ARCHIMEDEAN SHINTANI FUNCTIONS ON $GL(2)$

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1. Introduction
Shintani functions for $GL(n)$ was defined by Murase and Sugano in the study of automorphic $L$-functions [5]. They proved the uniqueness and the existence of this function over a non-archimedean local field and obtained new kinds of integral formula for the standard $L$-functions as an application. Our aim in this note is the case study of archimedean Shintani functions on $GL(2)$, which is not studied in [5]. In §3, we define archimedean Shintani functions on $GL(2)$ generalizing that of Murase and Sugano. Also, our definition of this function can be considered as a generalization of the $O_{\xi}$ model studied by Waldspurger [6].

Now we consider the following problems.

(1) Decide the dimension of the space of archimedean Shintani functions.
(2) Find an explicit formula of non-zero archimedean Shintani functions.

We will give an answer to these problems in §5.

2. Preliminaries
2.1. Groups and algebras. Throughout this note, $E$ means either the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. Let $G$ be the real reductive Lie group $GL(2, E)$ and $\theta$ be an involution defined by $\theta(g)={}^t\overline{g}^{-1}(g \in G)$. We denote the set of fixed points of $\theta$ by $K$. Then $K$ is a maximal compact subgroup of $G$ and

$$K \simeq \begin{cases} O(2, \mathbb{R}) & \text{for } E = \mathbb{R}, \\ U(2) & \text{for } E = \mathbb{C}, \end{cases}$$

Moreover we define an involutive automorphism $\sigma$ of $G$ by $\sigma(g) = JgJ$ ($g \in G$), where $J = \text{diag}(-1, 1)$. Then $\theta\sigma = \sigma\theta$ and the set $H$ of fixed points of $\sigma$ is equal to

$$H = \{g \in G|\sigma(g) = g\} = \{\text{diag}(h_1, h_2) \in G \mid h_i \in E^\times\} \simeq E^\times \times E^\times.$$

In particular, $H$ is abelian subgroup of $G$. 

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Let $\mathfrak{g} = \mathfrak{gl}(2, E)$ be the Lie algebra of $G$. If we denote the differentials of $\theta$ and $\sigma$, again by $\theta$ and $\sigma$, then we have $\theta(X) = -X$ and $\sigma(X) = JXJ(X \in \mathfrak{g})$. Let us write the eigenspaces of $\theta$ and $\sigma$ by

$$
\mathfrak{t} = \{X \in \mathfrak{g} | \theta(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\},
\mathfrak{h} = \{X \in \mathfrak{g} | \sigma(X) = X\}, \quad \mathfrak{q} = \{X \in \mathfrak{g} | \sigma(X) = -X\}.
$$

Therefore we have the decompositions $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. Remark that $\mathfrak{t}$ is the Lie algebra of $K$ and $\mathfrak{h}$ is that of $H$. Let

$$
A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \in G \right\} r \in \mathbb{R}, \quad a = \text{Lie}(A).
$$

Then $a$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$.

For a Lie algebra $\mathfrak{b}$, we denote by $\mathfrak{b}^\mathbb{C}$ the complexification $\mathfrak{b} \otimes_\mathbb{R} \mathbb{C}$ of $\mathfrak{b}$.

2.2. Representations. In this subsection, we recall parametrizations of the irreducible unitary representations of $K$, $H$ and $G$.

