Chandra's Plancherel formula for $G(F)$. The set of isomorphism classes of such representations is denoted by $\Pi_{\mathrm{disc}}(G(F))$.

Conjecture 3.2. *(1) $\Pi_{\mathrm{disc}}(G(F))$ is partitioned into a disjoint union of finite sets of representations Π_φ called (discrete) L-packets:*

$$\Pi_{\mathrm{disc}}(G(F)) = \coprod_{\varphi \in \Phi_{\mathrm{disc}}(G(F))} \Pi_\varphi.$$

(2) There exists a function $\delta(1, \bullet) : \Pi_\varphi \to \mathbb{C}^\times$ such that

$$\Theta_\varphi := \sum_{\pi \in \Pi_\varphi} \delta(1,\pi)\Theta_\pi$$

is a stable distribution.

An irreducible admissible representation of G is tempered if it contributes non-trivially to the Plancherel formula. Let $P = MU$ be a F-parabolic subgroup of G and $\tau \in \Pi_{\mathrm{disc}}(M(F))$. Then the induced representation $\mathrm{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}]$ is a direct sum of irreducible tempered representations of $G(F)$:

$$\mathrm{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}] \simeq \bigoplus_{i=1}^{\ell_\tau} \pi_i(\tau).$$

Moreover, any irreducible tempered representation of $G(F)$ is obtained in this way for some (M, τ) unique up to $G(F)$-conjugation. Regarding this, we define a *tempered L-packet* by

$$\Pi_\varphi := \coprod_{\tau \in \Pi_\varphi^M} \{\pi_i(\tau) \,|\, 1 \leq i \leq \ell_\tau\},$$

where Π_φ^M is a discrete L-packet of M. By putting $\delta(1, \pi_i(\tau)) := \delta(1,\tau)$, Conj.3.2 with $\Pi_{\mathrm{disc}}(G(F))$ replaced by the set $\Pi_{\mathrm{temp}}(G(F))$ of the isomorphism classes of irreducible tempered representations of $G(F)$ follows.

Finally we say that an irreducible admissible θ-stable representation π of $L(F)$ is *θ-discrete* if it is tempered and is not induced from a θ-stable tempered representation of a proper Levi subgroup. Note that each θ-discrete representation of $L(F)$ is generic.

Now we can define the twisted endoscopic lifting which we need.

Conjecture 3.3. *There should be a bijection ξ from the set $\Phi_{\mathrm{disc}}(G(F))$ of tempered L-packets of $G(F)$ to the set $\Pi_{\theta,\mathrm{disc}}(L(F))$ of isomorphism classes of θ-stable irreducible θ-discrete representations of $L(F)$, which should be characterized by*

$$\Theta_{\xi(\Pi),\theta}(f) = \Theta_\Pi(f^G),$$

for any $f \in C_c^\infty(L(F))$ and $f^G \in C_c^\infty(G(F))$ as in Conj.3.1.

3.2. **TE implies GPC.** Now we prove the following.

Theorem 3.4. *Suppose the Conj.3.3. Then the generic packet conjecture is valid for G.*

Write $\mathfrak{l} := \mathrm{Lie}\, L$. For $h \in C_c^\infty(\mathfrak{l}(F))$ and $t \in F^\times$, we put $h_t(X) := h(t^{-1}X)$, $(X \in \mathfrak{l}(F))$. We assume that the support of $f \in C_c^\infty(L(F))$ is sufficiently small so that there exists a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{l}(F)$, on which the exponential map is defined and injective, satisfying $\mathrm{supp} f \subset \exp(\mathcal{V})$. Then we can consider $f \circ \exp \in C_c^\infty(\mathfrak{l}(F))$. Taking t sufficiently small, we may define $f_t \in C_c^\infty(L(F))$ by $f_t \circ \exp := (f \circ \exp)_t$. Further we might take f and $\mathcal{V}$ so that the transferred function f^G satisfies the same condition. We define f_t^G in the same fashion. As in [Sh, Lem.9.7], one can prove:

Lemma 3.5. *Let $f \in C_c^\infty(L(F))$ and $f^G \in C_c^\infty(G(F))$ be as in Conj.3.1. Suppose that* $\mathrm{supp} f$ *is so small that we can define f_t and f_t^G for sufficiently small t. Then we have*

$$SO_\gamma(f_{t^2}^G) = \sum_\delta \Delta_{L/G}(\gamma, \delta) O_{\delta,\theta}(f_{t^2}),$$

for $t \in F^\times$ small enough.

Prove the theorem. Since $\mathrm{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}]$ is generic if τ is so, we are reduced to the case of a discrete L-packet Π. Then by Conj.3.3, we have

$$\Theta_{\xi(\Pi),\theta}(f) = \sum_{\pi\in\Pi} \delta(1,\pi)\Theta_\pi(f^G).$$

Suppose that $\mathrm{supp} f$ is sufficiently small. Then applying the asymptotic expansion (Th.2.3) to the both side, we have

$$\sum_{\mathfrak{O}\in\mathcal{N}(\mathfrak{l}^\theta(F))} c_{\mathfrak{O},\theta}(\xi(\Pi))\widehat{\mu}_{\mathfrak{O}}(f^\theta \circ \exp) = \sum_{\mathfrak{o}\in\mathcal{N}(\mathfrak{g}(F))} \sum_{\pi\in\Pi} c_{\mathfrak{o}}(\pi)\, \widehat{\mu}_{\mathfrak{o}}(f^G \circ \exp).$$

Here $f^\theta \in C_c^\infty(G^\theta(F))$ is the descent of f.

Let $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}(F))$ and $N \in \mathfrak{o}$. We say that $\mathfrak{o}$ is *r-regular* if the variety $\mathfrak{B}_N$ of Borel subalgebras of $\mathfrak{g}$ containing N is r-dimensional. It is a result of Harish-Chandra that

$$\widehat{\mu}_{\mathfrak{o}}(f_{t^2}^G \circ \exp) = |t|_F^{2r}\widehat{\mu}_{\mathfrak{o}}(f^G \circ \exp)$$

for an r-regular $\mathfrak{o}$. The same is true for $\mathfrak{l}^\theta$.

Now recall that $\xi(\Pi)$ is generic. That is, for any 0-regular nilpotent N and ϕ as in §2, we have $\mathcal{W}_{N,\phi}(\xi(\Pi)) \neq 0$. Noting the uniqueness of the Whittaker model, we deduce that $\mathrm{tr}(\xi(\Pi)(\theta)|\mathcal{W}_{N,\phi}(\xi(\Pi))) = 1$ and hence $c_{\mathfrak{O},\theta}(\xi(\Pi)) = 1$ for any regular $\mathfrak{O}$. Thus in the equality

$$\sum_{\mathfrak{O}\in\mathcal{N}(\mathfrak{l}^\theta(F))} c_{\mathfrak{O},\theta}(\xi(\Pi))\widehat{\mu}_{\mathfrak{O}}(f_{t^2}^\theta \circ \exp) = \sum_{\mathfrak{o}\in\mathcal{N}(\mathfrak{g}(F))} \sum_{\pi\in\Pi} c_{\mathfrak{o}}(\pi)\, \widehat{\mu}_{\mathfrak{o}}(f_{t^2}^G \circ \exp),$$

the terms of order 0 in $|t|_F$ on the left hand side is not zero. Thus $c_{\mathfrak{o}}(\pi)$ is not zero at least one regular $\mathfrak{o}$. This combined with Th.2.1 implies the genericity of Π.

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