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Lectures on Pleating Coordinates for Once Punctured Tori

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Preface

Pleating coordinate theory is a novel approach to understanding deformation spaces of holomorphic families of Kleinian groups, introduced in recent years by the author and Linda Keen. The key idea is to study deformation spaces via the internal geometry of the associated hyperbolic 3-manifold, in particular, the geometry of the boundary of its convex core. This allows one to relate combinatorial, analytical and geometrical data in hitherto unobserved ways. One important outcome is to give algorithms enabling one to compute the exact position of the deformation space, as a subset in $\mathbb{C}^n$. The idea is loosely similar to finding the Mandelbrot set by drawing its external rays. It is based on the observation that there is a close link between the geometry of boundary of the convex core and the complex analytic trace or length function of its bending lamination: a geodesic axis is a bending line implies that the corresponding group element has real trace. In these lectures, we develop the theory as it relates to once punctured tori. We show that,from this simple starting point, one can give a complete description of the position of the pleating varieties, that is, the loci on which the projective class of the bending measure of each of the two components of the convex hull boundary is fixed. We then discuss how this enables one to compute an arbitrarily accurate picture of the parameter space for various one dimensional families of groups, and conclude with a detailed description of how to compute the exact image of any embedding of the space of once punctured torus groups into PSL$(2,\mathbb{C})$.

The lectures on which these notes are based were given in Osaka City University in July 1998. They are an exposition of material which has been developed in a series of papers by the author and L.Keen. We have not altered the informal style of the lectures: this account is intended as a short user friendly guide. There are certainly many inaccuracies, some deliberate in the interests of brevity and some inadvertent. Detailed proofs are to be found in the papers of Keen and Series, especially in [KS98] and [KS93]. Useful background may also be found in an earlier series of lectures given by the author in Seoul, Korea [Se92]. Since these lectures were given we have revised the preprint [KS98] to correct a gap in the proof of the limit pleating theorem 3.11, and to give a shortened proof of the real length lemma 3.8. These changes have been incorporated into these notes. Since otherwise the two versions are largely the same, we refer mainly to the original version [KS98]. Where there is substantial difference, we refer to the revised version as [KS98a].

The computer graphics have been done at various times by various people, notably David Wright, Ian Redfern and Peter Liepa. We thank them for permission to include them here. The author would especially like to thank Yohei Komori for organizing the Osaka conference to give her the opportunity of presenting this work, and Hideki Miyachi, without whose help the notes would probably not have seen the light of day. Most of all, it is a pleasure to thank Komori for his untiring interest in all aspects of this work.

Contents

Lecture 1: Introduction and discussion of several examples

In this lecture we introduce quasifuchsian space for once punctured tori and describe the general problem we aim to solve in these notes. We give examples of some families of Kleinian groups we shall be studying and discuss the Mumford-Wright exploration of parameter space which provided the original motivation for our approach. We conclude with a brief introduction to the hyperbolic convex hull.

The general setting for these lectures is that of a *holomorphic family of Kleinian groups.* Recall that a Kleinian group G is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Its action on the Riemann sphere $\hat{\mathbb{C}}$ decomposes into the regular set Ω, on which the elements of G act properly discontinuously and form a normal family, and the limit set $\Lambda = \hat{\mathbb{C}} - \Omega$ on which the G-action is minimal, that is, on which every orbit is dense. By the *Ahlfors Finiteness Theorem,* if G is finitely generated then Ω/G is a finite union of Riemann surfaces of finite genus with finitely many punctures. In these lectures we concentrate especially on *quasifuchsian once punctured torus groups.* For these groups Ω has exactly two connected components, Ω^+ and Ω^-, each of which is G-invariant and simply connected, such that $\Omega^{\pm}/G$ are both punctured tori. The limit set Λ is a topological circle. Such a group G is a free group on two generators A, B whose commutator $[A,B] = ABA^{-1}B^{-1}$ is necessarily parabolic. The generators are represented by generating loops α, β on $\Omega^{\pm}/G$ so that $\langle \alpha, \beta\rangle = \pi_1(\Omega^{\pm}/G)$. (Note however that the relative orientation of α and β on Ω^+/G and Ω^-/G is opposite.)

By *Bers' Simultaneous Uniformization Theorem,* given any two (marked) complex structures $\omega^{\pm}$ on a once punctured torus, there exists a quasifuchsian once punctured torus group G for which $\Omega^+/G = \omega^+$, $\Omega^-/G = \omega^-$. This group is unique up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$.

A *holomorphic family* of finitely generated Kleinian groups $G = G(\xi)$, $\xi \in \mathbb{C}^n$, is a family of Kleinian groups $G = \langle g_1(\xi), \dots, g_k(\xi)\rangle$ for which the generators $g_i(\xi)$ are holomorphic functions of ξ on some open set $U \subset \mathbb{C}^n$. By a result of Sullivan, if $U \subset \mathbb{C}^n$ is open and all the representations $G_0 \to G(\xi)$ are faithful (for some fixed group $G = \langle g_1^0, \dots, g_k^0\rangle)$, then $G(\xi)$ is quasi-conformally equivalent to G_0. In the case of quasifuchsian once punctured torus groups, after correct normalization, we find $n = 2$. This corresponds to the fact that the Bers parameters $\omega^{\pm}$ are each points in the upper half plane $\mathbb{H}$, the classical Teichmüller space of the unpunctured torus, which we shall always denote by $\mathcal{T}$. We denote a more general holomorphic family by $\mathrm{Def}(G)$.

Exercise Do a dimension count on $G = \langle A, B \mid [A, B] \text{ is parabolic } \rangle$ to "verify" $n = 2$ is correct.

The Problem In these notes, $\mathcal{QF}$ always refers to the space of once punctured torus groups. Our aim in these lectures is to solve the following problem:

Given some specific set of holomorphic parameters $\xi \in \mathbb{C}^2$ for groups

$$G = G(\xi) = \langle A, B \mid [A, B] \text{ is parabolic}\rangle,$$

describe exactly how to compute quasifuchsian space

$$\mathcal{QF} = \{\xi \in \mathbb{C}^2 \mid G(\xi) \text{ is a quasifuchsian once punctured torus group}\} \subset \mathbb{C}^2.$$

In particular, find $\partial\mathcal{QF} \subset \mathbb{C}^2$.

By Bers' theorem, we know that $\mathcal{QF}$ is biholomorphically equivalent to $\mathbb{H} \times \mathbb{H}$. However this gives no information about the shape of $\mathcal{QF}$ in $\mathbb{C}^2$. We have two further useful pieces of information, namely the position of *Fuchsian space* $\mathcal{F}$ for which $\omega^+ = \overline{\omega^-}$ (the complex conjugate of ω^-), $\Omega^\pm$ are round discs and Λ is a round circle; and the nature of a dense set of boundary points of $\mathcal{QF}$ called *cusps.* Before discussing these further, let us look at some specific examples of the kinds of holomorphic parameters we have in mind.

Jørgensen Parameters for $\mathcal{QF}$. One can normalize so that

$$A = \begin{pmatrix} u - v/w & v/w^2 \\ u & v/w \end{pmatrix} \quad B = \begin{pmatrix} v - u/w & -v/w^2 \\ -v & u/w \end{pmatrix} \quad [A, B] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

where $u, v, w \in \mathbb{C}$ with $u^2 + v^2 + w^2 = uvw$. This relation is called the *Markoff equation* and follows from the trace identities in $\mathrm{PSL}(2, \mathbb{C})$. In this case $G = \langle A, B\rangle$ is Fuchsian (and hence in particular discrete) if and only if $u, v, w \in \mathbb{R}$. Note $u = \mathrm{Tr}\, A$, $v = \mathrm{Tr}\, B$, $w = \mathrm{Tr}\, AB$.

Button Parameters. A variant on the above is the following parameterization (J.Button, Warwick Ph.D.thesis 1995).

$$A=\begin{pmatrix}(1+z^2)/w & z\\ z & w\end{pmatrix}\quad B=\begin{pmatrix}(1+w^2)/z & -w\\ -w & z\end{pmatrix}\quad [A,B]=\begin{pmatrix}-1 & -u\\ 0 & -1\end{pmatrix}$$

Here $u=2(1+z^2+w^2)/zw, z, w\in\mathbb{C}$ and once again, G is Fuchsian if and only if $z,w\in\mathbb{R}$.

