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On a theta integral

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1 Introduction.

Around the beginning of 70’s, Doi-Naganuma and Shimura discovered correspondences between certain spaces of modular forms, being compatible with the Hecke operators[Sh,DN]. Shimura’s correspondence was investigated by Shintani and Niwa, using the Weil representations[Sn,Nw]. The construction of holomorphic cusp forms by Niwa was generalized by many authors [Kd,Za,Od,RS,Kj], namely Oda and Rallis-Shiffman independently considered the case of orthogonal groups $O(2, N)$ for general N.(Zagier constructed the holomorphic kernel function for Doi-Naganuma’s correspondence.)

Recently, Borcherds discovered a family of automorphic forms with infinite product on orthogonal groups of signature $(2, N)$. At the same time, Physicists [AFGNT, HM] started to study and developed a variant of theta correspondence from another direction. Especially, Harvey and Moore found certain theta integral express the automorphic form of Borcherds[HM,Bo2].

The purpose of this note is to give a construction of meromorphic automorphic forms on orthogonal groups of the signature $(2, N), N \geq 2$, using the same kind of theta correspondence , (in section 4,) without proofs. This generalizes a result of Antoniadis, Ferrara, Gava, Narain and Taylor [AFGNT], which dealt with the case $N = 1, 2$.

We note that, in the classical (positive weight) case, Maass’ lifting [Ma,Gr,Su] and theta correspondence are known to be coincide up to non-zero scalar multiplication , (at least $S$ is maximal even, as far as I know.) In the negative weight case, the situation is different.

The following facts for $SL_2(\mathbb{R})$ are well known ($k \in \mathbb{Z}, k \geq 2, D_\tau = \frac{\partial}{\partial \tau}$, and $\mathcal{H}_1$ is the upper half plane):

(1.1) for $f \in C^\infty(\mathcal{H}_1)$ and $g \in SL_2(\mathbb{R}),$

$$D^{k-1}_\tau(f|_{2-k}g) = (D^{k-1}_\tau f)|_{kg}.$$

(1.2) A holomorphic function $f$ on $\mathcal{H}_1$ satisfies $D^{k-1}_\tau f = 0$ if and only if $f$ is a polynomial in $\tau$ of degree at most $k - 2$. 
For a holomorphic function $F$ on $\mathcal{H}$, set

$$f = \int_{\tau_0}^{\tau} F(\tau') \frac{(\tau - \tau')^{k-2}}{(k-2)!} d\tau'.$$

Then it satisfies $D_{\tau}^{k-1} f = F$.

In section 3, we will find analogous statements in the case of orthogonal groups $O(2, N)$. Then, by integrating the form constructed in section 4, one can obtain a function, which behaves like an automorphic form of negative weight, but is multi-valued, and analytic function with logarithmic singularities. This form is obtained from Maass' lifting. The same construction for $N = 2$ already appears in [AFGNT]. See also [FS]. Note that [HM] and [Kw] also consider the case $N > 2$, but take slightly different construction.

2  Basic definitions.

Let $N \geq 2$, and $S$ be an even integral symmetric matrix of degree $N + 2$ with signature $(2, N)$, and of the following form:

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (1)

The real orthogonal group $O(S)$ of $S$ acts on the domain $H^* = \{ z \in \mathbb{C}^N \mid \eta(z) := 2^t Imz S_0 Imz > 0 \}$ as follows: If we set

$$p(z) = {}^t (-^t z S_0 z / 2 \quad {}^t z \quad 1) \quad (z \in H^*),$$

then for any $g \in O(S)$ and $z \in H^*$, there exists unique $gz \in H^*$ and $\mu(g, z) \in \mathbb{C}^*$ satisfying

$$g \; p(z) = p(gz) \mu(g, z).$$

Take one of the two connected components $H \subset H^*$, and we denote by $O(S)^+$ the set of the elements in $O(S)$, fixing $H$. Then $O(S)^+$ holomorphically acts on $H$. The action is transitive and the stabilizer of a point is isomorphic to $SO(2) \times O(N)$.

