On a theta integral

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1 Introduction.

Around the beginning of 70's, Doi-Naganuma and Shimura discovered correspondences between certain spaces of modular forms, being compatible with the Hecke operators[Sh,DN]. Shimura's correspondence was investigated by Shintani and Niwa, using the Weil representations[Sn,Nw].

The construction of holomorphic cusp forms by Niwa was generalized by many authors[Kd,Za,Od,RS,Kj], namely Oda and Rallis-Shiffman independently considered the case of orthogonal groups $O(2, N)$ for general $N$. (Zagier constructed the holomorphic kernel function for Doi-Naganuma's correspondence.)

Recently, Borcherds discovered a family of automorphic forms with infinite product on orthogonal groups of signature $(2, N)$. At the same time, Physicists [AFGNT, HM] started to study and developed a variant of theta correspondence from another direction. Especially, Harvey and Moore found certain theta integral express the automorphic form of Borcherds[HM, Bo2].

The purpose of this note is to give a construction of meromorphic automorphic forms on orthogonal groups of the signature $(2, N), N \geq 2$, using the same kind of theta correspondence (in section 4) without proofs. This generalizes a result of Antoniadis, Ferrara, Gava, Narain and Taylor [AFGNT], which dealt with the case $N = 1, 2$.

We note that, in the classical (positive weight) case, Maass' lifting [Ma, Gr, Su] and theta correspondence are known to be coincide up to non-zero scalar multiplication (at least $S$ is maximal even, as far as I know.) In the negative weight case, the situation is different.

The following facts for $SL_2(\mathbb{R})$ are well known ($k \in \mathbb{Z}, k \geq 2$, $D_\tau = \frac{\partial}{\partial \tau}$, and $\mathcal{H}_1$ is the upper half plane):

(1.1) For $f \in C^\infty(\mathcal{H}_1)$ and $g \in SL_2(\mathbb{R})$,

$$D_\tau^{k-1}(f|_{2-k}g) = (D_\tau^{k-1}f)|_{kg}.$$  

(1.2) A holomorphic function $f$ on $\mathcal{H}_1$ satisfies $D_\tau^{k-1}f = 0$ if and only if $f$ is a polynomial in $\tau$ of degree at most $k - 2$. 
(1.3) For a holomorphic function $F$ on $\mathcal{H}_1$, set

$$f = \int_{\tau_0}^{\tau} F(\tau') \frac{(\tau - \tau')^{k-2}}{(k-2)!} d\tau'.$$

Then it satisfies $D_{\tau}^{k-1} f = F$.

In section 3, we will find analogous statements in the case of orthogonal groups $O(2, N)$. Then, by integrating the form constructed in section 4, one can obtain a function, which behaves like an automorphic form of negative weight, but is multi-valued, and analytic function with logarithmic singularities. This form is obtained from Maass’ lifting. The same construction for $N = 2$ already appears in [AFGNT]. (See also [FS]). Note that [HM] and [Kw] also consider the case $N > 2$, but take slightly different construction.

2 Basic definitions.

Let $N \geq 2$, and $S$ be an even integral symmetric matrix of degree $N + 2$ with signature $(2, N)$, and of the following form:

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1)$$

The real orthogonal group $O(S)$ of $S$ acts on the domain $H^* = \{z \in \mathbb{C}^N | \eta(z) := 2^t Imz S_0 Imz > 0\}$ as follows: If we set

$$p(z) = \begin{pmatrix} -^t z S_0 z/2 & ^t z & 1 \end{pmatrix} \quad (z \in H^*), \quad (2)$$

then for any $g \in O(S)$ and $z \in H^*$, there exists unique $gz \in H^*$ and $\mu(g, z) \in \mathbb{C}^*$ satisfying

$$g \ p(z) = p(gz) \mu(g, z).$$

Take one of the two connected components $H \subset H^*$, and we denote by $O(S)^+$ the set of the elements in $O(S)$, fixing $H$. Then $O(S)^+$ holomorphically acts on $H$. The action is transitive and the stabilizer of a point is isomorphic to $SO(2) \times O(N)$.