<u>**The Earle slice of $\mathcal{QF}$.**</u> ([KoS98a]) This is a one-complex dimensional slice of $\mathcal{QF}$ in which Ω^+, Ω^- are required to be conformally isomorphic under the rhombic symmetry Θ which sends $A\to B$, $B\to A$. It extends the rhombus line $|\tau|=1$ in the classical upper half plane picture of the Teichmüller space of a torus holomorphically into $\mathcal{QF}\subset\mathbb{C}^2$. The parameterisation is:

$$A=\begin{pmatrix}\frac{d^2+1}{d} & \frac{d^3}{2d^2+1}\\ \frac{2d^2+1}{d} & d\end{pmatrix}\qquad B=\begin{pmatrix}\frac{d^2+1}{d} & -\frac{d^3}{2d^2+1}\\ -\frac{2d^2+1}{d} & d\end{pmatrix}$$

Here $d\in\mathbb{C}$. The conformal involution Θ is normalised so that $\Theta(z)=-z$. We have $\Theta A\Theta^{-1}=B$ and once again, $G\in\mathcal{F}$ if and only if $d\in\mathbb{R}$. We shall come back to this example in lecture 4.

<u>**The Maskit embedding of $\mathcal{T}$.**</u> This is a 1-dimensional holomorphic slice on $\partial\mathcal{QF}$ consisting of groups for which the generator A is pinched to a parabolic (a so called *cusp* group). This is the slice whose study led to the first results on pleating coordinates in [KS93]. It was first introduced by David Wright in [Wr88].

$$A=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\qquad B=-\begin{pmatrix}i\xi & i\\ i & 0\end{pmatrix}\qquad \xi\in\mathbb{C}$$

Here Ω^+/G is a once punctured torus while Ω^-/G is a 3-times punctured sphere. Since the Teichmüller space of a 3-times punctured sphere is a single point, we have $\mathrm{Def}(G)=\mathcal{T}=\mathbb{H}$. The parameters were chosen so that the map $(\mathbb{H},\tau)\to(\mathrm{Def}(G),\xi)$ should take the simplest possible form. This is discussed further in 1.3 below.

1.1 Exploration of $\mathcal{QF}$ and the Mumford-Wright Programme.

In the early 1980's, David Mumford, David Wright and Curt McMullen embarked on computer explorations of $\mathcal{QF}$. In particular, they plotted many limit sets and looked for *cusp groups* on $\partial\mathcal{QF}$. A *cusp* is a group in which an element representing a simple (non-self intersecting) curve on the torus becomes parabolic. One can think as moving towards a cusp on $\partial\mathcal{QF}$ as the process of shrinking a simple closed loop on one or other of the surfaces $\Omega^{\pm}/G$. (This usage is not to be confused with a *cusp* in the sense of a puncture on a hyperbolic surface; in the one case it is a missing point and in the other, by extension, it refers to the whole group.) Later, McMullen proved that cusps are dense on the boundary of every Bers slice in $\mathcal{QF}$, [McM91]. David Wright made a more systematic study of the Maskit embedding $\mathcal{M}$ described above. His plan was:

- Enumerate homotopy classes of simple closed curves on the once punctured torus.
- Find representatives of these curves as elements in G and compute their traces as functions of ξ.
- Find points where the traces are ± 2 (parabolics).

Note the problem with the last point: there may be many places where an element is parabolic, but we cannot conclude that the group is necessarily on $\partial\mathcal{QF}$ or $\partial\mathcal{M}$. In general, such a group may not even be discrete.

Since Wright's enumeration of curves underpins much of what we are about to do, we explain it briefly here. Let S denote a (topological) unpunctured torus and Σ a torus with a puncture. Both have marked generators A, B. The fundamental group $\pi_1(S)$ is the free abelian group $\mathbb{Z}^2$ while $\pi_1(\Sigma)$ is F_2, the free group on two generators. For each $p/q \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ (we allow $q = 0 \leftrightarrow \infty \in \hat{\mathbb{Q}}$), the homotopy class $A^{-p}B^{q}$ represents a simple closed loop on S. This loop is also simple on Σ and hence corresponds to some element (conjugacy class) $W_{p/q}$ in $\pi_1(\Sigma)$. By considering the action of the mapping class group on S and Σ, one can show that all simple homotopy classes on Σ arise in this way. The arrangement of these loops is shown in figure 1, in which $\hat{\mathbb{Q}}$ is represented as S^1. The point p/q represents the curve

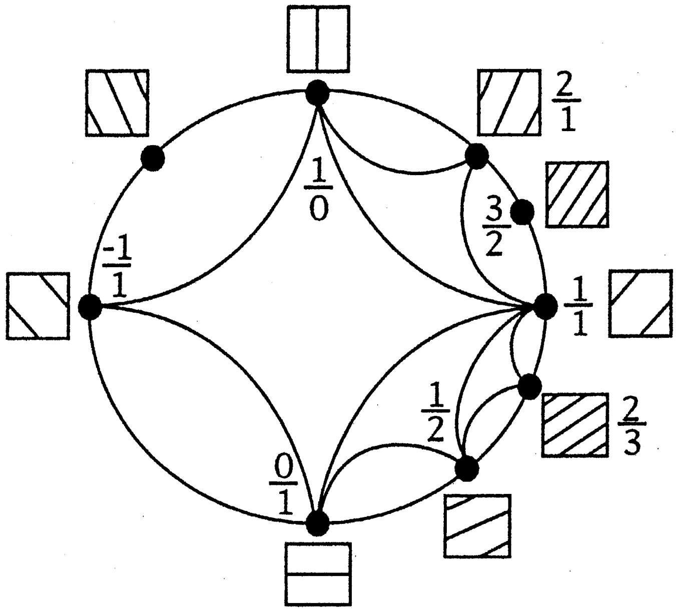


Figure 1:

$A^{-p}B^q$ on S, which we can think of as a line of rational slope in the plane projected onto S.

<u>**Exercise**</u> Find the slope on $\mathbb{R}^2$ of line which projects to $A^{-p}B^q$.

Remark It is well known that on a hyperbolic surface, each free homotopy class contains a unique geodesic. Therefore, given a hyperbolic metric on Σ, these classes represent exactly the simple closed geodesics of Σ.

Notice that successive p, q curves can be enumerated by *Farey addition*

$$\frac{p}{q} \oplus_F \frac{r}{s} = \frac{p+r}{q+s}, \text{ whenever } ps - rq = \pm 1.$$

Wright showed that cyclically reduced words in F_2 corresponding to $A^{-p}B^q$ could be found inductively by the following process, see also [KS93]).

$$W_{0/1} = B, \quad W_{1/1} = A^{-1}B, \quad W_{1/0} = W_\infty = A^{-1}.$$
$$W_{(p+r)/(q+s)} = W_{r/s}W_{p/q} \text{ if } ps - rq = -1.$$

Note the unexpected order in the definition of $W_{(p+r)/(q+s)}$.

Using the trace identity $\operatorname{Tr} XY = \operatorname{Tr} X \operatorname{Tr} Y - \operatorname{Tr} XY^{-1}$ (which holds for any three matrices in $\mathrm{SL}(2, \mathbb{C})$), one can show that:

- $\operatorname{Tr} W_{p/q}$ is a polynomial of degree q in ξ.
- $\operatorname{Tr} W_{p/q} = (-i)^q(\xi - 2p/q)^q + O(\xi^{q-2})$, where $O(\xi^{q-2})$ denotes terms of order $\leq q-2$.

<u>**Exercise**</u> Do this. (See [KS93, §3.2].)

Thus in general, the equation for the cusp group in which $W_{p/q}$ is pinched is $\operatorname{Tr} W_{p/q}(\xi) = \pm 2$. This has $2q$ roots, of which, however, only *one* is a discrete group on $\partial\mathcal{M}$ [KMS93]. (Actually two, since to get a unique copy of $\partial\mathcal{M}$ we should normalize with $\operatorname{Im}\xi > 0$, see 1.3 below.) In the special case $q = 1$, however, there is a unique root with $\operatorname{Im}\xi > 0$; these are the points $\xi = 2n + 2i$, $n \in \mathbb{Z}$ and correspond to cusps in which both A and $A^{-n}B$ are parabolic (so Ω^+/G and Ω^-/G are *both* 3-times punctured spheres). At the point $\xi = 2n + 2i$, notice that $\operatorname{Tr} W_{n/1}(\xi) = 2$.

Wright plotted these points and then proceeded to find roots of $\operatorname{Tr} W_{p/q}(\xi) = 2$ by Newton's method and interpolation, using the recursion described above. The result is shown in Figure I: it looks very like a boundary $\partial\mathcal{M}$!

