The asymptotic behavior of Eisenstein series and a comparison of the Weil-Petersson and the Zograf-Takhtajan metrics

Kunio Obitsu

Department of Mathematics, Tokyo Institute of Technology, Meguro, Tokyo 152-8551, Japan

Abstract. Of interest to us is the asymptotic behavior of Eisenstein series for degenerating hyperbolic surfaces with cusps. In order to investigate it we use integral representations of eigenfunctions of the Laplacian, the collar lemma, the interior Schauder estimates, the maximum principles for harmonic functions and the unique extension theorem of solutions to elliptic equations. As an application, we will compare the Weil-Petersson and the Zograf-Takhtajan metrics near the boundary of moduli spaces.

§0. Introduction

The Quillen metric defined for the determinant line bundle of Laplacian over the Teichmüller space \( T_{g,n} \) of compact hyperbolic surfaces with genus \( g \) has played an important role in moduli theory ([(14)]). The metric is described as the product of a special value of the Selberg zeta function and the usual \( L^2 \)-fibre metric with respect to Poincaré metric. The first Chern form of the metric is represented by the Weil-Petersson two-form for \( T_g \), which formula has been shown by various methods.

Zograf and Takhtajan proved the formula by quasiconformal deformation theory. Moreover they defined the regularized metric for the determinant bundle of Laplacian for Teichmüller space \( T_{g,n} \) of hyperbolic surfaces with cusps of type \((g,n)\) and calculated its first Chern form, which is described in terms of the Weil-Petersson metric and a new Kähler metric as called Zograf-Takhtajan metric. They showed in [16] that the Zograf-Takhtajan metric is Kählerian and invariant under the action of the mapping class group as the Weil-Petersson metric is. It has been recently shown that the Zograf-Takhtajan metric is incomplete for \( T_{g,n} \) as the Weil-Petersson metric is ([13]). The proof has been accomplished by showing that the length of a curve approaching the boundary of \( T_{g,n} \) with respect to the Zograf-Takhtajan metric is finite. The construction of the curve is due to Wolpert ([25] II).

On the other hand, recently Fujiki and Weng shed new light on the geometry of moduli space of punctured Riemann surfaces and the Zograf-Takhtajan metric ([4],[19]). From Arakelov geometric points of view, Weng found arithmetic Riemann-Roch theorem for singular metrics, and established a generalization of Mumford type isometries. Fujiki and Weng have observed that the Zograf-Takhtajan metric is algebraic, and Weng proposed a
general arithmetic factorization in terms of Weil-Petersson metric and Zograf-Takhtajan metric and Selberg zeta functions.

Therefore the asymptotic behavior of the Zograf-Takhtajan metric near the boundary of moduli space is of importance and of interest for studying compactification of moduli space of punctured Riemann surfaces. In the previous paper ([13]), we have observed that the metric is incomplete. In that proof we have obtained an estimate of Eisenstein series of index 2 just around pinching geodesics, which is regrettably very rough and far from the precise asymptotic behavior, and a rough estimate of the Zograf-Takhtajan metric ([13]).

In this paper we find the asymptotic behavior of Eisenstein series of index \( s \) with Re \( s > 1 \) in Theorem 1 and 2. As a simple application, we improve estimate of the Zograf-Takhtajan metric near the boundary of moduli space (Theorem 3). As a result it turns out that the magnitude of the Zograf-Takhtajan norm are less than or equal to the one of the Weil-Petersson norm.

We close this chapter with surveying and proposing some approaches to the asymptotic behavior of Eisenstein series. Wolpert has investigated Eisenstein series \( E(z, s) \) with \( \{ s \in \mathbb{C}\mid \text{Re} s = \frac{1}{2}, \ s \neq \frac{1}{2} \} \), while we shall investigate for Re \( s > 1 \) ([25] I). He has shown that a subsequence of \( \hat{E}(z, s) \), the normalized Eisenstein series so that \( L^2 \)-norm of that on a thick part of \( S_l \) could be constant, converges to a non-trivial sum of the Eisenstein series for \( S_0 \). Its beautiful proof has been accomplished by investigating Legendre functions and showing his original Schauder-type inequalities. It seems hard to apply our method directly to the case where Re \( s \leq 1 \). The reason for difficulty is that we can not use the maximum principles because \( E(z, s) \) with Re \( s \leq 1 \) is not subharmonic, and can not extend Lemma 1 to the case of Re \( s \leq 1 \), and \( E(z, s) \) has poles on \( \{ s \in [0, 1] \} \) (For example, we have observed that the constants \( M_i(\text{Re} s, a) \) takes infinity at Re \( s = 1 \). What we have to study seems to investigate the behavior of the scattering matrix \( \Phi(s) \) in (1.4).