Write $D_z = \begin{pmatrix} \frac{\partial}{\partial z_1} & \cdots & \frac{\partial}{\partial z_N} \end{pmatrix}$ for $z = \begin{pmatrix} z_1 & \cdots & z_N \end{pmatrix} \in H$. Following Shimura[Sh1], we define

$$q(z) = \begin{pmatrix} \eta(z) D_z (\eta(z)^{-1} \begin{pmatrix} p(z) \end{pmatrix}) \end{pmatrix} \quad (z \in H) \quad (3)$$
Then for any $g \in O(S)^+$ and $z \in H$, there exists $\lambda(g, z) \in O(S_0)_{\mathbb{C}}$ which satisfies the following:

\[ g \cdot q(z) = q(gz) \lambda(g, z), \]

\[ D_z(f \circ g) = ^t \lambda(g, z)(D_z f) \circ g) \mu(g, z)^{-1} \]

\[ (f \in C^\infty(H), g \in O(S)^+, z \in H), \]

and we find that, $\mu(g, z)$ and $\lambda(g, z)$ are the holomorphic automorphic factor.

### 3 Differential calculus.

Set $V = \mathbb{C}^N$ and $V_0 = \{a \in \mathbb{C}^N \mid ^t a S_0 a = 0\}$. The symmetric algebra $S(V) = \bigoplus_{l=0}^\infty S(V)_l$ of $V$ possess a bilinear form $\langle , \rangle$ satisfying $\langle a^l, b^l \rangle = (^t ab)^l$. We denote by $H(V)_l$ the subspace in $S(V)_l$ generated by $a^l$ ($a \in V_0$) over $\mathbb{C}$. Then the representation of $O(S_0)_{\mathbb{C}}$ on $H(V)_l$ is irreducible, equivalent to the one on the space of harmonic polynomial of degree $l$, under the isomorphism between $S(V)$ and the ring of polynomial map on $V$. The latter is identified with the symmetric algebra $S(V^*)$ of the dual space of $V$.

We have an isomorphism

\[ H(V)_l = S(V)_l / Q S(V)_{l-2} \quad (l \geq 0), \]

where $Q$ is the element in $S(V)_2$ corresponding to $^t S_0 x \in S(V^*)_2$. We denote by $\pi_l$ the projection of $S(V)_l$ on $H(V)_l$.

**Proposition 3.1.** Assume $r > 0$.

(1) For $f \in C^\infty(H)$, $g \in O(S)^+$ and $a \in V_0$,

\[ \langle a^r, D_z^r (\mu(g, z)^{-1} f(gz)) \rangle = \mu(g, z)^{-1} \langle \lambda(g, z) a^r, (D_z^r f)(gz) \rangle \]

(2) A holomorphic function $f \in C^\infty(H)$ satisfies $(^t a D_z)^r f(z) = 0$ for all $a \in V_0$, if and only if $f$ is a polynomial in $z_1, \cdots, z_N$, $^t z S_0 z$ of degree at most $r - 1$ (as a polynomial in $N + 1$ variable.)

(3) Suppose $\Phi : H \to Hom_{\mathbb{C}}(H(V)_r \bigoplus_{l=0}^r S(V)_l, \mathbb{C})$ is a holomorphic function satisfying the following condition:

\[ ^t b D_z \Phi(z, h) = \Phi(z, bh) \quad (b \in V_0, h \in H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l). \]
Then there exists a holomorphic function $f$ on $H$, such that

$$(t^*aD_z)^r f(z) = \Phi(z, a^r) \quad (a \in V_0).$$

In fact, if we set

$$\omega(z, z') = \sum_{l=0}^{r-1} b_l \frac{(Q/2)^{t}(z-z')S_0(z-z')/2)^l}{l!} \pi_{r-l} (\frac{(z-z')^{r-l-1}}{(r-l-1)!} dz'),$$

$$b_l = \prod_{j=1}^{l} (r-j+N/2-1)^{-1} \quad (r > l \geq 0)$$

then $\omega$ satisfies $d\omega = -dz' \wedge \omega$, $\Phi(z', \omega)$ is a closed 1-form, and the function

$$f(z) = \int_{z_0}^{z} \Phi(z', \omega(z, z')).$$

hold the required properties.

4 The construction of meromorphic automorphic forms.

Hereafter, we assume that $S$ is unimodular, for simplicity (so it follows $N \equiv 2 \text{ (mod 8)}$). Set $M = \mathbb{Z}^{N+2} = \begin{pmatrix} \mathbb{Z} & L \end{pmatrix}$, and $O(M)^+ = O(S)^+ \cap GL_{N+2}(\mathbb{Z})$.

