GENERALIZED ISOMETRIC SPHERES OF ELEMENTS OF $PU(1,n; \mathbb{C})$

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Let $G$ be a discrete subgroup of $PU(1,n; \mathbb{C})$. For a boundary point $y$ of the Siegel domain, we define the generalized isometric sphere $I_y(f)$ of an element $f$ of $PU(1,n; \mathbb{C})$. By using the generalized isometric spheres of elements of $G$, we construct a fundamental domain $P_y(G)$ for $G$, which is regarded as a generalization of the Ford domain. And we show that the Dirichlet polyhedron $D(w)$ for $G$ with center $w$ converges to $P_y(G)$ as $w \to y$.

1. First let us recall some definitions and notation. Let $\mathbb{C}$ be the field of complex numbers. Let $V = V^{1,n}(\mathbb{C})$ denote the vector space $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(z^*, w^*) = -(\overline{z_0}w_1^* + \overline{z_1}w_0^*) + \sum_{j=2}^{n} \overline{z_j}w_j^*$$

for $z^* = (z_0^*, z_1^*, z_2^*, ..., z_n^*), w^* = (w_0^*, w_1^*, w_2^*, ..., w_n^*)$ in $V$. An automorphism $g$ of $V$, that is a linear bijection such that $\tilde{\Phi}(g(z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*)$ for $z^*, w^*$ in $V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n; \mathbb{C})$. Set $PU(1,n; \mathbb{C}) = U(1,n; \mathbb{C})/(\text{center})$. Let $V_0 = \{w^* \in V | \tilde{\Phi}(w^*, w^*) = 0\}$ and $V_- = \{w^* \in V | \tilde{\Phi}(w^*, w^*) < 0\}$. It is clear that $V_0$ and $V_-$ are invariant under $U(1,n; \mathbb{C})$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi : V^* \longrightarrow \pi(V^*)$ be the projection map defined by $\pi(0, ..., 0, w_1, w_2, ..., w_n) = (w_1, w_2, ..., w_n)$, where $w_j = w_j^*/w_0^*$ for $j = 1, 2, ..., n$. We write $\infty$ for $\pi(0, 0, ..., 0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^n = \{w = (w_1, w_2, ..., w_n) \in \mathbb{C}^n | \Re(w_1) > \frac{1}{2} \sum_{j=2}^{n} |w_j|^2\}.$$ 

An element $g$ in $PU(1,n; \mathbb{C})$ acts on the Siegel domain $H^n$ and its boundary $\partial H^n$. In $H^n$, we can introduce the hyperbolic metric $d$ (see [3] and [6]). An element of $PU(1,n; \mathbb{C})$ is an isometry of $H^n$ with respect to $d$. Denote $H^n \cup \partial H^n$ by $\overline{H^n}$. The $H$-coordinates of a point $(w_1, w_2, ..., w_n) \in \overline{H^n} - \{\infty\}$ are defined by $k, t, w') \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C}^{n-1}$ such that $k = \Re(w_1) - \frac{1}{2} \sum_{j=2}^{n} |w_j|^2, t = \Im(w_1)$ and $w' = (w_2, ..., w_n)$. The Cygan metric $\rho(p, q)$ for $p = (k_1, t_1, w')_H$ and $q = (k_2, t_2, W')_H$ is given by

$$\rho(p, q) = \left[\frac{1}{2} ||W' - w'||^2 + |k_2 - k_1|\right] + i\{t_1 - t_2 + \Im(\overline{w'}W')\}^{\frac{1}{2}},$$

where $\overline{w'}W' = \sum_{j=2}^{n} \overline{w_j}W_j$. 

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Let \( f = (a_{ij})_{1 \leq i, j \leq n+1} \in PU(1, n; \mathbb{C}) \) with \( f(\infty) \neq \infty \). We define the isometric sphere \( I(f) \) of \( f \) by

\[
I(f) = \{ w = (w_1, w_2, \ldots, w_n) \in \overline{H^n} | |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))| \},
\]

where \( Q = (0, 1, 0, \ldots, 0) \), \( W = (1, w_1, w_2, \ldots, w_n) \) in \( V^* \) (see [5]). It follows that the isometric sphere \( I(f) \) is the sphere in the Cygan metric with center \( f^{-1}(\infty) \) and radius \( R_f = \sqrt{1/|a_{12}|} \), that is,

\[
I(f) = \left\{ w = (k, t, w)_{H}/\mathbb{R}^+ \{0\} \in (\cup) \times \mathbb{R} \times \mathbb{C}^{n-1} | \rho(w, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.
\]