He also made pictures of the limit sets of these special groups, see Figure II. Notice the two families of black and white circles, which correspond to the two thrice punctured sphere subgroups in Ω^+/G and Ω^-/G. These pictures were the starting point of [KS93]. After much computation and exploration, Keen and the author proposed plotting the branches of $\operatorname{Tr} W_{p/q} > 2$, $\operatorname{Tr} W_{p/q} \in \mathbb{R}$ moving away from the cusp. The result is shown in Figure III. Corresponding limit sets are shown in Figures IV and V in which you can see that the tangent circles in Figure II have opened so they now overlap. Notice that the real trace lines of Figure III have remarkable properties, which would certainly not be expected of the real loci of an arbitrary family of polynomials (or even this family if the lines through other solutions to Trace $= \pm 2$ were chosen.) In particular:

1. they are pairwise disjoint;
2. they end in "cusps";
3. they contain no critical points;
4. they are asymptotic to a fixed direction at ∞;
5. they appear to be dense in the presumed parameter space $\mathcal{M}$.

At this stage none of these properties could be either explained or proven. The key turned out to be to study the action of G on hyperbolic 3-space $\mathbb{H}^3$, in particular, on the boundary of the convex hull. This also eventually led to our method of drawing the parameter space $\mathcal{QF}$.

For the rest of this lecture we shall discuss this convex hull.

1.2 The Boundary of the Hyperbolic Convex Hull

Recall that a Kleinian group G acts not only on the Riemann sphere $\hat{\mathbb{C}}$ but also on on hyperbolic 3-space $\mathbb{H}^3$, which can be regarded as the interior $\mathbb{B}^3$ of the Riemann sphere $\hat{\mathbb{C}}$. The quotient $\mathbb{H}^3/G$ is a hyperbolic 3-manifold; in the case of a quasifuchsian once punctured torus group, it is homeomorphic to $\Sigma \times (0,1)$. The surfaces $\Omega^{\pm}/G$ compactify the 2-ends of $\mathbb{H}^3/G$ so that $(\Omega \cup \mathbb{H}^3)/G \simeq \Sigma \times [0,1]$.

The *convex hull* or *convex core* $\mathcal{C}$ (Nielsen region) of $\mathbb{H}^3/G$ is the smallest hyperbolic closed set containing all closed geodesics of $\mathbb{H}^3/G$. If G is Fuchsian, all of these are contained in a single flat plane, otherwise we get the picture shown in figure 2.

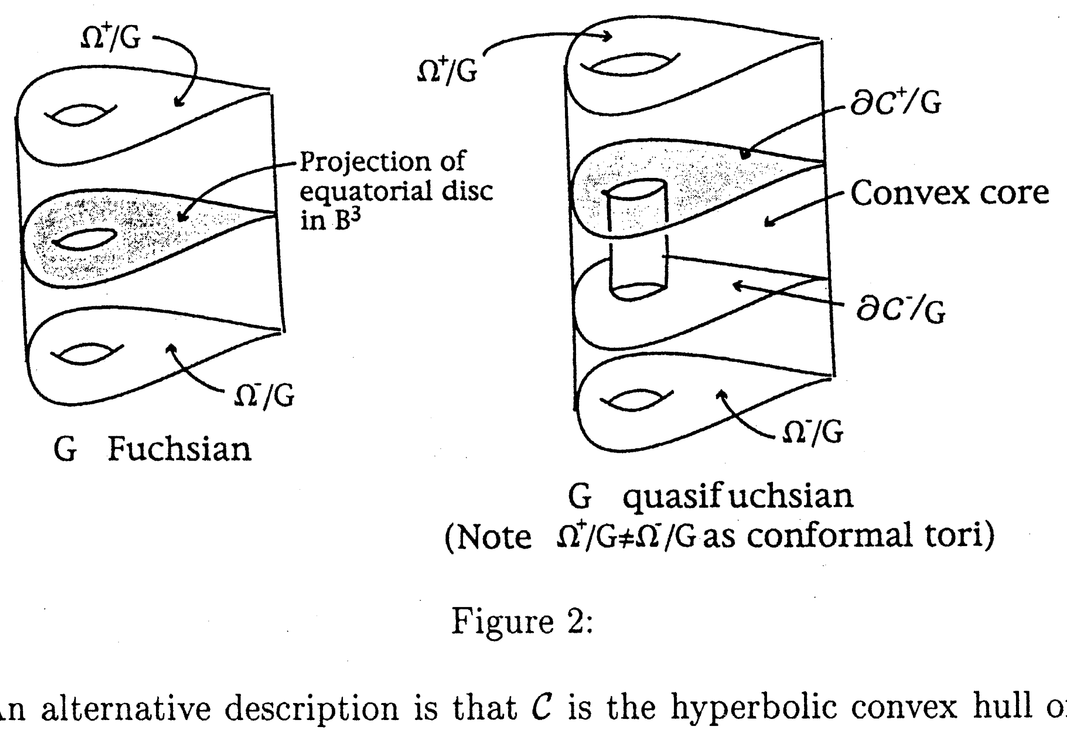


Figure 2:

An alternative description is that $\mathcal{C}$ is the hyperbolic convex hull of the limit set Λ, shown in figure 3.

We see from either picture that $\partial\mathcal{C}$ has two components $\partial\mathcal{C}^{\pm}$ which "face" the components $\Omega^{\pm}$ of Ω; one proves that $\partial\mathcal{C}^{\pm}/G$ are each homeomorphic to

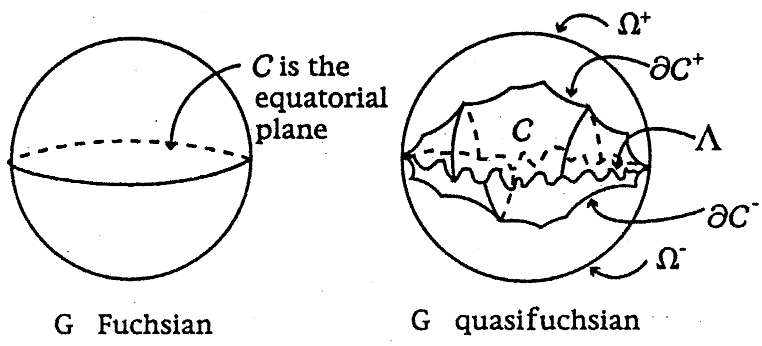


Figure 3:

punctured tori, [KS95].

Since C is convex, ∂C is made up of convex pieces of flat hyperbolic planes which meet along geodesics called *pleating* or *bending lines.* Since C is the convex hull of $\Lambda \subset \hat{\mathbb{C}}$, the flat faces are ideal polygons and the bending lines continue out to $\hat{\mathbb{C}}$. The bending lines are mutually disjoint. For more details about all this, see [EM87] and also lecture 3. As described in more detail in lecture 3, the bending lines project to a *geodesic lamination* on $\partial C^{\pm}/G$ which carries a transverse measure, called the *bending measure*, denoted $p\ell^{\pm}(G)$. We shall be especially interested in the case in which the bending lines all project to one simple closed curve on the torus; by the discussion in 1.1 this must be the projection of the axis of $W_{p/q}$ for some $p/q \in \hat{\mathbb{Q}}$. In this case the bending measure is given by the bending angle θ between the planes which meet along $\mathrm{Ax}(W_{p/q})$:

$$p\ell^{\pm}(G)(T) = i(\gamma_{p/q}, T)\theta$$

where $\gamma_{p/q}$ is the closed geodesic in question, T is a transversal and $i(\gamma_{p/q}, T)$ its intersection number with $\gamma_{p/q}$. This is the case to keep in mind.

Key Lemma 1.1. ([KS93, lemma 4.6]) *Suppose that the axis of $g \in G$ is a bending line of $\partial C^{\pm}(G)$. Then* $\mathrm{Tr}(g) \in (-\infty, -2) \cup (2, \infty)$*; in other words, g is* purely hyperbolic.