Write $D_z = {}^t \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N} \right)$ for $z = {}^t (z_1, \ldots, z_N) \in H$. Following Shimura[Sh1], we define

$$q(z) = {}^t (\eta(z) D_z (\eta(z)^{-1} {}^t p(z))). \quad (z \in H)$$

(3)
Then for any $g \in O(S)^{+}$ and $z \in H$, there exists $\lambda(g, z) \in O(S_0)_{\mathbb{C}}$ which satisfies the following:

$$
\begin{align*}
&g \ q(z) = q(gz) \lambda(g, z), \\
&D_z(f \circ g) = {}^t \lambda(g, z)((D_z f) \circ g) \mu(g, z)^{-1} \\
&(f \in C^\infty(H), g \in O(S)^{+}, z \in H),
\end{align*}
$$

(4)

and we find that, $\mu(g, z)$ and $\lambda(g, z)$ are the holomorphic automorphic factor.

3 Differential calculus.

Set $V = \mathbb{C}^N$ and $V_0 = \{a \in \mathbb{C}^N | {}^t a S_0 a = 0\}$. The symmetric algebra $S(V) = \bigoplus_{l=0}^{\infty} S(V)_l$ of $V$ possess a bilinear form $\langle, \rangle$ satisfying $\langle a^l, b^l \rangle = ({}^t ab)^l$. We denote by $H(V)_l$ the subspace in $S(V)_l$ generated by $a^l$ ($a \in V_0$) over $\mathbb{C}$. Then the representation of $O(S_0)_{\mathbb{C}}$ on $H(V)_l$ is irreducible, equivalent to the one on the space of harmonic polynomial of degree $l$, under the isomorphism between $S(V)$ and the ring of polynomial map on $V$. The latter is identified with the symmetric algebra $S(V^*)$ of the dual space of $V$. We have an isomorphism

$$
H(V)_l = S(V)_l/Q \ S(V)_{l-2} \ (l \geq 0),
$$

where $Q$ is the element in $S(V)_2$ corresponding to $^t x S_0 x \in S(V^*)_2$. We denote by $\pi_l$ the projection of $S(V)_l$ on $H(V)_l$.

Proposition 3.1. Assume $r > 0$.

(1) For $f \in C^\infty(H)$, $g \in O(S)^{+}$ and $a \in V_0$,

$$
\begin{align*}
&\langle a^r, D_z^r(\mu(g, z)^r-1 f(gz)) \rangle = \mu(g, z)^{-1} \langle \lambda(g, z) a^r, (D_z^r f)(gz) \rangle \\
&(f \in C^\infty(H), g \in O(S)^{+}, z \in H),
\end{align*}
$$

(6)

(2) A holomorphic function $f \in C^\infty(H)$ satisfies $^t a D_z^r f(z) = 0$ for all $a \in V_0$, if and only if $f$ is a polynomial in $z_1, \cdots, z_N$, $^t z S_0 z$ of degree at most $r - 1$ (as a polynomial in $N + 1$ variable.)

(3) Suppose $\Phi : H \to Hom_\mathbb{C}(H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l, \mathbb{C})$ is a holomorphic function satisfying the following condition:

$$
{}^t b D_z \Phi(z, h) = \Phi(z, bh) \quad (b \in V_0, h \in H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l).
$$

(7)
Then there exists a holomorphic function $f$ on $H$, such that
\[
(i^r aD_z)^r f(z) = \Phi(z, a^r) \quad (a \in V_0). \tag{8}
\]

In fact, if we set
\[
\omega(z, z') = \sum_{l=0}^{r-1} b_{l} \frac{(Q/2^t(z-z')S_0(z-z')/2)^l}{l!} \pi_{r-l} \left( \frac{(z-z')^{r-l-1}}{(r-l-1)!} dz' \right), \tag{9}
\]
\[
b_{l} = \prod_{j=1}^{l} (r-j+N/2-1)^{-1} \quad (r > l \geq 0) \tag{10}
\]
then $\omega$ satisfies $d\omega = -dz' \wedge \omega$, $\Phi(z', \omega)$ is a closed 1-form, and the function
\[
f(z) = \int_{z_0}^{z} \Phi(z', \omega(z, z')). \tag{11}
\]
hold the required properties.

4 The construction of meromorphic automorphic forms.

Hereafter, we assume that $S$ is unimodular, for simplicity (so it follows $N \equiv 2 \pmod{8}$). Set $M = \mathbb{Z}^{N+2} = \begin{pmatrix} \mathbb{Z} & L & \mathbb{Z} \end{pmatrix}$, and $O(M)^+ = O(S)^+ \cap GL_{N+2}(\mathbb{Z})$.