Let us denote by $\hat{K}$ the set of the equivalence classes of irreducible finite dimensional representations of $K$. Since $K$ is compact, the highest weight theory (cf. Knapp [4; Theorem 4.28]) gives a parametrization of $\hat{K}$ by the set

$$
\Lambda = \left\{ \begin{array}{ll}
\{ (0, \varepsilon) | \varepsilon = 0, 1 \} \cup \mathbb{N}, & \text{for } E = \mathbb{R}, \\
\{ \lambda = (\lambda_1, \lambda_2) | \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \}, & \text{for } E = \mathbb{C}.
\end{array} \right.
$$

Let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the corresponding representation to $\lambda \in \Lambda$. Then we have

$$
\dim V_\lambda = \left\{ \begin{array}{ll}
1, & \text{if } \lambda = (0, \varepsilon) \\
2, & \text{if } \lambda \in \mathbb{N} \\
\lambda_1 - \lambda_2 + 1, & \text{for } E = \mathbb{C}.
\end{array} \right.
$$

Next let us parametrize the totality $\hat{H}$ of the equivalence classes of irreducible unitary representations of $H$. To do this, we put

$$
\mathcal{N}_E = \left\{ \begin{array}{ll}
\{0,1\}^2, & \text{for } E = \mathbb{R}, \\
\mathbb{Z}^2, & \text{for } E = \mathbb{C}.
\end{array} \right.
$$

For every $s = (s_1, s_2) \in \mathbb{C}^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$, we define a representation $\eta_{s}^k$ of $H$ by

$$
\eta_{s}^k(\text{diag}(h_1, h_2)) = h_1^{k_1} h_2^{k_2} |h_1|^{s_1-k_1} |h_2|^{s_2-k_2} \text{diag}(h_1, h_2) \in H.
$$

Clearly $\hat{H} = \{ \eta_{s}^k | s = (s_1, s_2) \in (\sqrt{-1}\mathbb{R})^2, \ k = (k_1, k_2) \in \mathcal{N}_E \}$.

Let $P = N_P A_P M_P$ be the Langlands decomposition of the upper triangular subgroup $P$ of $G$. For every $z = (z_1, z_2) \in \mathbb{C}^2$ and $l = (l_1, l_2) \in \mathcal{N}_E$, we define $\sigma_l$ on $M_P$ and $\nu_z$ on $a_P = \text{Lie}(A_P)$ by

$$
\sigma_l(\text{diag}(\varepsilon_1, \varepsilon_2)) = \varepsilon_1^{l_1} \varepsilon_2^{l_2}, \quad \text{diag}(\varepsilon_1, \varepsilon_2) \in M_P, \quad \varepsilon_i \in E^{(1)}
$$

$$
\nu_z(\text{diag}(t_1, t_2)) = z_1 t_1 + z_2 t_2, \quad \text{diag}(t_1, t_2) \in a_P, \quad t_i \in \mathbb{R}.
$$
Then we can construct a representation $\pi_z^l = \text{Ind}_P^G(1_{N_P} \otimes \exp \nu_z \otimes \sigma_l)$ of $G$ which we call the non-unitary principal series representation. A dense subspace of the representation space is

$$\{ f \in C^\infty(G) \mid f(namx) = e^{(\nu_z + \rho_E) \log a \sigma_l(m)} f(x) \}$$

with norm

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

and $G$ acts by $\pi_z^l(g)f(x) = f(xg)$. Here $\rho_E$ is the half sum of the roots of $(a_P, g)$ positive for $N_P$.

If $z_i \in \sqrt{-1}\mathbb{R}$, then the representation $\pi_z^l$ is irreducible and unitary. This representation is usually called the unitary principal series representation $P_z^l$ of $G$. Now we put $\rho_{E,0} = \rho_{E}(\text{diag}(1, -1))$, i.e. $\rho_{\mathbb{R},0} = 1$ and $\rho_{\mathbb{C},0} = 2$. If the parameter $(z, l)$ satisfies $z_1 + z_2 \in \sqrt{-1}\mathbb{R}$, $-\rho_{E,0} < z_1 - z_2 < 0$ and $l_1 = l_2$, $\pi_z^l$ is irreducible and infinitesimally unitary. The unitary version of this representation is called the complementary series representation $C_z^l$ of $G$.