The author would like to thank Professors T. Ohsawa and K. Yoshikawa for valuable comments that encouraged him to study the theme of this paper. He would express his gratitude to Professor T. Oda for explaining to him A. Selberg’s work on Eisenstein series. He is also grateful to Professors T. Nakanishi and H. Shiga for interest in his work and some discussions.

\section{1. Preliminaries}

**1.1 Eisenstein series.**

Let \( S \) be a punctured hyperbolic surface of type \((g, n)\) \((n > 0)\). It can be represented as a quotient \( H/\Gamma \) of the upper half plane \( H = \{ z \in \mathbb{C}\mid \text{Im} z > 0 \} \) by the action of a torsion free finitely generated Fuchsian group \( \Gamma \in \text{PSL}_2(\mathbb{R}) \). The group is generated by \( 2g \) hyperbolic transformations \( A_1, B_1, \ldots, A_g, B_g \) and parabolic transformations \( P_1, \ldots, P_n \) satisfying the relation

\[
A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} P_1 \ldots P_n = 1.
\]

The fixed points of the parabolic elements \( P_1, \ldots, P_n \) will be denoted by \( z_1, z_2, \ldots, z_n \in \mathbb{R} \cup \{ \infty \} \) respectively and called inequivalent cusps. The projection of the cusps \( z_1, z_2, \ldots, z_n \) are the punctures \( p_1, p_2, \ldots, p_n \) of \( S \). For each \( i = 1, \ldots, n \), denote by \( \Gamma_i \) the stabilizer of \( z_i \) in \( \Gamma \) that is the cyclic subgroup of \( \Gamma \) generated by \( P_i \). Pick \( \sigma_i \in \text{PSL}_2(\mathbb{R}) \)
ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

such that \( \sigma_i \infty = z_i \) and \( \langle \sigma_i^{-1} P_i \sigma_i \rangle = \langle z \mapsto z + 1 \rangle \). Then, for \( a > 1 \), the \( a \)-cusp region \( C_i(a) \) associated to \( p_i \) is represented as a quotient \( \langle \sigma_i^{-1} P_i \sigma_i \rangle \setminus \{ z \in H | \text{Im} z > a \} \simeq \Gamma \setminus \{ z \in H | \text{Im} z > a \} \).

\[ C_i(a) \simeq [a, \infty) \times S^1, \text{ equipped with the metric } ds^2 = (dy^2 + dx^2)/y^2. \]

Let \( \Delta : C^\infty(S) \to C^\infty(S) \) be the negative hyperbolic Laplacian of \( S \). Regarded as an operator in \( L^2(S) \) with domain \( C^\infty_0(S) \), \( \Delta \) is essentially self-adjoint. Denote by \( \bar{\Delta} \) the unique self-adjoint extension (that is, Friedrichs extension). Then the continuous spectrum of \( \bar{\Delta} \) can be described in terms of Eisenstein series ([6] Chap. Seven, [10] Chap. V, [17] §3.2).

The Eisenstein series attached to \( z_i \) is defined by

\[ E_i(z, s) = \sum_{\gamma \in \langle P_i \rangle \setminus \Gamma} \text{Im}(\sigma_i^{-1}\gamma z)^s, \quad \text{Re } s > 1. \]

The series is absolutely convergent in the half-plane \( \text{Re } s > 1 \) and in the upper half-plane, it satisfies

\[ \Delta E_i(z, s) = s(s - 1)E_i(z, s). \tag{1.1} \]

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex \( s \)-plane, holomorphic on \( \{ \text{Re } s = \frac{1}{2} \} \) and satisfies a system of functional equations ([15] §7). Several mathematicians also verified it by the various methods ([3], [6] Th.11.6, [10] pp.23–46, [12]). \( E_i(z, s) \) has Fourier expansions at punctures \( p_j \), ([6] Prop. 8.6, [10] §2.2, [11] §8, [17] §3.1)

\[ E_i(\sigma_j z, s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s} + \sum_{m \neq 0} c_m(s) y^\frac{1}{2} K_{s-\frac{1}{2}}(2\pi |m| y) e^{2\pi i m x}, \tag{1.2} \]

\( K_{s-\frac{1}{2}} \) the MacDonald-Bessel function ([18], p.78), that has the following asymptotics ([18], p.202)

\[ y^\frac{1}{2} K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}} e^{-y}, \text{ as } y \to \infty, \quad \text{for any complex } s. \tag{1.3} \]

The scattering matrix \( \Phi(s) = (\phi_{ij}(s)) \) enters in the functional equations ([6] Th.11.8, [10] Th. 4.4.2, [15] (7.36), [17] Th. 3.5.1),

\[ E(z, s) = \Phi(s) E(z, 1-s), \quad \Phi(s) \Phi(1-s) = 1, \tag{1.4} \]

where \( E(z, s) \) is the vector of Eisenstein series.