Proof. Use the fact that the two planes in $\partial C^{\pm}$ which meet along $\mathrm{Ax}(g)$ are invariant under translation by g. See also lecture 2 for another picture. □

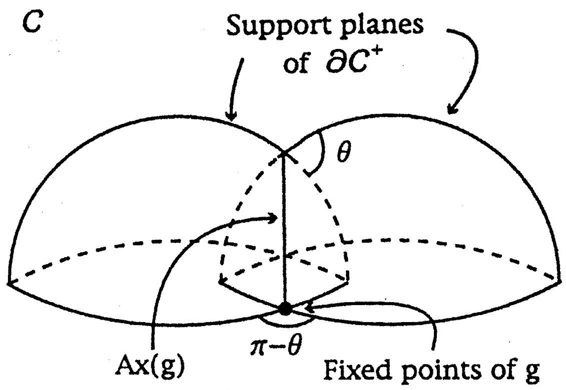


Figure 4:

Key Definition 1.2. *The $(p/q, r/s)$- pleating ray or pleating variety $\mathcal{P}_{p/q,r/s}$ is*

$$\mathcal{P}_{p/q,r/s} = \{\xi \in \mathcal{QF} \mid |p\ell^+(\xi)| = p/q, |p\ell^-(\xi)| = r/s\}.$$

Thus $\mathcal{P}_{p/q,r/s}$ is the set of groups in $\mathcal{QF}$ for which the support $|p\ell^{\pm}(\xi)|$ of the bending measures (i.e. the bending lines) are the geodesics $\gamma_{p/q}$ and $\gamma_{r/s}$ which correspond to the special words $W_{p/q}$, $W_{r/s}$. (Notice here p/q and r/s are arbitrary points in $\hat{\mathbb{Q}}$; we are not assuming $ps - rq = \pm 1$.) The terminology "plane" will be justified by the picture of $\mathcal{QF}$ we establish in these lectures: $\mathcal{P}_{p/q,r/s}$ is indeed a 2-real dimensional submanifold in $\mathbb{C}^2 \simeq \mathbb{R}^4$.

In the special case of the Maskit embedding, the accidental parabolic A acts as the bending line on Ω^-/G, so that $|p\ell^-(\xi)| \equiv \infty$. In this case we define

$$\mathcal{P}_{p/q} = \{\xi \in \mathcal{M} \mid \partial C^+/G \text{ is pleated (bent) along } p/q\},$$

Clearly, since $\xi \in \mathcal{P}_{p/q} \Rightarrow \operatorname{Tr} W_{p/q} \in (-\infty, -2) \cup (2, \infty)$, we have $\mathcal{P}_\infty = \emptyset$. In general, from the above discussion we have learned:

- $\operatorname{Tr} W_{p/q}(\xi)$ is a polynomial of degree q in ξ $(q \neq 0)$.
- $\mathcal{P}_{p/q}$ is contained in the real locus of $\operatorname{Tr} W_{p/q}$.

Theorem 1.3. ([KS93, theorems 5.1 and 7.1]) *The "real trace" lines described in the Wright picture of $\mathcal{M}$ above, are exactly the pleating rays $\mathcal{P}_{p/q}$ for $q \neq 0$. These lines have all the properties described above; in particular, they contain no critical points of* $\operatorname{Tr} W_{p/q}$ *and they are dense in $\mathcal{M}$. They terminate in the (unique) cusp groups at which* $\operatorname{Tr} W_{p/q} = \pm 2$.

This fully justifies Wright's construction of the boundary of $\mathcal{M}$ described above. Furthermore, if the space of simple closed curves is completed to the Thurston space of projective measured laminations S^1 (see lecture 3), then the above results extended to the irrational pleating varieties $\mathcal{P}_\nu$, $\nu \in S^1$.

The proof of all these claims will be given in lecture 4.

1.3 Appendix

The reason for David Wright's choice of parameterization for the Maskit slice $\mathcal{M}$, and the explanation of our statement that the map $(\mathbb{H}, \tau) \to (\mathcal{M}, \xi)$ is "nice", can be understood with the help of Maskit combination theorems. With Wright's normalization, the matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad B^{-1}AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a Fuchsian group representing 2 thrice punctured spheres, one the quotient of the upper half plane $\mathbb{H}$ and the other of the lower half plane $\mathbb{L}$. Adjoining the element $B : z \mapsto \xi + 1/z$ makes a "handle" on one side (this is Kra's plumbing construction, see section 6.3 of [Kr90]). If $\operatorname{Im} \xi >> 0$, we get the following picture:

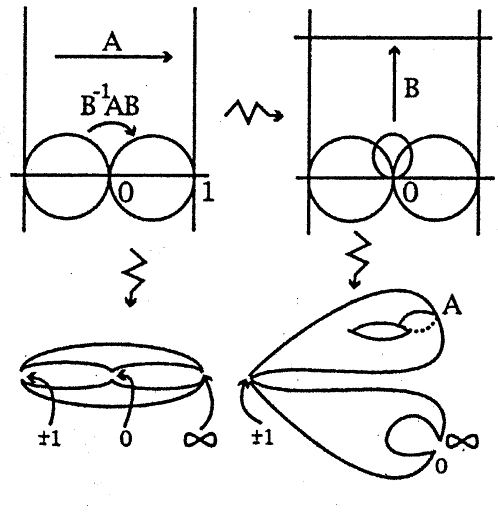


Figure 5:

One verifies that B carries the horocycle of Euclidian radius $1/2t$ to the horizontal horocycle of $\operatorname{Im} \xi = t$. We get an obvious fundamental domain for G if $\operatorname{Im} \xi > 2$. Moreover, if $\operatorname{Im} \xi < 1$, one can show that G is not discrete [Wr88]. Thus the boundary of $\mathcal{M}$ is contained in the strip $1 \leq \operatorname{Im} \xi \leq 2$.

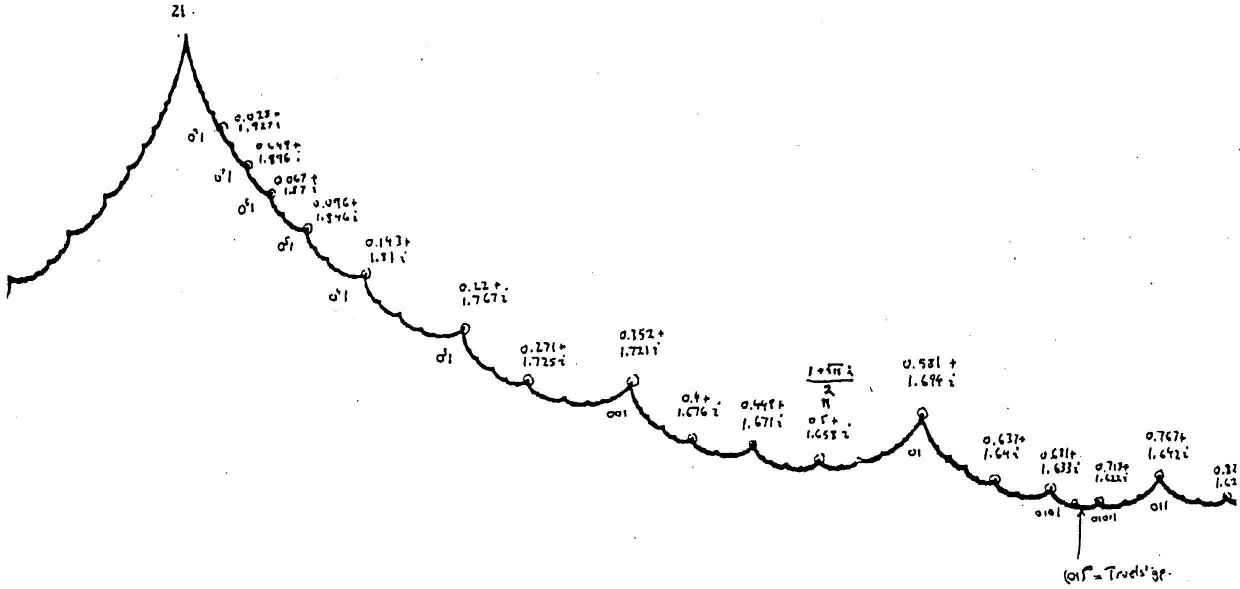


Figure I. The original Mumford-Wright picture of $\partial\mathcal{M}$.

