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Fourier expansion of holomorphic Siegel modular forms of genus 3 along the minimal parabolic subgroup

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ABSTRACT. We construct a certain type of Fourier expansion of holomorphic Siegel modular forms of genus 3, different from the two expansions already known, i.e. classical Fourier expansion and Fourier Jacobi expansion(cf. [2],[6],[9]). More precisely, it is along the minimal parabolic subgroup of $Sp(3;\mathbb{R})$, while the other two are along the Siegel parabolic subgroup or Jacobi parabolic subgroup. We already obtained such Fourier expansion for the case of genus 2(cf. [5]). In these days, we have constructed that expansion for the case of genus 3. From this work, we hope to obtain some hints to get the expansion for the case of arbitrary genus. We are also interested in relations among our expansion and the other two Fourier expansions. In the case of genus 2, we got some relations in terms of their Fourier coefficients. For the case of genus 3, we also do the same work after the construction of our Fourier expansion.

1. Notations for Lie group and Lie algebra.

Let $G = Sp(3;\mathbb{R})$ be the real symplectic group of degree 3 and K a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. The Cartan involution θ (i.e. $\theta(X) = -{}^tX$) induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{p}$ is the eigenspace of $\mathfrak{g}$ with the eigenvalue -1 and $\mathfrak{k}$ coincides with that with the eigenvalue 1.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, specified by

$$\left\{ \begin{pmatrix} A & \\ & -A \end{pmatrix} \,\middle|\, A = \mathrm{diag}(t_1, t_2, t_3),\ t_i \in \mathbb{R} \right\}.$$

Let $\{e_i\}_{1\le i\le 3}$ the standard basis of 3-dimensional Euclidean space. The set $\Delta(\mathfrak{g},\mathfrak{a}) = \{\pm e_i \pm e_j,\ \pm 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ gives the restricted root system. Let E_α denote the root vector corresponding to a root α. The set $\Delta(\mathfrak{g},\mathfrak{a})^+ = \{e_i \pm e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ forms a set of positive roots of $\Delta(\mathfrak{g},\mathfrak{a})$. Then we have the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{n} = \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})^+} \mathbb{R}E_\alpha$.

Let E_{ij} denote the ij-th matrix unit and set $\mathfrak{h} = \bigoplus_{1\le i\le 3} \mathbb{R}(E_{i,i+3} - E_{i+3,i})$, which is the Lie algebra of a compact Cartan subgroup. We think of the root space decomposition of $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$. The root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is of the same type as the restricted root system. The set $\Delta^+ = \{e_i \pm e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ give the standard positive root system, $\Delta_n^+ = \{e_i + e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le$

$k \leq 3\}$ the set of non-compact positive roots, and $\Delta_c^+ = \{e_i - e_j \mid 1 \leq i < j \leq 3\}$ the set of compact positive roots. Let $F_\alpha \in \mathfrak{g}_{\mathbb{C}}$ be the root vector corresponding to a root α, $\mathfrak{p}^+ = \bigoplus_{\alpha \in \Delta_n^+} \mathbb{C}F_\alpha$ and $\mathfrak{p}^- = \bigoplus_{\alpha \in \Delta_n^+} \mathbb{C}F_{-\alpha}$. Then, we have a following well-known decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

2. Representation of the maximal compact subgroup K

The maximal compact subgroup K is isomorphic to $U(3)$, so the complexifications of K and $\mathfrak{k}$ are isomorphic to $GL(3;\mathbb{C})$ and $\mathfrak{gl}(3;\mathbb{C})$ respectively. In terms of highest weight theory, the equivalence classes of irreducible finite dimensional representations of $GL(3;\mathbb{C})$ can be parametrized by the set of the dominant weights, which is given by

$$D(3) = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \lambda_1 \geq \lambda_2 \geq \lambda_3\}.$$

We denote by τ_λ the irreducible finite dimensional representation of $GL(3;\mathbb{C})$ with highest weight $\lambda \in D(3)$

Here, for the irreducible representation $(\tau_\lambda, V_\lambda)$ of $GL(3;\mathbb{C})$, we explicitly give the infinitesimal actions of generators of $\mathfrak{gl}(3;\mathbb{C})$ by the differential $d\tau_\lambda$ of τ_λ. For that purpose, we introduce the notion of Gel'fand Tsetlin scheme. The following argument and formulas are given in [7], §§18.11.

It can be shown that there is a basis of V_λ parametrized by the following diagrams:

$$Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_{12} & & \lambda_{22} & \\ & & \lambda_{11} & & \end{pmatrix},$$

where $(\lambda_{12}, \lambda_{22}, \lambda_{11}) \in \mathbb{Z}^{\oplus 3}$ is such that $\lambda_1 \geq \lambda_{12} \geq \lambda_2 \geq \lambda_{22} \geq \lambda_3$ and $\lambda_{12} \geq \lambda_{11} \geq \lambda_{22}$. We call these diagrams the *Gel'fand Tsetlin schemes* and the basis $\{v_Q\}$ parametrized by the diagrams $\{Q\}$ the *Gel'fand Tsetlin basis.* Using this basis, we give the explicit formulas of infinitesimal action of $\mathfrak{gl}(3,\mathbb{C})$ by $d\tau_\lambda$. The Lie algebra is generated by the ij-th matrix units E_{ij} with $1 \leq i,j \leq 3$. First we write the formulas for $E_{j,j+1}$ and E_{jj}:

$$d\tau_\lambda(E_{j,j+1})v_Q = \sum_{i=1}^{j} a_{ij}(Q) v_{Q^+_{(ij)}}, \quad d\tau_\lambda(E_{jj})v_Q = (\sum_{i=1}^{j} \lambda_{ij} - \sum_{i=1}^{j-1} \lambda_{i,j-1}) v_Q,$$

where $a_{ij}(Q) = \sqrt{\left| \frac{\prod_{k=1}^{j+1}(\lambda_{k,j+1} - \lambda_{ij} - k + i) \prod_{k=1}^{j-1}(\lambda_{k,j-1} - \lambda_{ij} - k + i - 1)}{\prod_{k=1,\neq i}^{j}(\lambda_{kj} - \lambda_{ij} - k + i) \prod_{k=1,\neq i}^{j}(\lambda_{kj} - \lambda_{ij} - k + i - 1)} \right|}$, and $Q^+_{(ij)}$ is the diagram with $\lambda_{ij} \to \lambda_{ij} + 1$ and $\lambda_{kl} \to \lambda_{kl}$ for $(k,l) \neq (i,j)$. Furthermore, since any E_{ij} with $i \leq j$ can be expressed by the bracket product of $E_{j,j+1}$'s, we can compute $d\tau_\lambda(E_{ij})$ from these formulas.