The representations $\pi_z^l$ belonging to these two series have the following $K$-types from the Frobenious reciprocity theorem;

$$\pi_z^l|_K = \begin{cases} \tau_0^l \oplus \sum_{n \in \mathbb{N}} \tau_{2n}, & \text{if } l_1 + l_2 \equiv 0 \pmod{2}, \\ \sum_{n \in \mathbb{N}} \tau_{2n-1}, & \text{if } l_1 + l_2 \equiv 1 \pmod{2}, \\ \sum_{j=0}^{\infty} \tau_{(l_1+j, l_2-j)}, & \text{if } l_1 \geq l_2, \\ \sum_{j=0}^{\infty} \tau_{(l_2+j, l_1-j)}, & \text{if } l_1 < l_2, \end{cases}$$

for $E = \mathbb{R}$,

$$\pi_z^l|_K = \begin{cases} \sum_{j=0}^{\infty} \tau_{(l_1+j, l_2-j)}, & \text{if } l_1 \geq l_2, \\ \sum_{j=0}^{\infty} \tau_{(l_2+j, l_1-j)}, & \text{if } l_1 < l_2, \end{cases}$$

for $E = \mathbb{C}$.

In the case of $E = \mathbb{R}$, $\pi_z^l$ contains the discrete series representation $D_{j, z_1+z_2}^l$ as a subrepresentation if the parameters satisfy $z_1 + z_2 \in \sqrt{-1}\mathbb{R}$, $z_1 - z_2 = -j - 1$ for $j \in \mathbb{Z}_{\geq 0}$, and $l_1 + l_2 \equiv j \pmod{2}$. The $K$-types of $D_{j, z_1+z_2}^l$ are given by

$$D_{j, z_1+z_2}^l|_K = \sum_{n \in \mathbb{N}} \tau_{j+2n}.$$
on which $G$ acts by the right translation. Then $C^\infty_\eta(H \setminus G)$ has structure of a smooth $G$-module and of a $(g^C, K)$-module.

On the other hand, let us take an irreducible Harish-Chandra module $\Pi^* \in \hat{\mathcal{G}}$, and consider the intertwining space

$$T_{\eta, \Pi} = \text{Hom}(g^C, K)(\Pi^*, C^\infty \text{Ind}_K^G(\eta))$$

and its image

$$S_{\eta, \Pi} = \bigcup_{T \in T_{\eta, \Pi}} \text{Image}(T).$$

Here $*$ means the contragredient $(g^C, K)$-module. We call $\varphi \in S_{\eta, \Pi}$ a Shintani function of type $(\eta, \Pi)$.

For any finite dimensional $K$-module $(\tau, V_\tau)$, we define $C^\infty_{\eta, \Pi}(H \setminus G/K)$ by the space of smooth functions $F : G \to V_\tau$ with the property

$$F(hgk) = \eta(h)\tau(k)^{-1}F(g), \quad (h, g, k) \in H \times G \times K.$$ 

Now let us take a finite dimensional $K$-module $(\tau, V_\tau)$ and a $K$-equivariant map $i : \tau^* \to \Pi^*|_K$. Here $\tau^*$ is the contragredient representation of $\tau$. Moreover let $i^*$ be the pullback via $i$. Then the map

$$T_{\eta, \Pi} \xrightarrow{i^*} \text{Hom}_K(\tau^*, C^\infty_{\eta}(H \setminus G)) \cong C^\infty_{\eta, \tau}(H \setminus G/K)$$

gives the restriction of $T \in T_{\eta, \Pi}$ to $\tau^*$ which we denote by $T_i \in C^\infty_{\eta, \tau}(H \setminus G/K)$. Set

$$S_{\eta, \Pi}(\tau) = \bigcup_i T_i, \quad T \in T_{\eta, \Pi},$$

and we call $\varphi \in S_{\eta, \Pi}(\tau)$ a Shintani function of type $(\eta, \Pi; \tau)$.

