<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>校正</td>
<td>数理解析研究所講究録 1999年 1103号 39-45</td>
</tr>
<tr>
<td>作者</td>
<td>九宮浩一</td>
</tr>
<tr>
<td>出版社</td>
<td>京都大学</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63204">http://hdl.handle.net/2433/63204</a></td>
</tr>
</tbody>
</table>

On a theta integral (Automorphic Forms and $L$-Functions)

Ninomiya, Hirohito

数理解析研究所講究録 1999年 1103号 39-45

九宮浩一

京都大学

http://hdl.handle.net/2433/63204
On a theta integral

Hirohito Ninomiya (Kyushu University)

1 Introduction.

Around the beginning of 70's, Doi-Naganuma and Shimura discovered correspondences between certain spaces of modular forms, being compatible with the Hecke operators [Sh, DN]. Shimura's correspondence was investigated by Shintani and Niwa, using the Weil representations [Sn, Nw].

The construction of holomorphic cusp forms by Niwa was generalized by many authors [Kd, Za, Od, RS, Kj], namely Oda and Rallis-Shiffman independently considered the case of orthogonal groups \(O(2, N)\) for general \(N\). (Zagier constructed the holomorphic kernel function for Doi-Naganuma's correspondence.)

Recently, Borcherds discovered a family of automorphic forms with infinite product on orthogonal groups of signature \((2, N)\). At the same time, Physicists [AFGNT, HM] started to study and developed a variant of theta correspondence from another direction. Especially, Harvey and Moore found certain theta integral express the automorphic form of Borcherds [HM, Bo2].

The purpose of this note is to give a construction of meromorphic automorphic forms on orthogonal groups of the signature \((2, N)\), \(N \geq 2\), using the same kind of theta correspondence, (in section 4,) without proofs. This generalizes a result of Antoniadis, Ferrara, Gava, Narain and Taylor [AFGNT], which dealt with the case \(N = 1, 2\).

We note that, in the classical (positive weight) case, Maass' lifting [Ma, Gr, Su] and theta correspondence are known to coincide up to non-zero scalar multiplication (at least \(S\) is maximal even, as far as I know.) In the negative weight case, the situation is different.

The following facts for \(SL_2(\mathbb{R})\) are well known \((k \in \mathbb{Z}, k \geq 2, D_\tau = \frac{\partial}{\partial \tau}, \text{and} \ \mathcal{H}_1 \text{is the upper half plane}):\)

(1.1) for \(f \in C^\infty(\mathcal{H}_1)\) and \(g \in SL_2(\mathbb{R})\),

\[
D_\tau^{k-1}(f|_{2-k}g) = (D_\tau^{k-1}f)|_{kg}.
\]

(1.2) A holomorphic function \(f\) on \(\mathcal{H}_1\) satisfies \(D_\tau^{k-1}f = 0\) if and only if \(f\) is a polynomial in \(\tau\) of degree at most \(k - 2\).
(1.3) For a holomorphic function $F$ on $\mathcal{H}_1$, set
\[
f = \int_{\tau_0}^{\tau} F(\tau') \frac{(\tau - \tau')^{k-2}}{(k-2)!} d\tau'.
\]
Then it satisfies $D_{\tau}^{k-1} f = F$.

In section 3, we will find analogous statements in the case of orthogonal groups $O(2, N)$. Then, by integrating the form constructed in section 4, one can obtain a function, which behaves like an automorphic form of negative weight, but is multi-valued, and analytic function with logarithmic singularities. This form is obtained from Maass’ lifting. The same construction for $N = 2$ already appears in [AFGNT] (See also [FS]). Note that, [HM] and [Kw] also consider the case $N > 2$, but take slightly different construction.

2 Basic definitions.

Let $N \geq 2$, and $S$ be an even integral symmetric matrix of degree $N + 2$ with signature $(2, N)$, and of the following form:
\[
S = \begin{pmatrix}
0 & 0 & 1 \\
0 & S_0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\] (1)

The real orthogonal group $O(S)$ of $S$ acts on the domain $H^* = \{ z \in \mathbb{C}^N | \eta(z) := 2^t \text{Im} z S_0 \text{Im} z > 0 \}$ as follows: If we set
\[
p(z) = ^t(-^t z S_0 z/2 \quad ^t z \quad 1) \quad (z \in H^*),
\] (2)
then for any $g \in O(S)$ and $z \in H^*$, there exists unique $gz \in H^*$ and $\mu(g, z) \in \mathbb{C}^\times$ satisfying
\[
g p(z) = p(gz) \mu(g, z).
\]

Take one of the two connected components $H \subset H^*$, and we denote by $O(S)^+$ the set of the elements in $O(S)$, fixing $H$. Then $O(S)^+$ holomorphically acts on $H$. The action is transitive and the stabilizer of a point is isomorphic to $SO(2) \times O(N)$.