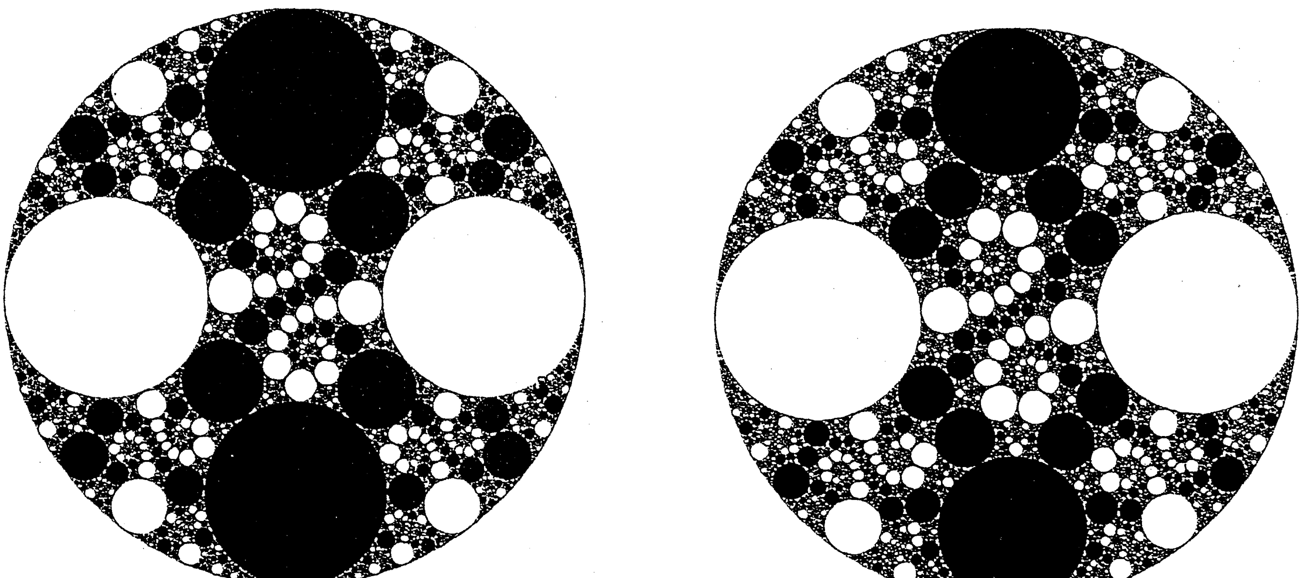


Figure II. Limit sets of cusp groups. The two different circle packings correspond to pinching different pairs of curves.

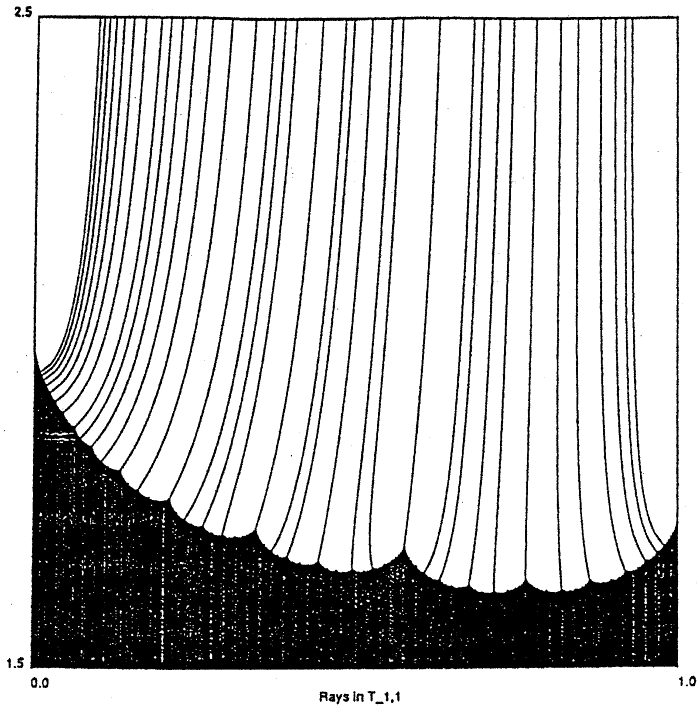
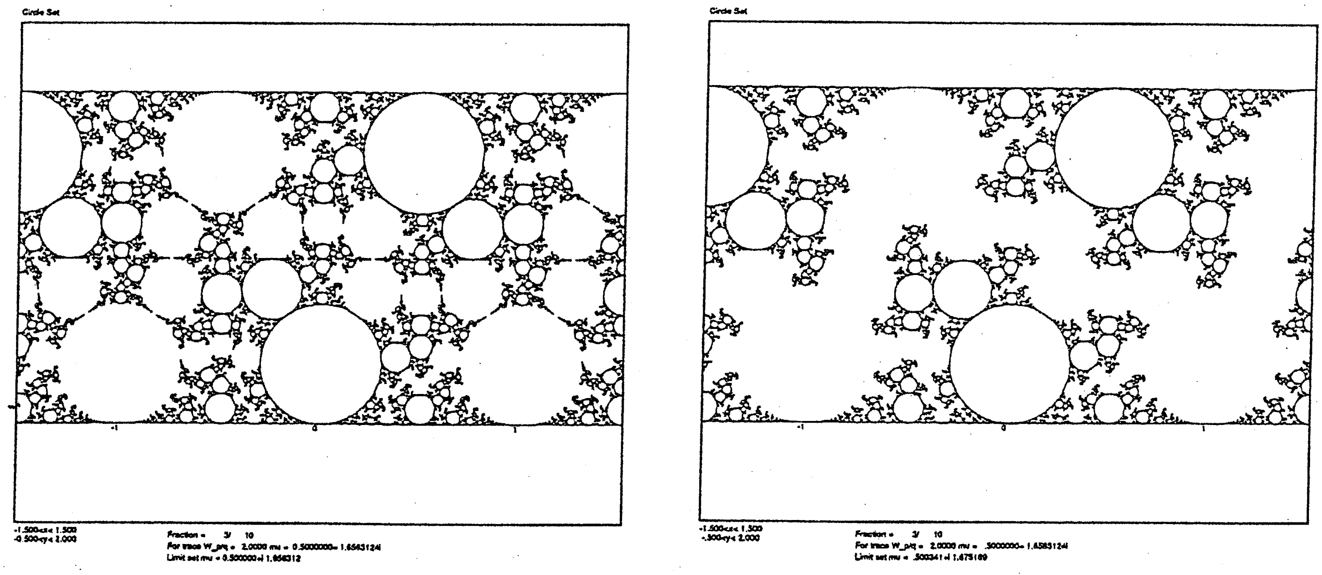


Figure III. The real trace lines.



Figures IV and V. Opening up cusps along the 3/10 ray.

Lecture 2: Convex Hull Boundaries with Rational Pleating Locus

In this lecture we look at the convex hull boundary in the special case in which the bending lines are simple closed geodesics. We review Fenchel Nielsen coordinates for the once punctured torus and their extension to complex Fenchel Nielsen coordinates for $\mathcal{QF}$. We discuss the important bending away theorem *which allows one to determine the bending or pleating locus for groups obtained by small quakebends away from Fuchsian space $\mathcal{F}$.*

Suppose G is a quasifuchsian group and Σ is a hyperbolic surface such that $\mathbb{H}^3/G \sim \Sigma \times (0,1)$. Fix Ω_0 a component of the regular set Ω and $\partial\mathcal{C}_0$ the component of the convex hull boundary facing Ω_0. Then Ω_0/G and $\partial\mathcal{C}_0/G$ are both homeomorphic to the surface Σ.

Let $\mathcal{S}(\Sigma)$ denote the set of (free homotopty classes of) simple closed non-boundary parallel curves on Σ. Assume that the pleating locus of $\partial\mathcal{C}_0$ consists entirely of curves (geodesics) in $\mathcal{S}(\Sigma)$. We call this a *rational pleating lamination.* Generically such a lamination decomposes $\partial\mathcal{C}_0/G$ into pairs of pants Π_i. Each Π_i is flat and so lifts to a piece of hyperbolic plane $\tilde{\Pi}_i$ whose extension meets $\hat{\mathbb{C}} = \partial\mathbb{H}^3$ in a circle $C(\tilde{\Pi}_i)$. If a geodesic γ is in the boundary of Π_i then $\mathrm{Ax}(\tilde{\gamma})$ is in the boundary of (a conjugate of) $C(\tilde{\Pi}_i)$ and hence fixes this circle and is purely hyperbolic, i.e. $\mathrm{Tr}\,\tilde{\gamma} \in (-\infty,-2) \cup (2,\infty)$. c.f. the key lemma 1.1 in lecture 1.

Lemma 2.1. *1. In the above situation, one or other of the two open discs bounded by $C(\tilde{\Pi}_i)$ has empty intersection with the limit set $\Lambda = \Lambda(G)$.*

2. Let $\Gamma_i = \pi_1(\Pi_i)$. Then $\Lambda(G) \cap C(\tilde{\Pi}_i) = \Lambda(\Gamma_i)$. In fact, $\tilde{\Pi}_i$ is just the Nielsen region of Γ_i.