Fix \( y \in \partial H^n \) such that \( f(y) \neq y \). Let \( \gamma \) be an element of \( PU(1, n; \mathbb{C}) \) with \( \gamma(y) = \infty \). We define the generalized isometric sphere \( I_{y}(f) \) at \( y \) of \( f \) as

\[
I_{y}(f) = \gamma^{-1}(I\gamma f \gamma^{-1}) = \{ Z \in \overline{H^n} | p(\gamma(z), \gamma f^{-1}(\infty)) = R_{\gamma f \gamma^{-1}} \}.
\]

We can write \( I_{y}(f) \) as

\[
I_{y}(f) = \left\{ z \in \overline{H^n} | \alpha_{y}(f, z) = 1 \right\}.
\]

Put

\[
Ext I_{y}(f) = \{ z \in \overline{H^n} | \alpha_{y}(f, z) < 1 \},
\]

\[
Int I_{y}(f) = \{ z \in \overline{H^n} | \alpha_{y}(f, z) > 1 \},
\]

respectively.

Just as in the case of isometric spheres, we have

Proposition 1.1.

(1) \( I_{f(y)}(f) = f(I_{y}(f)) = I_{y}(f^{-1}) \);
(2) \( f(Ext I_{y}(f)) \subset Int I_{y}(f^{-1}) \);
(3) \( f(Int I_{y}(f)) \subset Ext I_{y}(f^{-1}) \).

Next we consider the location of fixed points of elements.

Proposition 1.2. Let \( f \) be an element of \( PU(1, n; \mathbb{C}) \) with fixed point \( x \). If \( f \) is elliptic or parabolic, then \( x \) lies on the isometric sphere \( I(f^{-1}) \) of \( f^{-1} \). If \( f \) is loxodromic, then \( I(f^{-1}) \) does not contain \( x \).
Replacing isometric spheres by generalized isometric spheres leads to the same conclusion as in Proposition 1.2.

Proposition 1.3. Let $f$ be an element of $PU(1,n;\mathbb{C})$ with fixed point $x$. If $f$ is elliptic or parabolic, then $x$ lies on $I_y(f)$. If $f$ is loxodromic, then $I_y(f)$ does not contain $x$.

2. Let $z_1, z_2$ be two different points in $H^n$. Let $E(z_1, z_2)$ be the bisector of $\{z_1, z_2\}$, that is,

$$E(z_1, z_2) = \{w \in H^n \mid d(z_1, w) = d(z_2, w)\}$$

(see [5] for details). Let $G$ be a discrete subgroup of $PU(1,n;\mathbb{C})$ and let $w$ be any point of $H^n$ that is not fixed by any element of $G$ except the identity. The Dirichlet polyhedron $D(w)$ for $G$ with center $w$ is defined by

$$D(w) = \bigcap_{g \in G - \{id\}} H_g(w),$$

where $H_g(w) = \{z \in H^n \mid d(z, w) < d(z, g(w))\}$. We observe that

1. $D(w)$ is not necessarily convex,
2. $D(w)$ is star-shaped about $w$,
3. $D(w)$ is locally finite

(see [2], [4], [11] and [12]).

Let $\Omega(G)$ be the ordinary set of $G$. Assume that $\infty \in \Omega(G)$ and its stability subgroup $G_\infty = \{\text{identity}\}$. Then there is a positive constant $M$ such that $\rho(0,g(\infty)) \leq M$ for any element $g$ of $G$. The same argument as in [4] leads to the following results.

1. The radii of isometric spheres are bounded above.
2. The number of isometric spheres with radii exceeding a given positive quantity is finite.
3. Given any infinite sequence of distinct isometric spheres of elements of $G$, the radii being $R_{g_1}, R_{g_2}, \ldots$, then $\lim_{m \to \infty} R_{g_m} = 0$.

We show that the generalized isometric sphere $I_y(f)$ is closely related to the bisector $E(z, f^{-1}(z))$.

Proposition 2.1. If $z \in H^n$ converges to $y \in \partial H^n$, then $E(z, f^{-1}(z))$ converges to $I_y(f)$.

By using generalized isometric spheres, we can construct a fundamental domain.

Theorem 2.2. Let $G$ be a discrete subgroup of $PU(1,n;\mathbb{C})$. Let $\infty$ be a point of $\Omega(G)$ and let $G_\infty = \{\text{identity}\}$. Suppose that $y$ is a point of $\Omega(G) \cap \partial H^n$ and that $G_y$ consists only of the identity. Then

$$P_y(G) = \bigcap_{f \in G - \{id\}} \text{Ext } I_y(f)$$
is a fundamental domain for $G$.

By Proposition 2.1 and Theorem 2.2, we obtain

**Theorem 2.3.** Let $G$ be a discrete subgroup of $PU(1,n;C)$. Let $z \in H^n$ and let $y \in \partial H^n \cap \Omega(G)$. Then $D(z) \rightarrow P_y(G)$ as $z \rightarrow y$.

From the manner of constructing $P_y(G)$, we have

**Corollary 2.4.** The fundamental domain $P_y(G)$ is locally finite.

**References**

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