3. Holomorphic discrete series of $Sp(3;\mathbb{R})$.

We give the notations of holomorphic discrete series representation of $Sp(3;\mathbb{R})$. From the Harish-Chandra's characterization of discrete series representation(cf. [4], Chap.IX, §7, Chap.XII, §5), holomorphic discrete series representations of $Sp(3;\mathbb{R})$ can be parametrized by strictly dominant weights $\Lambda \in \mathbb{Z}^{\oplus 3}$ such that $\Lambda_1 > 0$, $\Lambda_2 > 0$, $\Lambda_3 > 0$ and $\Lambda_1 > \Lambda_2 > \Lambda_3$. Such Λ's are called the Harish-Chandra parameters. We denote by π_Λ the holomorphic discrete series with the parameter Λ. The highest weight of the minimal K-type of π_Λ is given by the special weight $\lambda = \Lambda + \rho - 2\rho_c$, which we call the Blattner parameter. Here we denote by ρ (resp. ρ_c) half of the sum of positive roots (resp. compact positive roots). More precisely, $\lambda = (\Lambda_1 + 1, \Lambda_2 + 2, \Lambda_3 + 3)$ if $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$. On the other hand, we will also treat the contragredient π_Λ^* of π_Λ. Its Harish-Chandra parameter (resp. Blattner parameter) is given by $(-\Lambda_3, -\Lambda_2, -\Lambda_1)$ (resp. $(-\Lambda_3 - 3, -\Lambda_2 - 2, -\Lambda_1 - 1)$).

4. Representation of the maximal unipotent subgroup

Let $N = \exp(\mathfrak{n})$, which is the standard maximal unipotent subgroup of G. Every element $x \in N$ can be written as

$$x = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x'_{12}, x'_{13}, x'_{23})$$

$$= \left(\begin{array}{ccc|ccc} 1 & & & x_1 & x_{12} & x_{13} \\ & 1 & & x_{12} & x_2 & x_{23} \\ & & 1 & x_{13} & x_{23} & x_3 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \left(\begin{array}{ccc|ccc} 1 & x'_{12} & x'_{13} & & & \\ & 1 & x'_{23} & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & -x'_{12} & 1 & \\ & & & x'_{12}x'_{23} - x'_{13} & -x'_{23} & 1 \end{array}\right),$$

where $x_{ij}, x'_{ij}, x_k \in \mathbb{R}$ for $1 \le k \le 3$ and $1 \le i < j \le 3$. Let $\mathfrak{n}^*$ be the dual space of $\mathfrak{n}$ and $\{l_k, l_{ij}, l'_{ij}\}$ with $1 \le k \le 3$ and $1 \le i < j \le 3$ the dual basis of $\mathfrak{n}^*$, where l_k, l_{ij} and l'_{ij} are dual to $E_{2e_k}, E_{e_i+e_j}$ and $E_{e_i-e_j}$ respectively. We write every linear form l as $l = \sum_{1 \le i<j \le 3}(\xi_{ij} l_{ij} + \xi'_{ij} l'_{ij}) + \sum_{1 \le k \le 3} \xi_k l_k$ with ξ_{ij}, ξ'_{ij}, $\xi_k \in \mathbb{R}$

We denote by Ad^* the coadjoint actions of N on $\mathfrak{n}^*$. Using the Kirillov theory on the unitary representations of nilpotent Lie group (cf. [1], Chap.2), we have

Proposition 4.1. (1) *Any $\eta \in \hat{N}$ is of the form:*

$$\eta_l = \mathrm{L}^2\text{-}\mathrm{Ind}_{M_l}^N \chi_l$$

with some $l \in \mathfrak{n}^*$, where $M_l = \exp(\mathfrak{M}_l)$ with $\mathfrak{M}_l$ a polarization subalgebra for l, and χ_l is the character on M_l defined by

$$\chi_l(m) = \exp(2\pi\sqrt{-1}l(\log(m))) \quad m \in M_l.$$

(2) *Two representations η_l and $\eta_{l'}$ are equivalent if and only if $l' = \mathrm{Ad}^*(n) \cdot l$ with some $n \in N$. In other word, we have a bijection:*

$$\hat{N} \simeq \mathfrak{n}^*/\mathrm{Ad}^*(N).$$

5. Generalized Whittaker function for holomorphic discrete series.

5.1. Definition. In this subsection, we recall the definition of generalized Whittaker functions for holomorphic discrete series and give their explicit formulas. First, we recall the definition. For that purpose, we introduce the following two spaces associated to fixed $(\tau, V_\tau) \in \hat{K}$ and $(\eta, H_\eta) \in \hat{N}$:

$$\begin{aligned} C_\eta^\infty(N\backslash G) :=&\{f : \text{smooth } H_\eta^\infty\text{-valued function on } G \\ &\mid f(xg) = \eta(x)f(g) \quad (x,g) \in N \times G\}, \\ C_{\eta,\tau}^\infty(N\backslash G/K) :=&\{F : \text{smooth } H_\eta^\infty \otimes V_\tau\text{-valued function on } G \\ &\mid F(xgk) = \eta(x) \otimes \tau^{-1}(k)F(g) \quad (x,g,k) \in N \times G \times K\}, \end{aligned}$$

where H_η^∞ denotes the space of C^∞-vectors in H_η.

Definition 5.1. *For the holomorphic discrete series π_Λ, consider the restriction map of* $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G))$ *to the minimal K-type τ_λ of π_Λ:*

$$res_{\tau_\lambda} : \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G)) \ni F \mapsto F \cdot \iota \in \mathrm{Hom}_K(\tau_\lambda, C_\eta^\infty(N\backslash G)),$$

where ι denotes the inclusion of τ_λ into π_Λ. A generalized Whittaker function with K-type τ_λ for π_Λ is defined to be an element of images by res_{τ_λ}.

Note that there is a canonical identification:

$$\mathrm{Hom}_K(\tau_\lambda, C_\eta^\infty(N\backslash G)) \simeq C_{\eta,\tau_\lambda^*}^\infty(N\backslash G/K),$$

where τ_λ^* denotes the contragredient of τ_λ. Furthermore, from the Iwasawa decomposition of G, one obtains a bijection $C_{\eta,\tau_\lambda^*}^\infty(N\backslash G/K) \simeq C^\infty(A; V_\lambda^* \otimes H_\eta^\infty)$(the space of smooth $V_\lambda^* \otimes H_\eta^\infty$-valued functions). The space of generalized Whittaker functions for π_Λ is under the bijection with

$$\{F \in C_{\eta,\tau_\lambda^*}^\infty(N\backslash G/K) \mid dR_X \cdot F = 0 \quad \forall X \in \mathfrak{p}^+\},$$

where dR denotes the differential of the right translation R (cf. [8], Proposition 10.1). The condition characterizing this space is called the *Cauchy Riemann condition.*

5.2. Explicit formulas of the Whittaker functions. Let $\Lambda = (\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3)$ in π_Λ. Then $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ gives the Blattner parameter. And let $W(a) = \sum w_Q(a) \cdot v_Q$ be the restriction of a generalized Whittaker function for π_Λ to the radial part A, where $\{v_Q\}$ denotes the Gel'fand Tsetlin basis for $(\tau_\lambda^*, V_\lambda^*)$. Note that the highest weight of τ_λ^* is $(-\lambda_3, -\lambda_2, -\lambda_1)$. By solving the differential equations arising from the Cauchy Riemann condition, we obtain

Theorem 5.2. *(I) For every $\eta \in \hat{N}$,*

$$\dim_\mathbb{C} \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G)) \leq 1.$$

In particular, the equality holds if and only if $\eta \in \hat{N}$ is one of the following four:

(1) *unitary character* η *corresponding to* $l \in \{[\mathfrak{n},\mathfrak{n}]\}^{\perp}$ *such that* $\xi'_{12} = \xi'_{23} = 0$, *which has* $\underline{l = \xi_3 l_3}$ *as a representative of its coadjoint orbit. Here* $\{*\}^{\perp}$ *means the annihilator of* $\{*\}$ *in* $\mathfrak{n}^*$.