Thanks to Colin de Verdière's keen observation, the Eisenstein series turns out to have the following characterization ([3], [12]).

Claim. For \( \text{Re } s > \frac{1}{2} \), \( s \not\in (\frac{1}{2}, 1) \), \( E_i(z, s) \) is a unique solution of the equation \( \Delta E_i(z, s) = s(s - 1)E_i(z, s) \) such that \( E_i(z, s) - \text{Im}(\sigma_i^{-1}z)^s \) is square integrable on \( C_i(1) \).

Remark. We will use the above Claim just for the case where \( \text{Re } s > 1 \) (Theorem 1 and 2).
1.2 The Weil-Petersson and the Zograf-Takhtajan metrics.

Denote by \( T_{g,n} \) Teichmüller space of hyperbolic-Takhtajan surfaces of type \((g, n)\). Now we consider the tangent and cotangent spaces at \( S \) of \( T_{g,n} \). The cotangent space is \( Q(S) \), the integrable holomorphic quadratic differentials on \( S \). Let \( B(S) \) be the \( L^\infty \)-closure of \( \Gamma \)-invariant, bounded, \((-1, 1)\)-forms, i.e. the Beltrami differentials for \( S \). For \( \mu \in B(S), \varphi \in Q(S) \), the integral \( \int_S \mu \varphi \) defines a paring, let \( Q(S)^\perp \) be the annihilator of \( Q(S) \). The tangent space at \( S \) to \( T_{g,n} \) is \( B(S)/Q(S)^\perp \simeq HB(S) \), the Serre dual space of \( Q(S) \), i.e. the harmonic Beltrami differentials on \( S \). Then for \( \mu, \nu \in HB(S) \), the Weil-Petersson and the Zograf-Takhtajan metrics are defined as follows ([16]),

\[
\langle \mu, \nu \rangle_{\text{WP}} = \int \int_S \mu(z)\overline{\nu(z)}y^{-2}dxdy
\]

\[
\langle \mu, \nu \rangle_{(i)} = \int \int_S E_i(z, 2)\mu(z)\overline{\nu(z)}y^{-2}dxdy
\]

\[
= \int_0^\infty \int_0^1 \mu(\sigma_i z)\overline{\nu(\sigma_i z)}dxdy
\]

\[
\langle \mu, \nu \rangle_{\text{ZT}} = \sum_{i=1}^n \langle \mu, \nu \rangle_{(i)}.
\]

Both Weil-Petersson and Zograf-Takhtajan metric are Kählerian and incomplete ([13], [16]).

1.3 Degenerating parameters, infinite-energy harmonic maps.

In this part, we consider degeneration of hyperbolic surfaces. Denote by \((S_l(l > 0), \rho_l(w)|dw|^2)\) a degenerating family of hyperbolic surfaces of type \((g, n)\). We assume that several disjoint simple closed geodesics \( l_1, l_2, \ldots, l_m \) on \( S_l \) will be pinched (We denote their hyperbolic lengths by the same notations). Let \( \Delta_l \) be the negative Laplacian of \( S_l \). To compare functions on the limit surface \((S_0, \rho(z)|dz|^2)\) and \((S_l, \rho_l(w)|dw|^2)\), we use infinite-energy harmonic maps \( w^l : S_0 \rightarrow S_l \setminus \{l_1, l_2, \ldots, l_m\} \) constructed by M. Wolf ([9], [21], [26]).

Let regular quadratic differentials \( \Psi_j dz^2 \) \((j = 1 \ldots m)\) for a surface with nodes \((S_0, \rho(z)|dz|^2)\) be holomorphic quadratic differentials that at j-th node have second-order poles, with equal residues, and at remaining cusps and nodes have at most simple poles. When we set \( l = (l_1, l_2, \ldots, l_m) \) and \( \Psi(l)dz^2 = \sum_{j=1}^m i_j^2 \Psi_j dz^2 \), the precise real-analytic parameterization are obtained by Wolf ([21], [26])

\[
(w^l)^* \rho_l|dw|^2 = \Psi(l)dz^2 + (H(l) + L(l))\rho|dz|^2 + \Psi(l)dz^2,
\]

\[
\Psi(l)dz^2 = \rho_l w^l\overline{w^l}, \quad H(l) = [\rho_l(w^l(z)/\rho(z))|w^l_z|^2, \quad L(l) = [\rho_l(w^l(z)/\rho(z))|w^l_z|^2].
\]