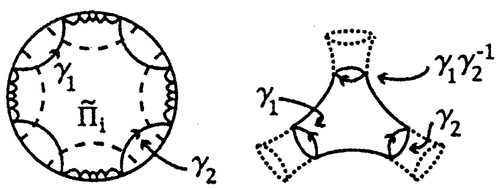


Figure 6:

Proof. Exercise, see [KS98, §4.3] and [KS94, §3]. The second part is illustrated in figure 6. □

What about the converse? The following is an easy exercise, see [KS98, lemma 4.1] and [KS94, lemma 3.2].

Lemma 2.2. *If* $\operatorname{Tr}\gamma_1$, $\operatorname{Tr}\gamma_2$ *and* $\operatorname{Tr}\gamma_1\gamma_2$ *are all real, then* $\Gamma = \langle\gamma_1, \gamma_2\rangle$ *is Fuchsian.*

Lemma 2.3. *Suppose that* $\Gamma \subset G$ *is Fuchsian with the limit set* $\Lambda(\Gamma)$ *contained in a round circle* $C(\Gamma)$. *Then* $\partial\mathcal{C}(\Lambda(\Gamma))$, *the boundary of the convex hull of* $\Lambda(\Gamma)$, *is a component of* $\partial\mathcal{C}(\Lambda(G))$ *iff one of the two discs bounded by* $C(\Gamma)$ *has empty intersection with* $\Lambda(\Gamma)$.

Proof. Exercise, same references as above. □

Definition 2.4. *We call a Fuchsian subgroup as in lemma 2.3* F-peripheral.

An example of a non-peripheral Fuchsian subgroup is shown in Figure VI.

Question How can one tell when a given Fuchsian subgroup is F-peripheral? This is not so easy to answer; a large part of these lectures will involve investigating exactly this point.

2.1 Special Case Example

Besides illustrating what is going on, the following example will come up repreatedly and is essential to the proof of some of our main results.

Let $G = \langle A, B\rangle$ be a once punctured torus group. The complex distance $\delta(A, B)$ between $\operatorname{Ax}(A)$ and $\operatorname{Ax}(B)$ is given by;

$$\sinh^2(\lambda_A/2)\sinh^2(\lambda_B/2)\sinh^2(\delta(A,B)) = -1.$$

Here λ_A is the complex translation length of A and $\operatorname{Tr} A = 2\cosh(\lambda_A/2)$. The proof is an exercise with trace identities, see [PS95, §2].

Thus

$$\operatorname{Tr} A, \operatorname{Tr} B \in \mathbb{R} \Rightarrow -\sinh^2(\delta(A,B)) > 0$$

which imples

$$\operatorname{Re}\delta(A,B) = 0 \text{ or } \operatorname{Im}\delta(A,B) = \pi/2.$$

In the first case $\operatorname{Tr} A, \operatorname{Tr} B$ are coplanar and G is Fuchsian (Why?); in the second the axes do not meet but are perpendicular. In this case G is a degenerate Schottky group obtained by identifying opposite circles as shown; the four points of tangency lie on a rectangle. This is shown in Figure VII, which shows a fundamental domain and how the limit set is formed in this case.

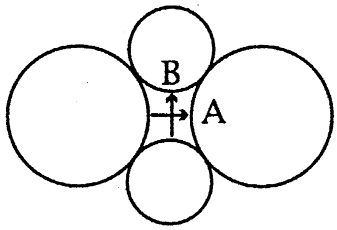


Figure 7:

If $\partial\mathcal{C}^+$ is pleated along $\mathrm{Ax}(A)$, then cutting $\partial\mathcal{C}^+/G$ along the projection of $\mathrm{Ax}(A)$, we obtain a punctured annulus. Lifting to $\mathbb{H}^3$ we get a piece of plane with boundary curves $\mathrm{Ax}(A)$, some conjugate of $\mathrm{Ax}(A)$, and the puncture; and similarly for $\partial\mathcal{C}^-$ and $\mathrm{Ax}(B)$. In fact one can show directly, by studying fundamental domains for G and how they cover Ω, that in this case $\Gamma^+ = \langle A, B^{-1}AB\rangle$ and $\Gamma^- = \langle B, A^{-1}BA\rangle$ are F-peripheral. The details are in [PS95, prop. 6.2]; Figure VII shows the idea.

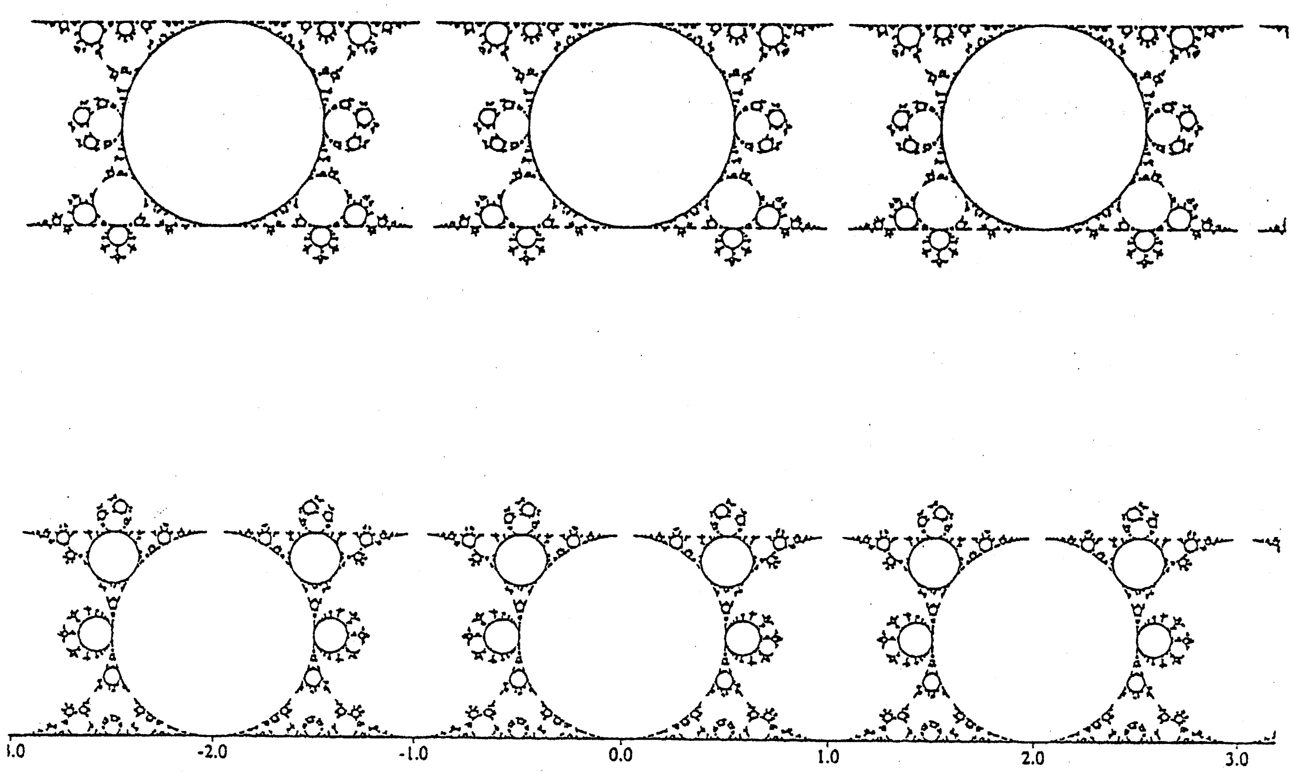


Figure VI. A non-peripheral Fuchsian subgroup. *Graphics by Ian Redfern*
This limit set corresponds to a surface group of genus 2.