(2) η *corresponding to* $l \in \mathfrak{n}^*$ *such that* $\xi_2 > 0$, $\xi_1 = \xi_{13} = \xi'_{12} = \xi'_{13} = 0$, *which has* $\underline{l = \xi_2 l_2 + \xi_3 l_3 \text{ with } \xi_2 > 0}$, *as a representative of its coadjoint orbit. This representation has* $L^2(\mathbb{R})$ *as one of its representation space.*

(3) η *corresponding to* $l \in \mathfrak{n}^*$ *such that* $\xi_1 > 0$ *and* $\xi_2 = \xi_{12} = \xi_{23} = \xi'_{23} = 0$, *which has* $\underline{l = \xi_1 l_1 + \xi_3 l_3 \text{ with } \xi_1 > 0}$, *as a representative of its coadjoint orbit. This representation has* $L^2(\mathbb{R}^2)$ *as one of its representation space.*

(4) η *corresponding to* $l \in \mathfrak{n}^*$ *such that* $\xi_1 > 0$, $\begin{vmatrix} 2\xi_1 & \xi_{12} \\ \xi_{12} & 2\xi_2 \end{vmatrix} > 0$ *and* $(\xi_2, \xi_{12}, \xi_{23}) \neq (0,0,0)$, *which has* $\underline{l = \xi_1 l_1 + \xi_2 l_2 + \xi_3 l_3 \text{ with } \xi_1 > 0 \text{ and } \xi_2 > 0}$, *as a representative of its coadjoint orbit. This representation has* $L^2(\mathbb{R}^3)$ *as one of its representation space.*

Furthermore, we set $\mathcal{A}_\eta(N\backslash G) := \{f \in C^\infty_\eta(N\backslash G) \mid f|_A \text{ is of moderate growth}\}$. *Then, for any* $\eta \in \hat{N}$ *as above,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, \mathcal{A}_\eta(N\backslash G)) = 1 \Leftrightarrow \xi_3 \geq 0 \textit{ in the representatives as above.}$$

(II) For these four cases, the explicit formulas of Whittaker functions are given as follows:
(i) When η *is as in* (1),

$$w_Q(\xi_3; a) = \begin{cases} C a_1^{\lambda_3} a_2^{\lambda_2} a_3^{\lambda_1} e^{-2\pi\xi_3 a_3^2} & Q = \begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_2 \\ & & -\lambda_2 \end{pmatrix}, \\ 0 & Q\text{:otherwise.} \end{cases}$$

(ii) When η *is as in* (2),

$$\begin{aligned} & w_Q(\xi_2, \xi_3; a, t) \\ &= \begin{cases} C a_1(Q) a_1^{\lambda_3} a_2^{\lambda_1 - l} a_3^{\lambda_2 + l} t^l & \\ \times \exp\{-2\pi(a_2^2\xi_2 + a_3^2\xi_3 + a_3^2\xi_2 t^2)\} & Q = \begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1 + l \\ & & -\lambda_3 \end{pmatrix}, \\ 0 & Q\text{:otherwise,} \end{cases} \end{aligned}$$

where t *denotes the coordinate of* $\mathbb{R}$.
(iii) When η *is as in* (3),

$$w_Q(\xi_1, \xi_3; a, s, u)$$

$$= \begin{cases} Ca_2(Q)a_1^{\lambda_1-l-m}a_2^{\lambda_3+m}a_3^{\lambda_2+l}s^m u^l \times \\ \exp\{-2\pi(a_1^2\xi_1+a_3^2\xi_3+a_2^2\xi_1 s^2+a_3^2\xi_1 u^2)\} & Q=\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1+l \\ & & -\lambda_1+l+m \end{pmatrix}, \\ 0 & Q\text{:}otherwise, \end{cases}$$

where (s,u) *denotes the coordinate of* $\mathbb{R}^2$.
(*iv*) *When* η *is as in* (4),

$$w_Q(\xi_1,\xi_2,\xi_3;a,s,t,u)$$
$$= \begin{cases} Ca_2(Q)a_1^{\lambda_1-l-m}a_2^{\lambda_3+m}a_3^{\lambda_2+l}s^m u^l \times \\ \exp\{-2\pi(a_1^2\xi_1+a_2^2(\xi_1 s^2+\xi_2) \\ +a_3^2(\xi_1 u^2+\xi_2 t^2+\xi_3)\} & Q=\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1+l \\ & & -\lambda_1+l+m \end{pmatrix}, \\ 0 & Q\text{:}otherwise, \end{cases}$$

where (s,t,u) *denotes the coordinate of* $\mathbb{R}^3$.
Here C *denotes an arbitrary constant and, in* (*ii*) (*iii*) *and* (*iv*),
$a_1(Q)=(-1)^l\sqrt{\frac{\prod_{0<i<l}(\lambda_1-\lambda_2-i+1)}{l!}}$,
$a_2(Q)=(-1)^m\sqrt{\frac{(\lambda_1-\lambda_3)!\prod_{0<i<l}(\lambda_1-\lambda_2-i+1)(\lambda_1-\lambda_3-i-m+1)}{l!m!(\lambda_1-\lambda_3-m)!\prod_{0\le i\le l}(\lambda_1-\lambda_3-i+1)}}$.

6. Formulation of the Fourier expansion

Put $\Gamma=Sp(3;\mathbb{Z})$, $N_{\mathbb{Z}}=N\cap\Gamma$. Let π_Λ, τ_λ and $(\tau_\lambda^*,V_\lambda^*)$ be as in the previous argument. And let V_λ^*-valued function f be a holomorphic Siegel modular form of weight τ_λ with respect to Γ. For a fixed $g\in G$, $f(xg)$ $(x\in N)$ belongs to $L^2(N_{\mathbb{Z}}\backslash N)\otimes_{\mathbb{C}} V_\lambda^*$. Since $N_{\mathbb{Z}}\backslash N$ is compact, we have

$$L^2(N_{\mathbb{Z}}\backslash N)=\bigoplus_{\eta\in\hat{N}} m(\eta)\cdot H_\eta \simeq \bigoplus_{\eta\in\hat{N}} \mathrm{Hom}_N(\eta,L^2(N_{\mathbb{Z}}\backslash N))\otimes_{\mathbb{C}} H_\eta,$$

where $m(\eta)=\dim_{\mathbb{C}}\mathrm{Hom}_N(\eta,L^2(N_{\mathbb{Z}}\backslash N))<\infty$(cf. [3]). Let $\{\Phi_M^\eta\}_{1\le M\le m(\eta)}$ denote a basis of $\mathrm{Hom}_N(\eta,L^2(N_{\mathbb{Z}}\backslash N))$. According to this decomposition, we have

$$f(xg)=\sum_{\{Q\}}\sum_{\eta}\sum_{M=1}^{m(\eta)}(\Phi_M^\eta\otimes W_f^{(\eta,Q)}(g))(x)\otimes v_Q,$$

where $\{Q\}$ denotes the set of Gel'fand Tsetlin schemes for τ_λ^*, $\{v_Q\}$ the Gelfand Tsetlin basis for V_λ^*, and $W_f^{(\eta,Q)}(g)\in H_\eta^\infty$ with $g\in G$. Set $W_f^\eta(g):=\sum_{\{Q\}}W_f^{(\eta,Q)}(g)\cdot v_Q$. Then we observe that $W_f^\eta\in C_{\eta,\tau_\lambda^*}^\infty(N\backslash G/K)$ and that this satisfies the Cauchy

Riemann condition since f does. Hence we see that W_f^η is a generalized Whittaker function with K-type τ_λ for π_Λ, whose explicit formula is given at §5.