Write $D_z = ^t(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_N})$ for $z = ^t(z_1, \cdots, z_N) \in H$. Following Shimura[Sh1], we define
\[
q(z) = ^t(\eta(z) D_z(\eta(z)^{-1} \quad ^t p(z))). \quad (z \in H)
\] (3)
Then for any $g \in O(S)^+$ and $z \in H$, there exists $\lambda(g, z) \in O(S_0)_C$ which satisfies the following:

\begin{align*}
g \ q(z) &= q(gz)\lambda(g, z), \\ D_z(f \circ g) &= ^t \lambda(g, z)((D_z f) \circ g)\mu(g, z)^{-1}
\end{align*}

(4) \hspace{2cm} (5)

and we find that, $\mu(g, z)$ and $\lambda(g, z)$ are the holomorphic automorphic factor.

3 Differential calculus.

Set $V = \mathbb{C}^N$ and $V_0 = \{a \in \mathbb{C}^N | ^t a S_0 a = 0\}$. The symmetric algebra $S(V) = \bigoplus_{l=0}^{\infty} S(V)_l$ of $V$ possess a bilinear form $\langle , \rangle$ satisfying $\langle a^l, b^l \rangle = (\text{det} a)^l$. We denote by $H(V)_l$ the subspace in $S(V)_l$ generated by $a^l (a \in V_0)$ over $\mathbb{C}$. Then the representation of $O(S_0)_C$ on $H(V)_l$ is irreducible, equivalent to the one on the space of harmonic polynomial of degree $l$, under the isomorphism between $S(V)$ and the ring of polynomial map on $V$. The latter is identified with the symmetric algebra $S(V^*)$ of the dual space of $V$. We have an isomorphism

$$H(V)_l = S(V)_l / Q \ S(V)_{l-2} \ (l \geq 0),$$

where $Q$ is the element in $S(V)_2$ corresponding to $^t x S_0 x \in S(V^*)_2$. We denote by $\pi_l$ the projection of $S(V)_l$ on $H(V)_l$.

Proposition 3.1. Assume $r > 0$.

(1) For $f \in C^\infty(H)$, $g \in O(S)^+$ and $a \in V_0$,

$$\langle a^r, D_z^r(\mu(g, z)^{r-1}f(gz)) \rangle = \mu(g, z)^{-1} \langle \lambda(g, z)a^r, (D_z^r f)(gz) \rangle$$

(6)

(2) A holomorphic function $f \in C^\infty(H)$ satisfies $(^t a D_z)^r f(z) = 0$ for all $a \in V_0$ if and only if $f$ is a polynomial in $z_1, \ldots, z_N$, $^t z S_0 z$ of degree at most $r - 1$ (as a polynomial in $N + 1$ variable.)

(3) Suppose $\Phi : H \rightarrow Hom_C(H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l, \mathbb{C})$ is a holomorphic function satisfying the following condition:

$$^t b D_z \Phi(z, h) = \Phi(z, bh) \quad (b \in V_0, h \in H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l).$$

(7)
Then there exists a holomorphic function $f$ on $H$, such that

$$(^t aD_z)^r f(z) = \Phi(z, a^r) \quad (a \in V_0).$$

(8)

In fact, if we set

$$\omega(z, z') = \sum_{l=0}^{r-1} b_l \frac{(Q/2 \cdot (z-z')^{r-l-1}/(r-l-1)! dz')}{2^l \cdot (r-l-1)!},$$

(9)

$$b_l = \prod_{j=1}^{l} (r-j+N/2-1)^{-1} \quad (r > l \geq 0)$$

(10)

then $\omega$ satisfies $d\omega = -dz' \wedge \omega$, $\Phi(z', \omega)$ is a closed 1-form, and the function

$$f(z) = \int_{z_0}^{z} \Phi(z', \omega(z, z')).$$

(11)

hold the required properties.

4 The construction of meromorphic automorphic forms.

Hereafter, we assume that $S$ is unimodular, for simplicity (so it follows $N \equiv 2 \pmod{8}$). Set $M = \mathbb{Z}^{N+2} = \begin{pmatrix} Z & L & \mathbb{Z} \end{pmatrix}$, and $O(M)^+ = O(S)^+ \cap GL_{N+2}(\mathbb{Z})$.