Define the Siegel theta function by

$$\theta_M(\tau, z, a^r) = \sum_{\lambda \in M} i^{r} \lambda S \frac{p(z)}{\eta(z)} \frac{t^* \lambda S q(z) a^r e \left( \overline{\tau} \frac{t^* \lambda S \lambda}{2} + 2iy \frac{|t^* \lambda Sp(z)|^2}{\eta(z)} \right)}$$

for $\tau = x + iy \in \mathcal{H}_1$, $z \in H$, and $a \in V_0 \ (r > 0)$. Here, we denote by $\mathcal{H}_1$ the upper half plane.

Let $C(\tau)$ be a modular form of weight $k = 2 - r - N/2$ for $SL_2(\mathbb{Z})$, holomorphic on $\mathcal{H}_1$, and meromorphic at the cusp $i\infty$. The Fourier expansion at $i\infty$ is given by

$$C(\tau) = \sum_{n \in \mathbb{Z}, n \geq -N_0} c(n) e(n\tau)$$
for some $c(n) \in C$, and $N_0 \geq 0$. We set $F = \{ \tau \in \mathcal{H}_1 \mid |\tau| \geq 1, |Re(\tau)| \leq 1/2 \}$, and $F_w = \{ \tau \in F \mid Im(\tau) \geq w \} \ (w \geq 1)$.

Further we assume that $r > 0$ is odd, and $c(0) = 0$ in case $r = 1$.

**Theorem 4.1.**

1. The integral

$$\Phi(z, a^r) = \lim_{w \to \infty} \int_{F_w} \frac{dx \, dy}{y^2} y^2 C(\tau) \overline{\theta_M(\tau, z, \overline{a}^r)} \quad (z \in H, a \in V_0) \quad (13)$$

converges outside the quadratic divisors, and defines a meromorphic function on $H$, satisfying

$$\Phi(z, a^r) = \mu(g, z)^{-1} \Phi(gz, \lambda(g, z)a^r) \quad (14)$$

for all $g \in O(M)^+$.

2. For any compactly supported open set $U$ in $H$, the singularities of $\Phi$ on $U$ are given by the finite sum

$$\frac{1}{4\pi} \sum_{\lambda \in M, \lambda S \lambda < 0, U \cap H_{\lambda} \neq \emptyset} c(\lambda S \lambda / 2) \left( ^{t}aD_{z}^{r} \lambda Sp(z) \right)^{r} / ^{t}\lambda Sp(z) \quad (15)$$

where $H_{\lambda} = \{ z \in H \mid ^{t}\lambda Sp(z) = 0 \}$. (Precise meaning of the word "singularity" is that, the difference of two functions is extended to the $C^\infty$ function on $U$.) Note that the inner expression of the sum can be rewritten as

$$( ^{t}aD_{z})^{r} \left\{ \frac{( ^{t}aD_{z}^{r} \lambda Sp(z))^{r-1}}{(r-1)!} \log( ^{t}aD_{z}^{r} \lambda Sp(z)) \right\}.$$

3. Set $C = H \cap \mathbb{R}^N$, and take a connected component $W_L$ of

$$C - \bigcup_{\mu \in L, ^{t}\mu S_0 \mu < 0, U \cap H_{\lambda} \neq \emptyset} \{ v \in C \mid ^{t}\mu S_0 v = 0 \}.$$

The Fourier expansion of $\Phi$ is given by

$$\Phi = -i \left( ^{t}aD_{z} \right)^{r} \left\{ h(z) + \sum_{\mu \in L, ^{t}\mu S_0 \mu < 0, c( ^{t}\mu S_0 / 2) \neq 0} c( ^{t}\mu S_0 \mu / 2) \sum_{n > 0} n^{-r} e( ^{t}\mu S_0 n z) \right\} \quad (16)$$

for sufficiently large $\eta(z) > 0, Imz \in W_L$, and a harmonic polynomial $h(z)$ of degree $r$. 
Example 4.2.\([E,AFGNT]\): Let \(N=2, S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r = 3, k = 2 - 3 - 2/2 = -2, C(\tau) = E_{10}(\tau)/\Delta(\tau).\) Then

\[
\Phi \left( \left( \frac{1}{\tau}, \frac{1}{\sigma} \right) \right) = i \frac{j'(\sigma)}{j(\sigma) - j(\tau)} \frac{C(\tau)}{C(\sigma)}. \tag{17}
\]

References


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