Consider the η-component of the decomposition as above. Let $\{h_i\}_{i\in I}$ be a complete orthogonal basis of H_η, and $W_f^{(\eta,Q)}(g) = \sum_{i\in I} c_i^{\eta,Q}(g)h_i$ the expansion of $W_f^{(\eta,Q)}$ by this basis. Then the η-component of the Fourier expansion is

$$\sum_{\{Q\}}\{\sum_{i\in I} c_i^{\eta,Q}(g)\cdot\Phi_M^\eta(h_i)(x)\}\cdot v_Q.$$

The remaining work for the construction of our Fourier expansion is to compute $c_i^{\eta,Q}$ and $\Phi_M^\eta(h_i)$ as above. The coefficient $c_i^{\eta,Q}(g)$ can be obtained by computing $\langle W_f^{(\eta,Q)}(g), h_i\rangle$ with $\langle *,*\rangle$ denoting the scalar product on H_η. Our H_η is isomorphic to $\mathbb{C}$ or $L^2(\mathbb{R}^n)$ with $n=1,2$ or 3. For η as in (2) (3) and (4) of Theorem 5.2, we take the totality of Hermite functions as the above $\{h_i\}_{i\in I}$ and the Hermite inner product as the scalar product on H_η. The explicit formula of $c_i^{\eta,Q}$ will be given in Theorem 8.1 (see also Remark 8.2). In the next section, we determine a basis of $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$ by giving the functions $\Phi_M^\eta(h_i)$ explicitly.

7. Generalized theta series.

Let $h_i(t) = e^{\frac{t^2}{2}}\frac{d^i}{dt^i}e^{-t^2}$ ($i\in\mathbb{Z}_{\geq 0}$) be the i-th Hermite function. The space $L^2(\mathbb{R}^n)$ has $\{h_{i_1}(t_1)\cdots h_{i_n}(t_n)\}_{i\geq 0,\dots,i_n\geq 0}$ as a complete orthogonal basis for it. We may consider the case $n=1,2,3$ now. Let $\eta\in\hat{N}$ be one of the four representations as in (1),(2),(3) and (4) of Theorem 5.2 (I). We find a basis of $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$ for them. It is settled by determining the images $\Phi(h_i)$ of Hermite functions (resp. $1\in\mathbb{C}$) by an intertwining operator $\Phi\in\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$, for η as in (2) (3) and (4) of Theorem 5.2 (I) (resp. the case (1)). Here we introduce the following ideal of the universal enveloping algebra $\mathfrak{u}(\mathfrak{n})$:

$$Ann(\eta) := \{X\in\mathfrak{u}(\mathfrak{n}) \mid d\eta(X)h = C_X h \quad \forall h\in H_\eta^\infty\}$$

for $\eta\in\hat{N}$, where C_X denotes a constant dependent only on X. Except for the case (1), $\Phi(h_i)$'s are characterized by

- differential equations coming from the actions $dr_N(X)$ for $X\in Ann(\eta)$,
- Hermite differential equations rewritten by the coordinate of N, via Φ.
- $N_\mathbb{Z}$-invariance.

As to the case (1), the image of $1\in\mathbb{C}$ is characterized by the differential equations arising from the infinitesimal actions of the generator of $\mathfrak{n}$ and the $N_\mathbb{Z}$-invariance. From calculating the above three conditions, we get

Proposition 7.1. (1) *When $\eta\in\hat{N}$ is as in* (1) *of Theorem* 5.2 (*I*),

$$\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N)) = \mathbb{C}\cdot\Phi_0,$$

where $\Phi_0 : \mathbb{C} \to \mathbb{C}\exp 2\pi\sqrt{-1}\xi_3 x_3$.
(2) *When* $\eta \in \hat{N}$ *is as in* (2) *of the theorem, we introduce a set*

$$\mathfrak{M}(\xi_2,\xi_3) = \{M \in \mathbb{Z} \mid \frac{M^2}{4\xi_1} + \xi_2 \in \mathbb{Z}\}/\sim,$$

where $M \sim M' \leftrightarrow M \equiv M' \mod 2\xi_1$. *For a* $M \in \mathfrak{M}(\xi_2,\xi_3)$, *we define* $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ *by*

$$\phi^i_{\xi_2,\xi_3}(M;x) := \Phi^\eta_M(h_i(t))(x) = \sum_{m\in\mathbb{Z}} h_i(x'_{23} + \frac{2\xi_2 m + M}{2\xi_2})$$
$$\times \exp 2\pi\sqrt{-1}(\xi_2 x_2 + \frac{(2\xi_2 m + M)^2 + 4\xi_2\xi_3}{4\xi_2}x_3 + (2\xi_2 m + M)x_{23}).$$

The set $\{\Phi^\eta_M\}_{M\in\mathfrak{M}(\xi_2,\xi_3)}$ *gives a basis of* $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$.
(3) *When* $\eta \in \hat{N}$ *is as in* (3) *of the theorem, we introduce a set*

$$\mathfrak{M}(\xi_1,\xi_3) = \{M = (M_{12}, M_{13}) \in \mathbb{Z}^2 \mid \frac{M_{12}^2}{4\xi_1} \in \mathbb{Z},\ \frac{M_{13}^2}{4\xi_1} + \xi_3 \in \mathbb{Z},\ \frac{M_{12}M_{13}}{2\xi_1} \in \mathbb{Z}\}/\sim,$$

where

$$M \sim M' = (M'_{12}, M'_{13}) \leftrightarrow$$
$$\begin{pmatrix} 1 & & \\ n'_{12} & 1 & \\ n'_{13} & n'_{23} & 1 \end{pmatrix}\begin{pmatrix} \xi_1 & M_{12}/2 & M_{13}/2 \\ M_{12}/2 & {M_{12}}^2/4\xi_1 & M_{12}M_{13}/4\xi_1 \\ M_{13}/2 & M_{12}M_{13}/4\xi_1 & \frac{{M_{13}}^2}{4\xi_1} + \xi_3 \end{pmatrix}\begin{pmatrix} 1 & n'_{12} & n'_{13} \\ & 1 & n'_{23} \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \xi_1 & M'_{12}/2 & M'_{13}/2 \\ M'_{12}/2 & {M'_{12}}^2/4\xi_1 & M'_{12}M'_{13}/4\xi_1 \\ M'_{13}/2 & M'_{12}M'_{13}/4\xi_1 & \frac{{M'_{13}}^2}{4\xi_1} + \xi_3 \end{pmatrix},$$

with some $(n'_{12}, n'_{13}, n'_{23}) \in \mathbb{Z}^3$.
For a $M = (M_{12}, M_{13}) \in \mathfrak{M}(\xi_1,\xi_3)$, *we define* $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ *by*

$$\phi^{i_1,i_2}_{\xi_1,\xi_3}(M;x) := \Phi^\eta_M(h_{i_1}(s)h_{i_2}(u))(x)$$
$$= \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3} h_{i_1}(x'_{12} + \frac{m'_{12}}{2\xi_1})h_{i_2}(x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1})$$
$$\times \exp 2\pi\sqrt{-1}(\xi_1 x_1 + \frac{{m'_{12}}^2}{4\xi_1}x_2 + (\frac{{m'_{13}}^2}{4\xi_1} + \xi_3)x_3 + m'_{12}x_{12} + m'_{13}x_{13} + \frac{m'_{12}m'_{13}}{2\xi_1}x_{23}).$$