Define the Siegel theta function by

$$\theta_M(\tau, z, a^r) = \sum_{\lambda \in \Lambda} \frac{1}{\eta(z)} \frac{(t\lambda S\overline{q(z)}a)^r e(\overline{\tau} t\lambda S\lambda/2 + 2iy \frac{|t\lambda S p(z)|^2}{\eta(z)})}{\eta(z)}$$

(12)

for $\tau = x + iy \in \mathcal{H}_1$, $z \in H$, and $a \in V_0 \quad (r > 0)$. Here, we denote by $\mathcal{H}_1$ the upper half plane.

Let $C(\tau)$ be a modular form of weight $k = 2 - r - N/2$ for $SL_2(\mathbb{Z})$, holomorphic on $\mathcal{H}_1$, and meromorphic at the cusp $i\infty$. The Fourier expansion at $i\infty$ is given by

$$C(\tau) = \sum_{n \in \mathbb{Z}, n \geq -N_0} c(n) e(n\tau).$$
for some $c(n) \in C$, and $N_0 \geq 0$. We set $F = \{\tau \in \mathcal{H}_1 \mid |\tau| \geq 1, |Re(\tau)| \leq 1/2\}$, and $F_w = \{\tau \in F \mid Im(\tau) \geq w\} \ (w \geq 1)$.

Further we assume that $r > 0$ is odd, and $c(0) = 0$ in case $r = 1$.

**Theorem 4.1.**

1. The integral

$$
\Phi(z, a^r) = \lim_{w \to \infty} \int_{F_w} \frac{dx \, dy}{y^2} y^2 C(\tau) \overline{\theta_M(\tau, z, \overline{a}^r)} \quad (z \in H, a \in V_0) \tag{13}
$$

converges outside the quadratic divisors, and defines a meromorphic function on $H$, satisfying

$$
\Phi(z, a^r) = \mu(g, z)^{-1} \Phi(gz, \lambda(g, z)a^r) \tag{14}
$$

for all $g \in O(M)^+$. 

2. For any compactly supported open set $U$ in $H$, the singularities of $\Phi$ on $U$ are given by the finite sum

$$
\frac{1}{4\pi} \sum_{\lambda \in M, \lambda Sp(z) < 0, U \cap H_{\lambda} \neq \emptyset} c(\lambda S \lambda/2) (t^aD_z)^r (\lambda Sp(z))^{r-1} \log(\lambda Sp(z)) \tag{15}
$$

where $H_{\lambda} = \{z \in H \mid \lambda Sp(z) = 0\}$. (Precise meaning of the word "singularity" is that, the difference of two functions is extended to the $C^\infty$ function on $U$.) Note that the inner expression of the sum can be rewritten as

$$
(t^aD_z)^r \left\{ (\lambda Sp(z))^{r-1} \frac{\log(\lambda Sp(z))}{(r-1)!} \right\}.
$$

3. Set $C = H \cap \mathbb{R}^N$, and take a connected component $W_L$ of

$$
C - \bigcup_{\mu \in L, \mu S_0 \mu < 0, \mu S_0 v = 0} \{v \in C \mid \mu S_0 v = 0\}.
$$

The Fourier expansion of $\Phi$ is given by

$$
\Phi = -i \left(\frac{t^aD_z}{2\pi i}\right)^r \left\{ h(z) + \sum_{\mu \in L, \mu S_0 \mu / 2 > 0} c(\mu S_0 \mu / 2) \sum_{n > 0} n^{-r} e(n^a \mu S_0 z) \right\} \tag{16}
$$

for sufficiently large $\eta(z) > 0$, $Im z \in W_L$, and a harmonic polynomial $h(z)$ of degree $r$. 

Example 4.2. [E,AFGNT]: Let \( N = 2, S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r = 3, k = 2 - 3 - 2/2 = -2, C(\tau) = E_{10}(\tau)/\Delta(\tau). \) Then

\[
\Phi \left( \left( \begin{array}{c} \sigma \\ \tau \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^3 \right) = i \frac{j'(\sigma)}{j(\sigma) - j(\tau)} \frac{C(\tau)}{C(\sigma)}. \tag{17}
\]

References


Graduate School of Mathematics, Kyushu University 33, Fukuoka 812 Japan
e-mail address: ninomiya@math.kyushu-u.ac.jp