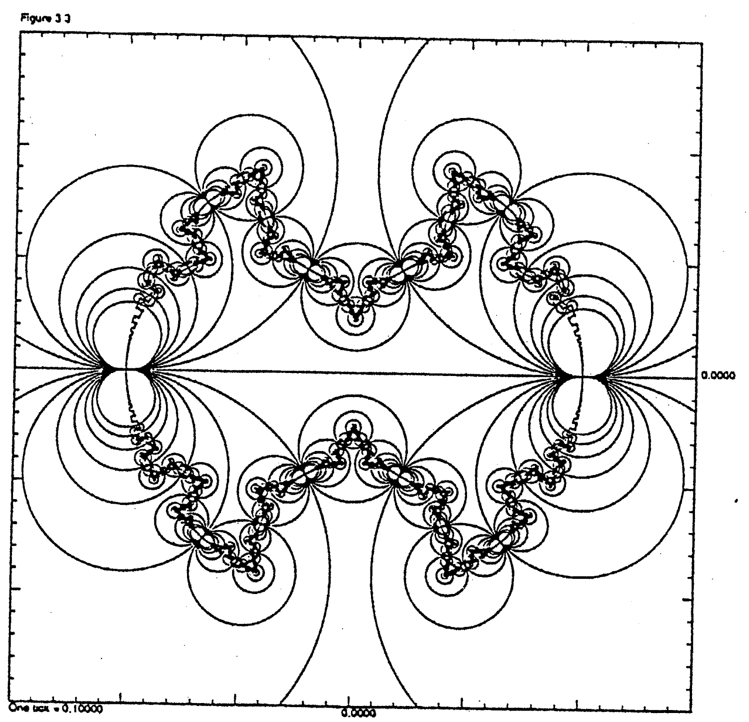


Figure VII. Limit set for the special case example.
Both generators have real trace.

2.2 Real and Complex Fenchel Nielsen coordinates.

For the rest of this lecture, we shall discuss a more general way to ensure that a given Fuchsian subgroup is F-peripheral. First we need to recall *Fenchel-Nielsen* and *complex Fenchel-Nielsen* coordinates for a once punctured torus. These coordinates (for Teichmüller space and quasifuchsian space respectively) are defined relative to a fixed generator pair (U, V) corresponding to geodesics (γ, δ) on the torus Σ. For more detail see [KS97, §4].

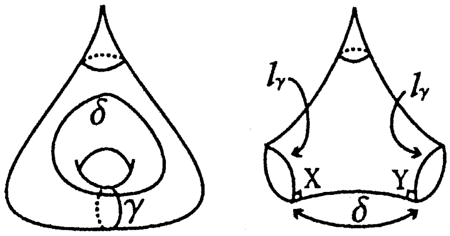


Figure 8:

The right side of figure 8 shows a punctured cylinder whose two boundary curves have equal lengths. This cylinder is shown lifted to $\mathbb{H}$ in figure 9. The conjugate axes of U and $V^{-1}UV$ project to the two boundary curves of the cylinder and are identified by the transformation V, whose axis projects to the curve δ on Σ. Cutting the cylinder along the perpendiculars from the cusp to the two boundary curves gives two pentagons with four right angles and one cusp, which can be thought of as two right angled hexagons with one degenerate side. From hyperbolic trigonometry, the length of the boundary curve l_γ determines such a pentagon up to isometry. The two boundary curves are glued with a twist $t_\gamma \in \mathbb{R}$. To understand the twist better we lift to $\mathbb{H}$; by definition t_γ is the *signed* distance $d(Y, V(X))$ as shown in figure 9.

Theorem 2.5. *The Fenchel Nielsen coordinates* (l_γ, t_γ) *determine* Σ *uniquely up to conjugation in* $\mathrm{PSL}(2, \mathbb{R})$.

Complex F-N coordinates for quasifuchsian once punctured tori [Ta94], [Ko94], [KS97], are made by exactly the same construction but with $\lambda_\gamma \in \mathbb{C}^+ = \{x + iy \mid x > 0\}$ and $\tau_\gamma \in \mathbb{C}$. (Remember that $\mathrm{Tr}\, g_\gamma = 2\cosh(\lambda_\gamma/2)$.) The transformation V glues $\mathrm{Ax}(V^{-1}UV)$ to $\mathrm{Ax}(U)$ with a shear of distance $\mathrm{Re}\, \tau_\gamma$ and a twist (bend) through angle $\mathrm{Im}\, \tau_\gamma$. Notice that the four endpoints of the axes are in general not concyclic.

<u>Exercise</u> Prove that the endpoints of $\mathrm{Ax}(U)$, $\mathrm{Ax}(V^{-1}UV)$ are concyclic iff $\lambda_\gamma \in \mathbb{R}$.

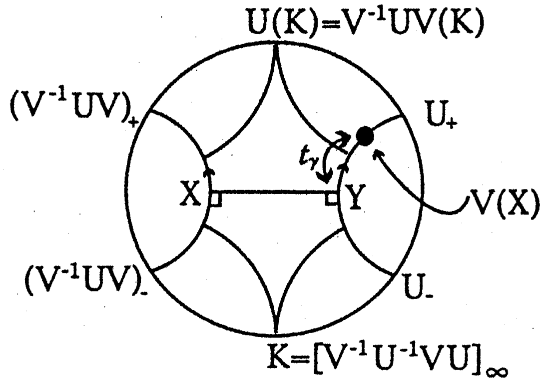


Figure 9:

Remark 2.6. Theorem 2.5 shows that for any $(\lambda_\gamma, \tau_\gamma) \in \mathbb{R}^+ \times \mathbb{R}$, we can write down generators A, B (or U, V) for a group G in which $[A, B]$ is parabolic. Part of the content of theorem 2.5 is that this group is automatically Fuchsian and represents a hyperbolic once punctured torus. In the case $(\lambda_\gamma, \tau_\gamma) \in \mathbb{C}^+ \times \mathbb{C}$, we can still write down generators A, B (or U, V) for the group G. However for general complex parameters, this group may be neither free, discrete, nor quasifuchsian.

Complex Fenchel Nielsen Twists or Quakebends ([KS97, §5]) For $t \in \mathbb{R}$, the time t F-N twist or *earthquake* $\mathcal{E}_\gamma(t)$ along γ is described in F-N coordinates by $(l_\gamma, t_\gamma) \mapsto (l_\gamma, t_\gamma + t)$. Likewise the time τ *complex F-N twist* or *quakebend* $\mathcal{Q}_\gamma(\tau)$ is described in complex F-N coordinates as $(\lambda_\gamma, \tau_\gamma) \mapsto (\lambda_\gamma, \tau_\gamma + \tau), \tau \in \mathbb{C}$. If $\operatorname{Re} \tau = 0$, it is called a *pure bend.* In what follows, we shall be exploring exactly what happens to the convex hull when we perform quakebends.

2.3 Developed Surfaces and the Bending Away Theorem Part 1

In this section we are going to discuss the following theorem, which is a slight variant on [KS97, prop. 7.2].

Theorem 2.7 (Bending away theorem Part 1). *Let (l_γ, t_γ) be the F-N coordinates of a Fuchsian once punctured torus group G_0. Then for small θ, the groups with complex F-N coordinates $(l_\gamma, t_\gamma + i\theta)$ are in $\mathcal{P}_\gamma$ (i.e. have pleating locus γ on one or other side of $\partial\mathcal{C}$.)*

To prove this theorem, we need to study the *developed surface* associated to a complex F-N twist.

The Developed Surface Say $\lambda_\gamma \in \mathbb{R}$ so the endpoints of $\mathrm{Ax}(U)$ and $\mathrm{Ax}(V^{-1}UV)$ are concyclic. The Nielsen region N of $\Gamma = \langle U, V^{-1}UV \rangle$ maps to a convex part of a hyperbolic plane in $\mathbb{H}^3$. The image $V(N)$ lies in another hyperbolic plane which meets N along $\mathrm{Ax}(U)$ at an angle $\theta = \mathrm{Im}\,\tau$. (This is the "Micky Mouse example" of [Th79, 8.7.3].) The shaded region above the hemisphere is $\mathcal{C}$.

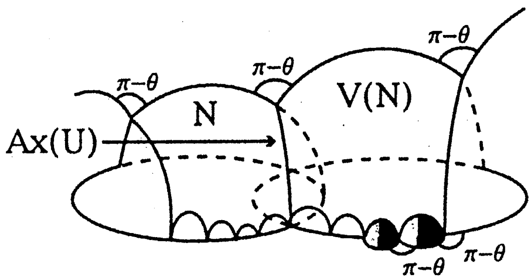


Figure 10:

Continuing in this way, we get a map $\phi_\tau^\gamma = \phi_\tau : (\mathbb{D}, G_0) \to (\mathbb{H}^3, G_\tau)$ which conjugates the actions of $G_0 = G(\lambda_\gamma, t_\gamma)$ on $\mathbb{D}$ with $G_\tau = G(\lambda_\gamma, t_\gamma + \tau)$ on $\phi_\tau^\gamma(\mathbb{D}) \subset \mathbb{H}^3$. We call ϕ_τ^γ the *developed image* of $\mathbb{D}$ under the quakebend $\mathcal{Q}_\gamma(\tau)$. Of course, for large $\theta = \mathrm{Im}\,\tau$, we do not expect $\phi_\tau(\mathbb{D})$ to be embedded in $\mathbb{H}^3$.