The Beltrami differential of \( w^l \) is \( \mu_l = w^l_z/\overline{w^l_z} = \Psi(l)/\rho H \). Furthermore Wolf finds the
ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

following expansions ([21], Cor. 5.4),

\begin{align*}
H(l) &= 1 + \sum_{j=1}^{2m} \frac{i l_j^2}{6} E_{q_j}(z, 2) + O(\sum_{j=1}^{m} l_j^2), \\
L(l) &= O(\sum_{j=1}^{2m} l_j^2),
\end{align*}

for \( q_1, \ldots, q_{2m} \) the new cusps that come from pinching geodesics, where \( E_{q_j}(z, 2) \) are Eisenstein series attached to \( q_j \) for the component containing \( q_j \).

Moreover the Beltrami differential \( \mu_l \) converges uniformly to zero on compact subsets of \( S_0 \). And the harmonic map \( w^l \) converges uniformly to \( id \) on compact subsets of \( S_0 \).

Of interest to us now is the behavior of \( \Delta_l \). We review the discussion of Wolpert ([24], [25], [26]). There exists a basic map \( \sigma_l : L^2(S_l) \to L^2(S_0) \), for \( f \in L^2(S_l) \), \( \sigma_l(f) = f(w^l) \). Define \( (w^l)^*\Delta_l = \sigma_l\Delta_l\sigma_l^{-1} \). Then \( (w^l)^*\Delta_l \) is real-analytic family ([24], p.450, [25], p.98, [26], pp.254–258). Especially from [24] Lemma 5.3, we easily see that for any \( k \in \mathbb{N} \), \( C^k \)-norm on compact subsets of components of \( S_0 \), \( (w^l)^*\Delta_l \) converges uniformly to \( \Delta_0 \) that is defined to be the formal sum of the hyperbolic Laplacians for components of \( S_0 \).

Many mathematicians investigated, by various parameterizations, degeneration of hyperbolic surfaces and the asymptotic behavior of several functions; for example, Green’s functions ([7], [8], [9], [20]), eigenfunctions of the Laplacian and eigenvalues ([7], [8], [9], [25], [26]), Riemannian matrix and Faltings invariant ([20]).

Remark. Basically those various parameterizations turn out to be almost the same powerful tools. But what we should pay attention to is that the infinite-energy parameters are independent of twist-angles around the pinching geodesics. Nevertheless, the family \( S_l \) with pairs of opened collars glued by adequate twist-angles agrees with \( \mathbb{R}^l \) constructed by Wolpert [25], pp.103–104 ([23], Appendix, [26], pp.251–252).

\S 2. SOME ELEMENTARY ESTIMATES OF EISENSTEIN SERIES

We give key lemmas which play important roles in the proof of the main theorem (cf. [13], Lemma 4).

Lemma 1. We set the same notations as in \S 1. Let the index of Eisenstein series \( \Re s > 1 \). For any \( i = 1, 2, \ldots, n \) and sufficiently large \( a > 0 \),

\[ |E_i(z, s)| < M_1(\Re s, a) a^{\Re s - 1}, \quad \text{on } \partial C_i(a). \]

Here \( M_1(\Re s, a) \) is a constant depending only on \( s, a \), independent of complex structure and topological type of the surface.

Let \( l_1, \ldots, l_m \) be pinching geodesics on \( S_l \). For \( 0 < k \leq 1 \) and \( j = 1, \ldots, m \), set

\[ N_{l_j}(k) = \{ p \in S_l | d(p, l_j) < k \sinh^{-1}(1/\sinh l_j^2) \} \]

, the collar neighborhood around \( l_j \) in \( S_l \) ([2], 4.1 ). Here we quote an important claim due to S. Wolpert ([25] II, Lemma 2.1).
Claim. Let $\rho(z)$ be the injectivity radius of $S_l$ at $z$. There is an absolute positive constant $C_0$ such that for $l < 2\sinh^{-1}1$, and any $z \in N_l(1)$, then $\rho(z)e^{d(z, \partial N_l(1))} \geq C_0$.

We improve the estimate of the Eisenstein series in [13], Lemma 4 just for the case where there exist separating pinching geodesics on $S_l$. Let $E^l_i(z, s)$ be the Eisenstein series attached to $p_i$ for $S_l$.

