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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1999.07.1104: 133-136</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63216">http://hdl.handle.net/2433/63216</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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GENERALIZED ISOMETRIC SPHERES OF ELEMENTS OF $PU(1,n;\mathbb{C})$

SHIGEYASU KAMIYA

神谷 茂保（岡山理大 工）

Let $G$ be a discrete subgroup of $PU(1,n;\mathbb{C})$. For a boundary point $y$ of the Siegel domain, we define the generalized isometric sphere $I_{y}(f)$ of an element $f$ of $PU(1,n;\mathbb{C})$. By using the generalized isometric spheres of elements of $G$, we construct a fundamental domain $P_{y}(G)$ for $G$, which is regarded as a generalization of the Ford domain. And we show that the Dirichlet polyhedron $D(w)$ for $G$ with center $w$ converges to $P_{y}(G)$ as $w \to y$.

1. First let us recall some definitions and notation. Let $\mathbb{C}$ be the field of complex numbers. Let $V = V^{1,n}(\mathbb{C})$ denote the vector space $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(z^{*}, w^{*}) = -(z_{0}^{*}w_{1}^{*} + \overline{z_{1}w_{0}^{*}}) + \sum_{j=2}^{n} \overline{z_{j}w_{j}^{*}}$$

for $z^{*} = (z_{0}^{*}, z_{1}^{*}, z_{2}^{*}, \ldots, z_{n}^{*}), w^{*} = (w_{0}^{*}, w_{1}^{*}, w_{2}^{*}, \ldots, w_{n}^{*})$ in $V$. An automorphism $g$ of $V$, that is a linear bijection such that $\tilde{\Phi}(g(z^{*}), g(w^{*})) = \tilde{\Phi}(z^{*}, w^{*})$ for $z^{*}, w^{*}$ in $V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n;\mathbb{C})$. Set $PU(1,n;\mathbb{C}) = U(1,n;\mathbb{C})/(\text{center})$. Let $V_{0} = \{w^{*} \in V | \tilde{\Phi}(w^{*}, w^{*}) = 0\}$ and $V_{-} = \{w^{*} \in V | \tilde{\Phi}(w^{*}, w^{*}) < 0\}$. It is clear that $V_{0}$ and $V_{-}$ are invariant under $U(1,n;\mathbb{C})$. Set $V^{*} = V_{-} \cup V_{0} - \{0\}$. Let $\pi : V^{*} \longrightarrow \pi(V^{*})$ be the projection map defined by $\pi(w_{0}^{*}, w_{1}^{*}, w_{2}^{*}, \ldots, w_{n}^{*}) = (w_{1}, w_{2}, \ldots, w_{n})$, where $w_{j} = w_{j}^{*}/w_{0}^{*}$ for $j = 1, 2, \ldots, n$. We write $\infty$ for $\pi(0,1,0,\ldots,0)$. We may identify $\pi(V_{-})$ with the Siegel domain

$$H^{n} = \{w = (w_{1}, w_{2}, \ldots, w_{n}) \in \mathbb{C}^{n} | \text{Re}(w_{1}) > \frac{1}{2} \sum_{j=2}^{n} |w_{j}|^{2}\}.$$

An element $g$ in $PU(1,n;\mathbb{C})$ acts on the Siegel domain $H^{n}$ and its boundary $\partial H^{n}$. In $H^{n}$, we can introduce the hyperbolic metric $d$ (see [3] and [6]). An element of $PU(1,n;\mathbb{C})$ is an isometry of $H^{n}$ with respect to $d$. Denote $H^{n} \cup \partial H^{n}$ by $\overline{H^{n}}$. The $H$-coordinates of a point $(w_{1}, w_{2}, \ldots, w_{n}) \in \overline{H^{n}} - \{\infty\}$ are defined by $(k, t, w')_{H} \in (\mathbb{R}^{+} \cup \{0\}) \times \mathbb{R} \times \mathbb{C}^{n-1}$ such that $k = \text{Re}(w_{1}) - \frac{1}{2} \sum_{j=2}^{n} |w_{j}|^{2}, t = \text{Im}(w_{1})$ and $w' = (w_{2}, \ldots, w_{n})$. The Cygan metric $\rho(p, q)$ for $p = (k_{1}, t_{1}, w')_{H}$ and $q = (k_{2}, t_{2}, w'')_{H}$ is given by

$$\rho(p, q) = \left\{ \frac{1}{2} ||w'' - w'||^{2} + |k_{2} - k_{1}| + i(t_{1} - t_{2} + \text{Im}(\overline{w'}w'')) \right\}^{\frac{1}{2}},$$

where $\overline{w'}w'' = \sum_{j=2}^{n} \overline{w_{j}}w_{j}$. 
Let $f = (a_{ij})_{1 \leq i,j \leq n+1} \in PU(1, n; \mathbb{C})$ with $f(\infty) \neq \infty$. We define the isometric sphere $I(f)$ of $f$ by

$$I(f) = \{ w = (w_1, w_2, \ldots, w_n) \in \overline{H^n} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))| \},$$

where $Q = (0,1,0,\ldots,0)$, $W = (1, w_1, w_2, \ldots, w_n)$ in $V^*$ (see [5]). It follows that the isometric sphere $I(f)$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \frac{1}{|a_{12}|}$, that is,

$$I(f) = \left\{ w = (k, t, w) \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C}^{n-1} \mid \rho(w, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Fix $y \in \partial H^n$ such that $f(y) \neq y$. Let $\gamma$ be an element of $PU(1, n; \mathbb{C})$ with $\gamma(y) = \infty$. We define the generalized isometric sphere $I_y(f)$ at $y$ of $f$ as

$$I_y(f) = \gamma^{-1}(I_{\gamma f \gamma^{-1}}) = \{ z \in \overline{H^n} \mid \rho(\gamma(z), \gamma f^{-1} \gamma^{-1}(\infty)) = R_{\gamma f \gamma^{-1}} \}$$

(see [1]). Note that if $y = \infty$, then $I_\infty(f)$ is the usual isometric sphere $I(f)$. The definition above does not depend on the choice of the element $\gamma$ such that $\gamma(y) = \infty$.