The set $\{\Phi^\eta_M\}_{M\in\mathfrak{M}'(\xi_1,\xi_3)}$ *gives a basis of* $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$.
(4) *When* $\eta \in \hat{N}$ *is as in* (4) *of the theorem, we introduce a set*

$$\mathfrak{M}(\xi_1,\xi_2,\xi_3) =$$
$$\{M = (M_{12}, M_{13}, M_{23}) \in \mathbb{Z}^3 \mid \frac{M_{12}^2}{4\xi_1} + \xi_2, \frac{M_{13}^2}{\xi_1} + \frac{(2\xi_1 M_{23} - M_{12}M_{13})^2}{16\xi_1^2\xi_2} + \xi_3 \in \mathbb{Z}\}/\sim,$$

where
$M \sim M' = (M'_{12}, M'_{13}, M'_{23}) \leftrightarrow$

$$\begin{pmatrix} 1 & & \\ n'_{12} & 1 & \\ n'_{13} & n'_{23} & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & M_{12}/2 & M_{13}/2 \\ M_{12}/2 & \frac{M_{12}{}^2}{4\xi_1} + \xi_2 & M_{23}/2 \\ M_{13}/2 & M_{23}/2 & \frac{(2\xi_1 M_{23} - M_{12}M_{13})^2 + 4\xi_1\xi_2 M_{13}{}^2}{16\xi_1^2\xi_2} + \xi_3 \end{pmatrix} \begin{pmatrix} 1 & n'_{12} & n'_{13} \\ & 1 & n'_{23} \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \xi_1 & M'_{12}/2 & M'_{13}/2 \\ M'_{12}/2 & \frac{M'_{12}{}^2}{4\xi_1} + \xi_2 & M'_{23}/2 \\ M'_{13}/2 & M'_{23}/2 & \frac{(2\xi_1 M'_{23} - M'_{12}M'_{13})^2 + 4\xi_1\xi_2 M'_{13}{}^2}{16\xi_1^2\xi_2} + \xi_3 \end{pmatrix}$$

with some $(n'_{12}, n'_{13}, n'_{23}) \in \mathbb{Z}^3$.

For a $M = (M_{12}, M_{13}, M_{23}) \in \mathfrak{M}(\xi_1, \xi_2, \xi_3)$, *we define* $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$ *by*

$$\begin{aligned} &\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}(M;x) := \Phi^\eta_M(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u))(x) \\ &= \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3} h_{i_1}(x'_{12} + \frac{m'_{12}}{2\xi_1})h_{i_2}(x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1}) \times \\ &h_{i_3}(x'_{23} + \frac{2\xi_1 m'_{23} - m'_{12}m'_{13}}{4\xi_1\xi_2}) \exp 2\pi\sqrt{-1}(\xi_1 x_1 + (\frac{m'_{12}{}^2}{4\xi_1} + \xi_2)x_2 \\ &+ (\frac{(2\xi_1 m'_{23} - m'_{12}m'_{13})^2 + 4\xi_1\xi_2 m'_{13}{}^2}{16\xi_1^2\xi_2} + \xi_3)x_3 + m'_{12}x_{12} + m'_{13}x_{13} + m'_{23}x_{23}). \end{aligned}$$

The set $\{\Phi^\eta_M\}_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)}$ *gives a basis of* $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$.

In the notations for (3) *and* (4), $m'_{12} = 2\xi_1 m_{12} + M_{12}$, $m'_{13} = 2\xi_1 m_{13} + M_{12}m_{23} + M_{13}$, $m'_{23} = 2\xi_1 m_{12}m_{13} + (\frac{M_{12}^2}{2\xi_1} + 2\xi_2)m_{23} + M_{12}(m_{13} + m_{12}m_{23}) + M_{13}m_{12} + M_{23}$.

Remark 7.2. From direct computation, we see that the equivalence relations on $\mathfrak{M}(\xi_2, \xi_3)$, $\mathfrak{M}(\xi_1, \xi_3)$ and $\mathfrak{M}(\xi_1, \xi_2, \xi_3)$ are well-defined and that these sets are finite.

8. Main result.

According to the formulation given at §6, we obtain our Fourier expansion in terms of the theta series computed at the previous section.

Theorem 8.1. *The Fourier expansion of a holomorphic Siegel modular form* f *of weight* τ_λ *on* G *is as follows:*

$$\begin{aligned} f(xa) = &\sum_{\xi_3\in\mathbb{Z}_{\geq 0}} C^f_{\xi_3} a_1^{\lambda_3} a_2^{\lambda_2} a_3^{\lambda_1} \exp 2\pi\sqrt{-1}(\xi_3(x_3 + \sqrt{-1}a_3^2)) \cdot v_{Q_H} \\ &+ \sum_{Q\in\Lambda_1} (\sum_{\substack{\xi_2\in\mathbb{Z}_{>0},\ \xi_3\geq 0 \\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)} C^{f,M}_{\xi_2,\xi_3} a_1(Q) a_1^{\lambda_3} a_2^{\lambda_1 - l} a_3^{\lambda_2 + l} e^{-2\pi(a_2^2\xi_2 + a_3^2\xi_3)} \end{aligned}$$

$$\times \sum_{i\geq 0} \alpha_i(l; \frac{1}{2} - 2\pi\xi_2 a_2^2)\phi^i_{\xi_2,\xi_3}(M;x))v_Q$$

$$+ \sum_{Q\in\Lambda_2} (\sum_{\substack{\xi_1\in\mathbb{Z}_{>0},\xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_3)} C^{f,M}_{\xi_1,\xi_3} a_2(Q) a_1^{\lambda_1-l-m} a_2^{\lambda_3+m} a_3^{\lambda_2+l} e^{-2\pi(a_1^2\xi_1+a_3^2\xi_3)}$$

$$\times \sum_{i_1\geq 0,\ i_2\geq 0} \alpha_{i_1}(m; \frac{1}{2} - 2\pi a_2^2\xi_1)\alpha_{i_2}(l; \frac{1}{2} - 2\pi a_3^2\xi_2)\phi^{i_1,i_2}_{\xi_1,\xi_3}(M;x))v_Q$$

$$+ \sum_{Q\in\Lambda_2} (\sum_{\substack{\xi_1\in\mathbb{Z}_{>0},\ \xi_2>0,\ \xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} a_2(Q) a_1^{\lambda_1-l-m} a_2^{\lambda_3+m} a_3^{\lambda_2+l}$$

$$\times e^{-2\pi(a_1^2\xi_1+a_2^2\xi_2+a_3^2\xi_3)} \sum_{i_1\geq 0,\ i_2\geq 0,\ i_3\geq 0} \alpha_{i_1}(m; \frac{1}{2} - 2\pi a_2^2\xi_1)\alpha_{i_2}(l; \frac{1}{2} - 2\pi a_3^2\xi_1)$$

$$\times \alpha_{i_3}(0; \frac{1}{2} - 2\pi a_3^2\xi_2)\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}(M;x))v_Q$$