Lemma 2. Let the index $\text{Re } s > 1$. Assume that there is only one pinching geodesic $l = l_1$ on $S_l$, separating $S_l$ into two parts; $S_{l,1}$ containing the puncture $p_i$ and the other component $S_{l,2}$. Then for $l < 2\sinh^{-1}1$,

$$|E^l_i(z, s)| < M_2(k, \text{Re } s) l^{\text{Res}(1+k)-2}, \quad \text{on } \partial N_l(k) \cap S_{l,2}.$$  

Here $M_2(k, \text{Re } s)$ is an absolute constant depending only on $k, \text{Re } s$.

Remark. For any $s$ with $\text{Re } s > 1$, there is $0 < k \leq 1$ such that $\text{Re } s (1 + k) - 2 > 0$.

§ 3. THE ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

Our aim is to prove one of the main theorems. From now on, the cusps of $S_0$ that arise from the cusps of $S_l$ are called the old cusps and the cusps of $S_0$ that arise from the pinching geodesics of $S_l$ are called the new cusps.

Theorem 1. We set the same notations as in §1. Let the index $\text{Re } s > 1$.

(1) If $\{l_1, \ldots, l_m\}$ do not separate $S_l$, then for any $i = 1, \ldots, n$, as $l_1, \ldots, l_m \to 0$,

$$E^l_i(z, s) \to E^0_i(z, s)$$

uniformly on any compact subset of $S_0$. Here $E^0_i(z, s)$ is the Eisenstein series attached to the old puncture $p_i$ for $S_0$.

(2) Assume that $\{l_1, \ldots, l_m\}$ separate $S_l$ and all components of $S_0$ have negative Euler numbers. Denote by $S^i_{0,1}$ and $S^i_{0,2}$ respectively the component of $S_0$ containing $p_i$ and the union of the components of $S_0$ not containing $p_i$. Let $q_j (j = 1, \ldots, m)$ be the new cusp arising from $l_j$. Denote by $C_j(b)$ ($b > 1$) be the cusp region around $q_j$ in $S_0$, each composed of usual two $b$–cusp regions. Then

(i) For any $i = 1, \ldots, n$, as $l_1, \ldots, l_m \to 0$,

$$E^l_i(z, s) \to E^0_i(z, s)$$

uniformly on any compact subset of $S^i_{0,1}$. Here $E^0_i(z, s)$ is the Eisenstein series attached to $p_i$ for $S^i_{0,1}$.

(ii) For any $i = 1, \ldots, n$ and any $b > 1$, as $l_1, \ldots, l_m \to 0$,

$$E^l_i(z, s) \to 0$$

uniformly on $S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b)$. 
ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

Theorem 2. We set the same assumption and notations as in Theorem 1 (2). Pick $R$ one of the components of $S^3_{0,2}$, which have the new cusps $q_1, q_2, \ldots, q_t$, arising from the pinching geodesics $l_1, \ldots, l_m$ on $S_1$, and have the old cusps which may be denoted by $p_1, \ldots, p_u$, differing from $p_i$, replacing the enumeration if necessary. Denote by $E_{q_j}(z, s)$ the Eisenstein series attached to $q_j$ for $R$ ($j = 1, \ldots, t$).

Then there exist some constants $K_l \rightarrow \infty$ and a subsequence $l_1^{(h)} = \ldots = l_m^{(h)} = l^{(h)} \rightarrow 0$,

$$K_{l^{(h)}}(w^{l^{(h)}}) * E^{l^{(h)}}_l(z, s) \rightarrow G_0(z, s)$$

on any compact subset of $R$, where $G_0(z, s)$ is a non-trivial smooth function on $R$ satisfying

$$(\Delta_0 - s(s - 1)) G_0(z, s) = 0 \quad \text{on } R.$$

And $\lim_{l \rightarrow 0} K_l l^{2(\text{Res} - 1) - \delta} = \infty$, for any $\delta > 0$.

Moreover $G_0(z, s)$ is of the form,

$$G_0(z, s) = \sum_{j=1}^{t} B_j E_{q_j}(z, s),$$

where $B_j$ ($j = 1, \ldots, t$) are some constants.


Theorem 3. The Weil-Petersson and the Zograf-Takhtajan metrics have the following behavior near the boundary of $T_{g,n}$, along the degenerating family of hyperbolic surfaces constructed by Wolpert. That is, let $\tau$ be the vector field formed by the degenerating family with only one pinching geodesic for simplicity. Then the norms of $\tau$ with respect to the Weil-Petersson and Zograf-Takhtajan metrics satisfy

$$\|\tau\|_{ZT} \leq n\tilde{c}\|\tau\|_{WP} \quad \text{as } l \rightarrow 0,$$

where $\tilde{c}$ is an absolute constant.

REFERENCES

K. OBITSU