Unless otherwise stated, we shall always take $f$, $g$, ... to be elements of $PU(1, n; \mathbb{C})$ fixing neither $y$ nor $\infty$. Set

$$\alpha_y(f, z) = \frac{R_f \rho(y, z)}{\rho(z, f^{-1}(y)) \rho(y, f(\infty))}.$$ 

We can write $I_y(f)$ as

$$I_y(f) = \{ z \in \overline{H^n} \mid \alpha_y(f, z) = 1 \}.$$

Put

$$Ext \ I_y(f) = \{ z \in \overline{H^n} \mid \alpha_y(f, z) < 1 \},$$

$$Int \ I_y(f) = \{ z \in \overline{H^n} \mid \alpha_y(f, z) > 1 \},$$

respectively.

Just as in the case of isometric spheres, we have

**Proposition 1.1.**

1. $I_{f(y)}(f) = f(I_y(f)) = I_y(f^{-1})$;
2. $f(Ext \ I_y(f)) \subset Int \ I_y(f^{-1})$;
3. $f(Int \ I_y(f)) \subset Ext \ I_y(f^{-1})$.

Next we consider the location of fixed points of elements.

**Proposition 1.2.** Let $f$ be an element of $PU(1, n; \mathbb{C})$ with fixed point $x$. If $f$ is elliptic or parabolic, then $x$ lies on the isometric sphere $I(f^{-1})$ of $f^{-1}$. If $f$ is loxodromic, then $I(f^{-1})$ does not contain $x$. 
Replacing isometric spheres by generalized isometric spheres leads to the same conclusion as in Proposition 1.2.

Proposition 1.3. Let \( f \) be an element of \( \text{PU}(1,n;\mathbb{C}) \) with fixed point \( x \). If \( f \) is elliptic or parabolic, then \( x \) lies on \( I_y(f) \). If \( f \) is loxodromic, then \( I_y(f) \) does not contain \( x \).

2. Let \( z_1, z_2 \) be two different points in \( H^n \). Let \( E(z_1, z_2) \) be the bisector of \( \{z_1, z_2\} \), that is,

\[
E(z_1, z_2) = \{w \in H^n \mid d(z_1, w) = d(z_2, w)\}
\]

(see [5] for details). Let \( G \) be a discrete subgroup of \( \text{PU}(1,n;\mathbb{C}) \) and let \( w \) be any point of \( H^n \) that is not fixed by any element of \( G \) except the identity. The Dirichlet polyhedron \( D(w) \) for \( G \) with center \( w \) is defined by

\[
D(w) = \bigcap_{g \in G \setminus \{\text{id}\}} H_g(w),
\]

where \( H_g(w) = \{z \in H^n \mid d(z, w) < d(z, g(w))\} \). We observe that

1. \( D(w) \) is not necessarily convex,
2. \( D(w) \) is star-shaped about \( w \),
3. \( D(w) \) is locally finite

(see [2], [4], [11] and [12]).

Let \( \Omega(G) \) be the ordinary set of \( G \). Assume that \( \infty \in \Omega(G) \) and its stability subgroup \( G_{\infty} = \{\text{identity}\} \). Then there is a positive constant \( M \) such that \( \rho(0, g(\infty)) \leq M \) for any element \( g \) of \( G \). The same argument as in [4] leads to the following results.

1. The radii of isometric spheres are bounded above.
2. The number of isometric spheres with radii exceeding a given positive quantity is finite.
3. Given any infinite sequence of distinct isometric spheres of elements of \( G \), the radii being \( R_{g_1}, R_{g_2}, \ldots \), then \( \lim_{m \to \infty} R_{g_m} = 0 \).

We show that the generalized isometric sphere \( I_y(f) \) is closely related to the bisector \( E(z, f^{-1}(z)) \).

Proposition 2.1. If \( z \in H^n \) converges to \( y \in \partial H^n \), then \( E(z, f^{-1}(z)) \) converges to \( I_y(f) \).

By using generalized isometric spheres, we can construct a fundamental domain.

Theorem 2.2. Let \( G \) be a discrete subgroup of \( \text{PU}(1,n;\mathbb{C}) \). Let \( \infty \) be a point of \( \Omega(G) \) and let \( G_{\infty} = \{\text{identity}\} \). Suppose that \( y \) is a point of \( \Omega(G) \cap \partial H^n \) and that \( G_y \) consists only of the identity. Then

\[
P_y(G) = \bigcap_{f \in G \setminus \{\text{id}\}} \text{Ext} I_y(f)
\]
is a fundamental domain for $G$.

By Proposition 2.1 and Theorem 2.2, we obtain

Theorem 2.3. Let $G$ be a discrete subgroup of $PU(1,n; C)$. Let $z \in H^n$ and let $y \in \partial H^n \cap \Omega(G)$. Then $D(z) \rightarrow P_y(G)$ as $z \rightarrow y$.

From the manner of constructing $P_y(G)$, we have

Corollary 2.4. The fundamental domain $P_y(G)$ is locally finite.

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Okayama University of Science
1-1 Ridai-cho, Okayama 700-0005 JAPAN
e-mail:kamiya@mech.ous.ac.jp