Define the Siegel theta function by
\[
\theta_M(\tau, z, a^r) = \sum_{\lambda \in M} \left( i^t \lambda \frac{S_0(z)}{\eta(z)} \right)^{tr} e \left( \frac{i^r \lambda S \lambda/2 + 2iy| \frac{i^r \lambda S p(z)}{\eta(z)} |^2}{\eta(z)} \right) \tag{12}
\]
for $\tau = x + iy \in \mathcal{H}_1$, $z \in H$, and $a \in V_0$ ($r > 0$). Here, we denote by $\mathcal{H}_1$ the upper half plane.

Let $C(\tau)$ be a modular form of weight $k = 2 - r - N/2$ for $SL_2(\mathbb{Z})$, holomorphic on $\mathcal{H}_1$, and meromorphic at the cusp $i\infty$. The Fourier expansion at $i\infty$ is given by
\[
C(\tau) = \sum_{n \in \mathbb{Z}, n \geq -N_0} c(n)e(n\tau)
\]
for some \( c(n) \in \mathbb{C} \), and \( N_0 \geq 0 \). We set \( F = \{ \tau \in \mathcal{H}_1 \mid |\tau| \geq 1, |Re(\tau)| \leq 1/2 \} \), and \( F_w = \{ \tau \in F \mid Im(\tau) \geq w \} \) \((w \geq 1)\).

Further we assume that \( r > 0 \) is odd, and \( c(0) = 0 \) in case \( r = 1 \).

**Theorem 4.1.**

1. The integral

\[
\Phi(z, a^r) = \lim_{w \to \infty} \int_{F_w} \frac{dx \, dy}{y^2} y^2 C(\tau) \overline{\theta_M(\tau, z, \bar{a}^r)} \quad (z \in H, a \in V_0)
\]

converges outside the quadratic divisors, and defines a meromorphic function on \( H \), satisfying

\[
\Phi(z, a^r) = \mu(g, z)^{-1} \Phi(gz, \lambda(g, z)a^r)
\]

for all \( g \in O(M)^+ \).

2. For any compactly supported open set \( U \) in \( H \), the singularities of \( \Phi \) on \( U \) are given by the finite sum

\[
\frac{1}{4\pi} \sum_{\lambda \in M, \lambda S\lambda < 0, U \cap H_{\lambda} \neq \emptyset} c(\lambda S\lambda/2) \sum_{7SW > 0} n^{-r} \mathrm{e}(n^t \lambda S_0 z)
\]

where \( H_{\lambda} = \{ z \in H \mid \lambda S\lambda = 0 \} \). (Precise meaning of the word "singularity" is that, the difference of two functions is extended to the \( C^\infty \) function on \( U \).) Note that the inner expression of the sum can be rewritten as

\[
(\lambda S\lambda - 1) \left\{ \frac{2\pi i}{(r - 1)!} \log(\lambda S\lambda) \right\}.
\]

3. Set \( C = H \cap \mathbb{R}^N \), and take a connected component \( W_L \) of

\[
C - \bigcup_{\mu \in L, \mu S_0 \mu < 0, \mu S_0 \mu /2 \neq 0} \{ v \in C \mid \mu S_0 v = 0 \}.
\]

The Fourier expansion of \( \Phi \) is given by

\[
\Phi = -i \left( \frac{\lambda S\lambda}{2\pi i} \right)^r \left\{ h(z) + \sum_{\mu \in L, \mu S_0 \mu < 0, \mu S_0 \mu /2 \neq 0} c(\mu S_0 \mu /2) \sum_{n > 0} n^{-r} \mathrm{e}(n^t \mu S_0 z) \right\}
\]

for sufficiently large \( \eta(z) > 0 \), \( Imz \in W_L \), and a harmonic polynomial \( h(z) \) of degree \( r \).
Example 4.2. [E,AFGNT]: Let $N = 2, S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r = 3, k = 2 - \frac{3 - 2}{2} = -2, C(\tau) = E_{10}(\tau)/\Delta(\tau)$. Then

$$\Phi \left( \left( \begin{array}{c} \sigma \\ \tau \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^3 \right) = i \frac{j'(\sigma)}{j(\sigma) - j(\tau)} \frac{C(\tau)}{C(\sigma)}. \quad (17)$$

References


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