Notations for this:
(1) $C^f_{\xi_3}, C^{f,M}_{\xi_2,\xi_3}, C^{f,M}_{\xi_1,\xi_3}$ *and* $C^{f,M}_{\xi_1,\xi_2,\xi_3}$ *are Fourier coefficients,*
(2) $\alpha_i(k;\rho) = \{(-1)^{k+i+[\frac{i}{2}]} + (-1)^{[\frac{i}{2}]}\}2^{2[\frac{i}{2}]+\delta-1}\left(\frac{\delta-k}{2}\right)_{[\frac{i}{2}]}\Gamma\left(\frac{k+1-\delta}{2}\right) \times$
${}_2F_1\left(\frac{k+1}{2}, \frac{k+2}{2}, \frac{k+1-\delta}{2} - [\frac{i}{2}]; \rho\right)$, *where* $\delta = 0$ *or* 1 *when* i *is even or odd respectively.*
(3) $Q_H = \begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_2 & \\ & -\lambda_3 & \end{pmatrix}$,
(4) $Q \in \Lambda_1$ *means that* Q *run through Gel'fand Tsetlin schemes of the form*
$\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_1+l & \\ & -\lambda_3 & \end{pmatrix}$ *with* $0 \leq l \leq \lambda_1 - \lambda_2$,
(5) $Q \in \Lambda_2$ *means that* Q *run through Gel'fand Tsetlin schemes of the form*
$\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_1+l & \\ & -\lambda_1+l+m & \end{pmatrix}$ *with* $0 \leq l \leq \lambda_1 - \lambda_2$ *and* $0 \leq m \leq \lambda_1 - \lambda_3 - l$.

Remark 8.2. The coefficient $c_i^{\eta,Q}$ mentioned in §6 is explicitly given as the coefficients of $\phi^{i_1}_{\xi_2,\xi_3}$, $\phi^{i_1,i_2}_{\xi_1,\xi_3}$ and $\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}$ when η is not a character.

By giving a certain change of the summation to the expansion, we obtain another expansion in terms of generalized Whittaker functions:

Theorem 8.3.

$$f(xa) = \sum_{\xi_3\in\mathbb{Z}_{\geq 0}} C^f_{\xi_3} w_Q(\xi_3, a)\exp 2\pi\sqrt{-1}(\xi_3 x_3)\cdot v_{Q_H}$$

$$+ \sum_{Q\in\Lambda_1} (\sum_{\substack{\xi_2\in\mathbb{Z}_{>0},\ \xi_3\geq 0\\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)} C^{f,M}_{\xi_2,\xi_3} \sum_m w_Q(\xi_2,\xi_3; a, x'_{23} + \frac{2\xi_2 m + M}{2\xi_1})$$

$$\times \exp 2\pi\sqrt{-1}(\xi_2 x_2 + (\frac{(2\xi_1 m + M)^2}{4\xi_2} + \xi_3)x_3 + (2\xi_2 m + M)x_{23}))v_Q$$

$$+ \sum_{Q\in\Lambda_2}(\sum_{\substack{\xi_1\in\mathbb{Z}_{>0},\xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_3)} C^{f,M}_{\xi_1,\xi_3} \sum_{(m_{12},m_{13})\in\mathbb{Z}^2} w_Q(\xi_1,\xi_3;a,x'_{12}+\frac{m'_{12}}{2\xi_1},$$

$$x'_{13}+\frac{m'_{12}}{2\xi_1}x'_{23}+\frac{m'_{13}}{2\xi_1})\exp 2\pi\sqrt{-1}(\xi_1 x_1 + \frac{{m'_{12}}^2}{4\xi_1}x_2 + (\frac{{m'_{13}}^2}{4\xi_1}+\xi_3)x_3$$

$$+ m'_{12}x_{12} + m'_{13}x_{13} + \frac{m'_{12}m'_{13}}{4\xi_1}x_{23})v_Q$$

$$+ \sum_{Q\in\Lambda_2}(\sum_{\substack{\xi_1\in\mathbb{Z}_{>0},\ \xi_2>0,\ \xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} w_Q(\xi_1,\xi_2,\xi_3;a,x'_{12}+\frac{m'_{12}}{2\xi_1},x'_{13}$$

$$+\frac{m'_{12}}{2\xi_1}x'_{23}+\frac{m'_{13}}{2\xi_1},x'_{23}+\frac{2\xi_1 m'_{23}-m'_{12}m'_{13}}{4\xi_1\xi_2})\exp 2\pi\sqrt{-1}(\xi_1 x_1 + (\frac{{m'_{12}}^2}{4\xi_1}+\xi_2)x_2$$

$$+(\frac{(2\xi_1 m'_{23}-m'_{12}m'_{13})^2+4\xi_1\xi_2{m'_{13}}^2}{16\xi_1^2\xi_2}+\xi_3)x_3 + m'_{12}x_{12}+m'_{13}x_{13}+m'_{23}x_{23}))v_Q.$$

9. Relation with the classical Fourier expansion.

Here, let f be a $\mathbb{C}$-valued form on Siegel upper half space and of weight $l=\lambda_1=\lambda_2=\lambda_3$. And let $z=\begin{pmatrix} z_1 & z_{12} & z_{13}\\ z_{12} & z_2 & z_{23}\\ z_{13} & z_{23} & z_3\end{pmatrix}$ be an element of the Siegel upper half space of degree 3, and $T=\begin{pmatrix} t_1 & t_{12}/2 & t_{13}/2\\ t_{12}/2 & t_2 & t_{23}/2\\ t_{13}/2 & t_{23}/2 & t_3\end{pmatrix}$ a semi-integral matrix of degree 3. The classical Fourier expansion of the form f can be written as

$$f(z)=\sum_{T\geq 0} C^f_T \exp 2\pi\sqrt{-1}(tr(Tz)),$$

where $T\geq 0$ means that T is positive semi-definite and C^f_T denotes the Fourier coefficient for T. By lifting f to a function on G, we can rewrite this expansion using the coordinate of G, and compare it with our expansion. Then we will obtain the relation of the Fourier coefficients C^f_T, $C^f_{\xi_3}$, $C^{f,M}_{\xi_2,\xi_3}$, $C^{f,M}_{\xi_1,\xi_3}$ and $C^{f,M}_{\xi_1,\xi_2,\xi_3}$. As a preparation for it, we give a following lemma:

Lemma 9.1. *Let T be as above. Using the notations of Proposition 7.1, we have the following:*
(1) *If $t_1=0$, T can be written as*

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & t_2 & t_{23}/2\\ 0 & t_{23}/2 & t_3\end{pmatrix}\quad (t_2\neq 0)\quad \textit{or}\quad \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & t_3\end{pmatrix}\quad (t_2=0).$$

If T is of the former one, it can be expressed as

$$T^{M}_{\xi_2,\xi_3}(m) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & m'_{12}/2 \\ 0 & m'_{12}/2 & \frac{{m'_{12}}^2}{4\xi_2} + \xi_3 \end{pmatrix}$$

with $\xi_2 \in \mathbb{Z}_{>0}$, $\xi_3 \geq 0$ *such that* $\mathfrak{M}(\xi_2, \xi_3) \neq \emptyset$.