3.2. Radial part. Let us write the centralizer and the normalizer of $a$ in $K \cap H$ by $Z_{K \cap H}(a)$ and $N_{K \cap H}(a)$, respectively. If we put $w_0 = \text{diag}(1, -1)$, then the quotient group $W = N_{K \cap H}(a)/Z_{K \cap H}(a)$ has the unique nontrivial element $w_0 Z_{K \cap H}(a)$.

For each pair of $\eta \in \hat{H}$ and a finite dimensional $K$-module $(\tau, V_\tau)$, let us denote by $C^\infty_W(A; \eta, \tau)$ the space of smooth functions $\varphi : A \to V_\tau$ satisfying the following conditions;

\begin{align*}
(1) \quad (\eta(m)\tau(m))\varphi(a) &= \varphi(a), \quad m \in Z_{K \cap H}(a), \quad a \in A, \\
(2) \quad (\eta(w_0)\tau(w_0))\varphi(a) &= \varphi(a^{-1}), \quad a \in A, \\
(3) \quad (\eta(l)\tau(l))\varphi(1) &= \varphi(1), \quad l \in K \cap H.
\end{align*}

**Lemma 3.1.** (Flensted-Jensen [1; Theorem 4.1])

1. $G = HAK = HA^+K$, where $A^+ = \{a_\tau \in A | \tau > 0\}$.
2. The set $C^\infty_{\eta, \tau}(H \setminus G/K)$ is in bijective correspondence, via restriction $A$, with the set $C^\infty_W(A; \eta, \tau)$.

Let $(\tau, V_\tau)$ and $(\tau', V_{\tau'})$ be two finite dimensional $K$-modules. For each $C$-linear map $u : C^\infty_{\eta, \tau}(H \setminus G/K) \to C^\infty_{\eta, \tau'}(H \setminus G/K)$, we have a unique $C$-linear map $\mathcal{R}(u) : C^\infty_W(A; \eta, \tau) \to C^\infty_W(A; \eta, \tau')$ with the property $(uf)|_A = \mathcal{R}(u)(f)|_A$ for $f \in C^\infty_W(A; \eta, \tau)$. We call $\mathcal{R}(u)$ the radial part of $u$. 
4. Characterization

4.1. Shift operator. The vector space $p^c$ becomes a $K$-module via the adjoint representation. Let $p^c = p_S \oplus p_Z$ be the irreducible decomposition of $p^c$ as a $K$-module, where $p_Z = (p \cap Z_g)^c$, $Z_g$ is the center of $g$, and $p_S \simeq V_\beta$ with $\beta = 2$ for $E = \mathbb{R}$ or $\beta = (1, -1)$ for $E = \mathbb{C}$.

Take an orthonormal basis $\{X_i\}$ of $p_S$ with respect to the Killing form. For a given $\eta^k_S \in \hat{H}$ and $(\tau_\lambda, V_\lambda) \hat{K}$, we define a first order gradient type differential operator $\nabla^S_{\eta^k_S, \tau_\lambda} : C^\infty_{\eta^k_S, \tau_\lambda}(H \backslash G/K) \to C^\infty_{\eta^k_S, \tau_\lambda} \otimes \text{Ad}_{p_S}(H \backslash G/K)$ by

$$\nabla^S_{\eta^k_S, \tau_\lambda} f = \sum_i R_{X_i} f \otimes X_i, \quad f \in C^\infty_{\eta^k_S, \tau_\lambda}(H \backslash G/K),$$

where

$$R_{X} f(g) = \frac{d}{dt} f(g \cdot \exp(tX))|_{t=0}, \quad \text{for } X \in g^c, \ g \in G.$$  

This differential operator $\nabla^S_{\eta^k_S, \tau_\lambda}$ is called the Schmid operator. Now let us assume that $\lambda \in \mathbb{N}_{\geq 3}$ for $E = \mathbb{R}$ or $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ with $\lambda_1 - \lambda_2 \geq 2$ for $E = \mathbb{C}$. Then we can define the minus shift operator

$$\nabla^-_{\eta^k_S, \tau_\lambda} : C^\infty_{\eta^k_S, \tau_\lambda}(H \backslash G/K) \to C^\infty_{\eta^k_S, \tau_{\lambda-\beta}}(H \backslash G/K)$$

as the compositions of $\nabla^S_{\eta^k_S, \tau_\lambda}$ with the projector from $V_\lambda \otimes p_S$ into an irreducible component $V_{\lambda-\beta}$.

4.2. System of differential equations. Let $\Pi^* \in \hat{G}_\infty$, and let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the minimal $K$-type of $\Pi$. Moreover, let $\eta^k_S \in \hat{H}$ with $s = (s_1, s_2) \in (\sqrt{-1}\mathbb{R})^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$. We consider a characterization of the space $S_{\eta^k_S, \Pi}(\tau_\lambda)$ of Shintani functions of type $(\eta^k_S, \Pi; \tau_\lambda)$ by some differential equations.

Let $Z(g^c)$ be the center of the universal enveloping algebra of $g^c$. It is well known that each element $u \in Z(g^c)$ acts on $\Pi^*$, hence on $S_{\eta^k_S, \Pi}(\tau_\lambda)|_A$, as a scalar operator $\chi_u$ called an infinitesimal character (cf. Knapp [4; Chap.VIII §6]). Therefore we have the differential equation

$$(4.1) \quad \mathcal{R}(u) \varphi(a_r) = \chi_u \varphi(a_r)$$

for each $\varphi \in S_{\eta^k_S, \Pi}(\tau_\lambda)|_A$ and $u \in Z(g^c)$.  

Now let us assume that $\Pi^* = D^l_{l_1, l_2}$ if $E = \mathbb{R}$ or $\Pi^* = P^l_l$ such that $|l_1 - l_2| \geq 2$ if $E = \mathbb{C}$. Since $\tau_\lambda$ is the minimal $K$-type of $\Pi$, then $\tau_{\lambda-\beta}$ does not occur in the $K$-type of $\Pi$. Thus any element in $S_{\eta^k_S, \Pi}(\tau_\lambda)$ is annihilated by the action of the minus shift operator $\nabla^-_{\eta^k_S, \tau_\lambda} : C^\infty_{\eta^k_S, \tau_\lambda}(H \backslash G/K) \to C^\infty_{\eta^k_S, \tau_{\lambda-\beta}}(H \backslash G/K)$, and hence, the differential equation

$$(4.2) \quad \mathcal{R}(\nabla^-_{\eta^k_S, \tau_\lambda}) \varphi(a_r) = 0$$

holds for each $\varphi \in S_{\eta^k_S, \Pi}(\tau_\lambda)|_A$.

The above differential equations for $\varphi \in C^\infty_W(\mathfrak{a}; \eta^k_S, \tau_\lambda)$ are necessary conditions for belonging to the space $S_{\eta^k_S, \Pi}(\tau_\lambda)|_A$. But we can prove the following theorem which says that the above equations are also sufficient conditions.
Theorem 4.1. ([2; Proposition 6.1], [3; Theorem 5.3], [8; Theorem 2.4])

Let $\eta^k_s \in \hat{H}$, $\Pi^* \in \hat{G}_\infty$, and let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the minimal $K$-type of $\Pi$. Then the following system of differential equations characterizes the space $S_{\eta^k_s,\Pi}(\tau_\lambda)|_A \subset C^\infty_W(A; \eta^k_s, \tau_\lambda)$ of Shintani functions of type $(\eta^k_s, \Pi; \tau_\lambda)$.