If T is of the latter one,

$$T_{\xi_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$$

with $\xi_3 \in \mathbb{Z}_{\geq 0}$.
(2) *If* $t_1 \neq 0$, *T has the following two expressions:*
(*i*) *If* $\begin{vmatrix} t_1 & t_{12}/2 \\ t_{12}/2 & t_2 \end{vmatrix} = 0$,

$$T^{M_{12},M_{13}}_{\xi_1,\xi_3}(m_{12}, m_{13}, m_{23}) = \begin{pmatrix} \xi_1 & m'_{12}/2 & m'_{13}/2 \\ m'_{12}/2 & {m'_{12}}^2/4\xi_1 & m'_{12}m'_{13}/2\xi_1 \\ m'_{13}/2 & m'_{12}m'_{13}/2\xi_1 & \frac{{m'_{13}}^2}{4\xi_1} + \xi_3 \end{pmatrix}$$

with $\xi_1 \in \mathbb{Z}_{>0}$, $\xi_3 \geq 0$ *such that* $\mathfrak{M}(\xi_1, \xi_3) \neq \emptyset$.
(*ii*) *Otherwise,*

$$T^{M_{12},M_{13},M_{23}}_{\xi_1,\xi_2,\xi_3}(m_{12}, m_{13}, m_{23}) = \begin{pmatrix} \xi_1 & m'_{12}/2 & m'_{13}/2 \\ m'_{12}/2 & \frac{{m'_{12}}^2}{4\xi_1} + \xi_2 & m'_{23}/2 \\ m'_{13}/2 & m'_{23}/2 & \frac{(2\xi_1 m'_{23} - m'_{12}m'_{13})^2 + 4\xi_1\xi_2 {m'_{13}}^2}{16\xi_1^2\xi_2} + \xi_3 \end{pmatrix}$$

with $\xi_1 \in \mathbb{Z}_{>0}$, $\xi_2 > 0$ *and* $\xi_3 \geq 0$ *such that* $\mathfrak{M}(\xi_1, \xi_2, \xi_3) \neq \emptyset$.

From this, we obtain

Theorem 9.2. (1) *If* $T = T_{\xi_3}$, $C^f_T = C^f_{\xi_3}$.
(2) *If* $T = T^M_{\xi_2,\xi_3}(m)$, $C^f_T = C^{f,M}_{\xi_2,\xi_3}$ *for* $\forall m \in \mathbb{Z}$.
(3) *If* $T = T^{M_{12},M_{13}}_{\xi_1,\xi_3}(m_{12}, m_{13}, m_{23})$, $C^f_T = C^{f,M}_{\xi_1,\xi_3}$ *for* $\forall(m_{12}, m_{13}, m_{23}) \in \mathbb{Z}^3$.
(4) *If* $T = T^{M_{12},M_{13},M_{23}}_{\xi_1,\xi_2,\xi_3}(m_{12}, m_{13}, m_{23})$, $C^f_T = C^{f,M}_{\xi_1,\xi_2,\xi_3}$ *for* $\forall(m_{12}, m_{13}, m_{23}) \in \mathbb{Z}^3$.
In (3) *or* (4), M *denotes* (M_{12}, M_{13}) *or* (M_{12}, M_{13}, M_{23}).

10. Relation with the Fourier Jacobi expansion.

The form f has two types of Fourier-Jacobi expansions as follows:

$$(1) \qquad f(z) = \sum_{t_1 \in \mathbb{Z}_{\geq 0}} \phi_{t_1}(z_{(23)}, w_{(23)}) \exp 2\pi\sqrt{-1}(t_1 z_1),$$

$$(2)\qquad f(z) = \sum_{T_{(12)}=\begin{pmatrix} t_1 & t_{12}/2 \\ t_{12}/2 & t_2 \end{pmatrix}\geq 0} \phi_{T_{(12)}}(z_3, w_{(3)}) \exp 2\pi\sqrt{-1}(tr(T_{(12)}z_{(12)})).$$

In (1), we use the notations $z_{(23)} = \begin{pmatrix} z_2 & z_{23} \\ z_{23} & z_3 \end{pmatrix}$, $w_{(23)} = (z_{12}, z_{13})$ and

$$\phi_{t_1}(z_{(23)}, w_{(23)}) = \sum_{T=\begin{pmatrix} t_1 & R_{(23)} \\ {}^tR_{(23)} & T_{(23)} \end{pmatrix}\geq 0} C_T^f \exp 2\pi\sqrt{-1}(tr(T_{(23)}z_{(23)}) + 2R_{(23)}\,{}^tw_{(23)})$$

with $R_{(23)} = (t_{12}/2, t_{13}/2)$ and $T_{(23)} = \begin{pmatrix} t_2 & t_{23}/2 \\ t_{23}/2 & t_3 \end{pmatrix}$.

In (2), we have the notations $w_{(3)} = \begin{pmatrix} z_{13} \\ z_{23} \end{pmatrix}$, $z_{(12)} = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}$ and

$$\phi_{T_{(12)}}(z_3, w_{(3)}) = \sum_{T=\begin{pmatrix} T_{(12)} & R_{(3)} \\ {}^tR_{(3)} & t_3 \end{pmatrix}\geq 0} C_T^f \exp 2\pi\sqrt{-1}(t_3 z_3 + 2tr(R_{(3)}\,{}^tw_{(3)}))$$

with $R_{(3)} = \begin{pmatrix} t_{13}/2 \\ t_{23}/2 \end{pmatrix}$.

10.1. Relation with the Fourier Jacobi expansion of type (1). In the first expansion, the contribution Φ_0 of ϕ_0 to our Fourier expansion is

$$\Phi_0(a_{(23)}, x) = \sum_{\xi_3\in\mathbb{Z}_{\geq 0}} C_{\xi_3}^f (a_2a_3)^l \exp 2\pi\sqrt{-1}(\xi_3 x_3) + \sum_{\substack{\xi_2\in\mathbb{Z}_{>0},\ \xi_3\geq 0 \\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)} C_{\xi_2,\xi_3}^{f,M}$$

$$\sum_m (a_2a_3)^l \exp(-2\pi(a_2^2\xi_2 + a_3^2\xi_3 + a_2^2\xi_2(x'_{23} + \frac{2\xi_2 m + M}{2\xi_2})^2))$$

$$\times \exp 2\pi\sqrt{-1}(\xi_2 x_2 + (\frac{(2\xi_1 m + M)^2}{4\xi_2} + \xi_3)x_3 + (2\xi_2 m + M)x_{23})),$$

and the contribution Φ_{ξ_1} of ϕ_{ξ_1} ($\xi_1 \in \mathbb{Z}_{>0}$) to our expansion is

$$\Phi_{\xi_1}(a_{(23)}, x) = \sum_{\substack{\xi_3\geq 0 \\ \mathfrak{M}(\xi_1,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_3)} C_{\xi_1,\xi_3}^{f,M} \sum_{(m_{12},m_{13})\in\mathbb{Z}^2} (a_2a_3)^l \exp(-2\pi(a_3^2\xi_3$$

$$+ a_2^2\xi_1(x'_{12} + \frac{m'_{12}}{2\xi_1})^2 + a_3^2\xi_1(x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1})^2) \times$$

$$\exp 2\pi\sqrt{-1}(\frac{{m'_{12}}^2}{4\xi_1}x_2 + (\frac{{m'_{13}}^2}{4\xi_1} + \xi_3)x_3 + m'_{12}x_{12} + m'_{13}x_{13} + \frac{m'_{12}m'_{13}}{4\xi_1}x_{23})$$

$$+\sum_{\substack{\xi_2>0,\ \xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}}\ \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3}(a_2a_3)^l$$

$$\exp(-2\pi(a_2^2\xi_2+a_3^2\xi_3+a_2^2\xi_1(x'_{12}+\frac{m'_{12}}{2\xi_1})^2+a_3^2\xi_1(x'_{12}+\frac{m'_{12}}{2\xi_1}x'_{23}+\frac{m'_{13}}{2\xi_1})^2$$

$$+a_3^2\xi_2(x'_{23}+m_{23}+\frac{2\xi_1M_{23}-M_{12}M_{13}}{4\xi_1\xi_2})^2)\exp 2\pi\sqrt{-1}((\frac{{m'_{12}}^2}{4\xi_1}+\xi_2)x_2+$$

$$(\frac{(2\xi_1m'_{23}-m'_{12}m'_{13})^2+4\xi_1\xi_2{m'_{13}}^2}{16\xi_1^2\xi_2}+\xi_3)x_3+m'_{12}x_{12}+m'_{13}x_{13}+m'_{23}x_{23}).$$

Using these, the Fourier Jacobi expansion of f of type (1) can be written as follows:

Theorem 10.1.

$$f(xa)=\sum_{\xi_1\in\mathbb{Z}_{\geq 0}}\Phi_{\xi_1}(a_{(23)},x_{(1)})w_{\xi_1}(a_1)\exp 2\pi\sqrt{-1}(\xi_1x_1),$$

where $w_{\xi_1}(a_1)=a_1^l\exp(-2\pi\xi_1a_1^2)$ *and* $x_{(1)}$ *means that* $x_1=0$ *in* $x\in N$.

10.2. Relation with the Fourier Jacobi expansion of type (2). Next, we treat the second Fourier Jacobi expansion of f. The matrix $T_{(12)}$ which gives the index of the Jacobi form, is one of the following 4:

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\0&\xi_2\end{pmatrix},\ \begin{pmatrix}\xi_1&m'_{12}/2\\m'_{12}/2&{m'_{12}}^2/4\xi_1\end{pmatrix},\ \begin{pmatrix}\xi_1&m'_{12}/2\\m'_{12}/2&\frac{{m'_{12}}^2}{4\xi_1}+\xi_2\end{pmatrix}.$$

(1) If $T_{(12)}$ is of the first form, the contribution $\Phi_{T_{(12)}}$ of $\phi_{T_{(12)}}$ to our expansion is

$$\Phi_{T_{(12)}}(a_3,x_{(12)})=\sum_{\xi_3\in\mathbb{Z}_{\geq 0}}a_3^lC^f_{\xi_3}\exp(-2\pi(a_3^2\xi_3))\exp 2\pi\sqrt{-1}(\xi_3x_3).$$

(2) If $T_{(12)}$ is of the second form, the contribution $\Phi_{T_{(12)}}$ is

$$\Phi_{T_{(12)}}(a_3,x_{(12)})=\sum_{\substack{\xi_3\geq 0\\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}}\ \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)}C^{f,M}_{\xi_2,\xi_3}\sum_m a_3^l\exp(-2\pi(a_3^2\xi_3+$$

$$a_2^2\xi_2(x'_{23}+\frac{2\xi_2m+M}{2\xi_2})^2))\exp 2\pi\sqrt{-1}((\frac{(2\xi_1m+M)^2}{4\xi_2}+\xi_3)x_3+(2\xi_2m+M)x_{23})).$$

(3) If $T_{(12)}$ is of the third form, the contribution is

$$\Phi_{T_{(12)}}=\sum_{\substack{\xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_3)\neq\emptyset}}\ \sum_{M\in\mathfrak{M}(\xi_1,\xi_3)}C^{f,M}_{\xi_1,\xi_3}\sum_{(m_{12},m_{13})\in\mathbb{Z}^2}a_3^l\exp(-2\pi(a_3^2\xi_3$$

$$+a_3^2\xi_1(x'_{13}+\frac{m'_{12}}{2\xi_1}x'_{23}+\frac{m'_{13}}{2\xi_1})^2)\exp 2\pi\sqrt{-1}((\frac{{m'_{13}}^2}{4\xi_1}+\xi_3)x_3+m'_{13}x_{13}+\frac{m'_{12}m'_{13}}{4\xi_1}x_{23}).$$

(4) If $T_{(12)}$ is of the fourth form, the contribution is

$$\sum_{\substack{\xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3} a_3^l \exp(-2\pi(a_3^2\xi_3$$

$$+ a_3^2\xi_1(x'_{12}+\frac{m'_{12}}{2\xi_1}x'_{23}+\frac{m'_{13}}{2\xi_1})^2 + a_3^2\xi_2(x'_{23}+m_{23}+\frac{2\xi_1M_{23}-M_{12}M_{13}}{4\xi_1\xi_2})^2)$$

$$\times \exp 2\pi\sqrt{-1}((\frac{(2\xi_1 m'_{23}-m'_{12}m'_{13})^2+4\xi_1\xi_2 {m'_{13}}^2}{16\xi_1^2\xi_2}+\xi_3)x_3+m'_{13}x_{13}+m'_{23}x_{23}).$$

With these functions, we can write the Fourier Jacobi expansion of type (2), in terms of ours:

Theorem 10.2.

$$f(xa) = \sum_{\xi_2\in\mathbb{Z}_{\geq 0}} \Phi_{\begin{pmatrix}0&0\\0&\xi_2\end{pmatrix}}(a_3, x_{(12)}(0)) w_{0,\xi_2}(a_1,a_2,x'_{12}:m'_{12}) \exp 2\pi\sqrt{-1}(\xi_2 x_2)$$

$$+ \sum_{\substack{\xi_1\in\mathbb{Z}_{>0},\xi_2\geq 0\\ \mathfrak{M}(\xi_1,\xi_2)\neq\emptyset}} \sum_{M_{12}\in\mathfrak{M}(\xi_1,\xi_2)} \sum_{m_{12}\in\mathbb{Z}} \Phi_{\begin{pmatrix}\xi_1 & m'_{12}/2\\ m'_{12}/2 & \frac{{m'_{12}}^2}{4\xi_1}+\xi_2\end{pmatrix}}(a_3, x_{(12)}(0))$$

$$\times w_{\xi_1,\xi_2}(a_1,a_2,x'_{12}:m'_{12}) \exp 2\pi\sqrt{-1}(\xi_1 x_1 + m'_{12}x_{12} + (\frac{{m'_{12}}^2}{4\xi_1}+\xi_2)x_2),$$

where $x_{(12)}(0)$ *means that* $x_1 = x_{12} = x_2 = 0$ *in* x *and* $w_{\xi_1,\xi_2}(a_1,a_2,x'_{12};m'_{12}) = (a_1a_2)^l \exp(-2\pi(a_1^2\xi_1 + a_2^2\xi_2 + a_2^2\xi_1(x'_{12}+\frac{m'_{12}}{2\xi_1})^2))$.

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