1. If $\Pi^* = P_z^l$ or $C_z^l$, the equations (4.1) for all $u \in Z(G^C)$.

2. If $E = R$ and $\Pi^* = D_{j,z_1+z_2}$, the equations (4.1) for $u = I$ and (4.2).

5. Results

In view of Theorem 4.1, the space $S_{\eta^k_s,\Pi}(\tau_\lambda)|_A$ of Shintani functions is the solution space of some system of differential equations in $C^\infty_W(A; \eta^k_s, \tau_\lambda)$. By the systems of equations in Theorem 4.1 and the constructions of Shintani functions via the Poisson integrals [3; §6], we can prove the following theorem.

Theorem 5.1. ([2; Theorem 6.2, 6.3], [3; Theorem 7.1])

Let $\eta^k_s \in \hat{H}$ and $\Pi^* \in \hat{G}_\infty$. Then the space $S_{\eta^k_s,\Pi}$ of Shintani functions of type $(\eta^k_s; \Pi)$ is non zero if and only if $\eta^k_s|_{Z_G} = \Pi|_{Z_G}$. Here $Z_G$ is the center of $G$. Moreover, for such pair of representations $(\eta^k_s, \Pi)$ we have

$$\dim S_{\eta,\Pi} = \begin{cases} 2, & \text{if } E = R, \ \Pi^* = P_z^l, \ \text{and } l_1 \neq l_2 \\ 1, & \text{otherwise.} \end{cases}$$

Moreover we can state an explicit formula of Shintani functions of type $(\eta^k_s, \Pi; \tau_\lambda)$ for some special cases.

Theorem 5.2. ([2; Theorem 6.2], [3; Theorem 7.2])

Let $\eta^k_s \in \hat{H}$ and $\Pi^* = P_z^l$ or $C_z^l \in \hat{G}_\infty$ with $l_1 = l_2$. Then the minimal $K$-type $(\tau_\lambda, V_\lambda) \in \hat{K}$ of $\Pi$ is 1-dimensional. If the parameters $s$, $z$, $k$, and $l$ satisfy the equations

$s_1 + s_2 = z_1 + z_2, \ \ k_1 + k_2 \equiv l_1 + l_2 \ (\text{mod} 2) \ (E = R), \ \ k_1 + k_2 = l_1 + l_2 \ (E = C),$

then the space $S_{\eta^k_s,\Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$x^4(1 - x)^{-\frac{s_1 + \rho E_0 + \delta}{4}} F_1 \left( \frac{z_1 + s_2 + \rho E_0 + \delta}{4}, \frac{z_1 - s_2 + \rho E_0 + \delta}{4}, \frac{\rho E_0 + \delta}{2}; x \right) v_0^\lambda,$$

with $\delta = 2|k_1 - l_1|$, $v_0^\lambda \in V_\lambda$, and the variable $x = \tanh^2 2r$. Here $F_1(a; b; c; x)$ is the Gauss's hypergeometric function.

Theorem 5.3. ([2; Theorem 6.3])

Let $\eta^k_s \in \hat{H}$ and $\Pi^* = D_{j,z_1+z_2} \in \hat{G}_\infty$. Then $(\tau_\lambda, V_\lambda) \in \hat{K}$ with $\lambda = j + 2$ is the minimal $K$-type of $\Pi$. If the parameters satisfy the equations

$s_1 + s_2 = z_1 + z_2, \ \ k_1 + k_2 \equiv l_1 + l_2 \ (\text{mod} 2),$

the space $S_{\eta^k_s,\Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$u_{j+2}(y)v_{j+2} + (-1)^k u_{-j-2}(y)v_{-j-2}. $$

Here $\{v_{j-2}, v_{j+2}\}$ is the standard basis of $V_{j+2}$, and

$$u_{\pm(j+2)}(r) = \left( \frac{\delta x - \sqrt{1 - x}}{4} \right)^{\frac{j+2}{2}} (1 - y)^{\frac{j-2}{4}}$$

with the variable $y = \left( \frac{e^x - \sqrt{1 - x}}{e^{2x} + \sqrt{1 - x}} \right)^2$.
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