Theorem 2.8. ([KS97, prop. 6.5]) *In the situation of theorem 2.7, with* $\mathrm{Re}\,\tau = 0$, $\mathrm{Im}\,\tau$ *small, then* ϕ_τ^γ *is an embedding which extends continuously to a map* $\partial\mathbb{D} \to \partial\mathbb{H}^3$.

The bending away theorem 2.7 is a corollary of 2.8 as follows:

(a) Show that $\phi_\tau(\mathbb{D})$ separates $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ into two half spaces.

(b) Show that one of these half spaces is convex. (Note that this uses that Σ is a torus so that all bending is in the same direction.)

(c) Show that $\phi_\tau(\partial\mathbb{D}) = \Lambda(G(\lambda_\gamma, t_\gamma + \tau))$ and that ϕ_τ conjugates the actions of G_0 and G_τ on $\partial\mathbb{D}$, $\Lambda(G_\tau)$ respectively.

(d) Conclude that $\phi_\tau(\mathbb{D})$ is a component of $\partial\mathcal{C}_0(G_\tau)$.

None of these points is very hard: all are explained in [KS97, §6 and 7].

Proof of theorem 2.8. The idea is to use *nested cones* in $\mathbb{H}^3$. We write $C_\delta(\alpha, x)$ for the *cone* with vertex x, angle α, and central axis δ, see figure 11. The key point in the proof is what we call the *cone lemma* [KS97, lemma 6.3].

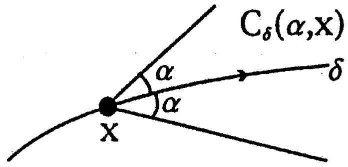


Figure 11:

Suppose that $s \mapsto \eta(s)$ is a geodesic on Σ. As shown in figure 12 its image under the developing map ϕ_τ is a "bent" geodesic in $\mathbb{H}^3$. For a point $\eta(s)$ on η, let $v(s)$ denote the $\mathbb{H}^3$-geodesic based at the point $\phi_\tau(\eta(s))$ and pointing in the forward direction along $\phi_\tau(\eta)$. If $\eta(s)$ is a bending point of $\phi_\tau(\eta)$, this doesn't quite make sense since there are two forward directions of $\phi_\tau(\eta)$ corresponding to the directions immediately before and immediately after the bend. For simplicity, we allow $v(s)$ to denote either. In all cases, $C_{v(s)}(\alpha, \phi_\tau(\eta(s)))$ is a cone of angle α based at a point on $\phi_\tau(\eta)$ and pointing in one or other of the forward directions along $\phi_\tau(\eta)$. The content of the cone lemma is that, provided we consider reasonably well spaced points along η, these cones are *nested*. At bending points, we have *two* cones and the lemma applies to them both.

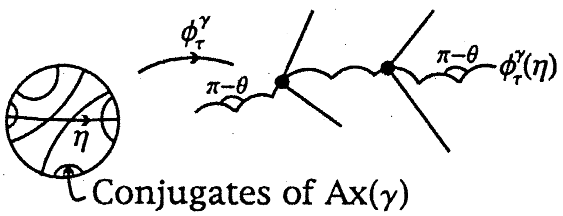


Figure 12:

Theorem 2.9 (Cone Lemma). ([KS97, lemma 6.3])
Let $s \mapsto \eta(s)$ be a geodesic on Σ, and let $\alpha \in (0, \pi/2)$. Then there exist $\epsilon = \epsilon(\ell_\gamma, \alpha) > 0$ and $d = d(\ell_\gamma, \alpha) > 0$ such that if $\mathcal{Q}_\gamma(i\theta)$ is a pure bend along γ with $|\theta| < \epsilon$, then

$$C_{v(s)}(\alpha, \phi_\tau(\eta(s))) \supset C_{v(s+s')}(\alpha, \phi_\tau(\eta(s+s')))$$

whenever $s' > d$.

We require the spacing condition $s' > d$ to take account of the two cones at the bending points. A cone obviously contains cones further out along its own axis; the point is that hyperbolic geometry allows us the freedom to make small bends.

Proof. The full proof is to be found in [KS97]. Here is simpler exercise, which contains the basic idea: Show that there exists $d > 0$ such that $C_{\delta_1}(\alpha, x) \supset C_{\delta_2}(\alpha, y)$ provided $\text{dist}(x, y) > d$, but that this fails as $d \to 0$. (Here δ_2 is a geodesic making an angle θ with δ_1 at y.) The set up is illustrated in figure 13. □

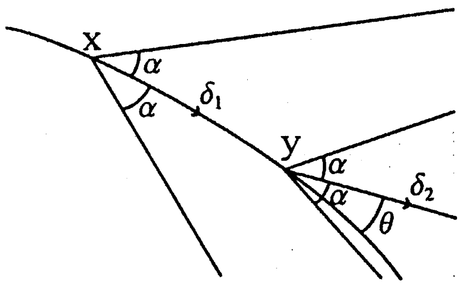


Figure 13:

Theorem 2.8 can now be proved by using nesting of cones to show that for any geodesic $\eta \in \mathbb{D}$, its developed image is embedded and $\phi^\gamma_\tau(\eta)$ has two limit points on $\partial\mathbb{H}^3$.

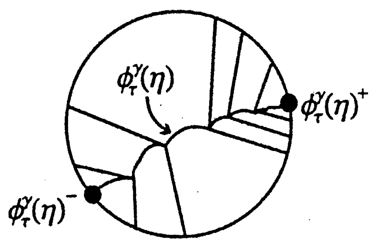


Figure 14:

Remarks

(a) The proof of theorem 2.7 has been done for a pure bend ($\operatorname{Re}\tau = 0$) but we can extend to general τ by first doing an earthquake (F-N twist) by $\operatorname{Re}\tau$ in $\mathbb{D}$ along γ. It would also be possible to give a direct proof.

(b) The same proof will clearly work more generally starting at some group $G \in \mathcal{P}_\gamma$ and bending a small amount on γ.

(c) Another proof of (b) can be given more elementary means, by showing that peripheral circles persist under small deformations. This was done in [KS93] and [KS94]. We need the ideas in the above proof later.

(d) Note the difficulty of extending to genus greater than 1. If we want to bend away from $\mathcal{F}$ along two disjoint simple closed curves simultaneously, we must ensure the bending angle is in the same direction along both, otherwise we loose convexity.

2.4 Controlling the pleating locus on both sides.

The bending away theorem 2.7 controls the pleating locus of one side, say $p\ell^+$ on $\partial\mathcal{C}^+$. (Which side is which depends on which way we bend.) Now we want to simultaneously control $p\ell^-$. Suppose $\gamma' \in \mathcal{S}(\Sigma)$ and we want $p\ell^- = \gamma'$. A necessary condition is that $\operatorname{Tr}\gamma' \in \mathbb{R}$. How can this be achieved near Fuchsian space $\mathcal{F}$?

To answer this question, we make use of the fact that $\operatorname{Tr}\gamma'$ is holomorphic on $\mathcal{QF}$ and real valued on $\mathcal{F}$. In particular, it is holomorphic on the quakebend plane

$$\mathcal{Q}_\gamma(\tau) = \{G(\lambda_\gamma, \tau) \mid \lambda_\gamma \text{ fixed}, \;\tau \in \mathbb{C}\}.$$

Notice that $\mathcal{Q}_\gamma(\tau) \cap \mathcal{F} = \mathcal{E}_\gamma(t)$ is exactly the *earthquake line* $\tau = t, t \in \mathbb{R}$. Now the real locus of a holomorphic function has a very special form: figure 15 illustrates the real locus of a holomorphic function f which is real on the real axis in the τ-plane. The only branching can be at a critical point of f.

Now we use a famous result due to Kerckhoff and Wolpert, see [Ke83] and [Wo82].

Theorem 2.10. *On $\mathcal{E}_\gamma(t)$, $\lambda_{\gamma'} = \lambda_{\gamma'}(t_\gamma)$ has a* unique *critical point t^0_γ which is a minimum, in addition $\lambda''_{\gamma'}(t^0_\gamma) > 0$.*

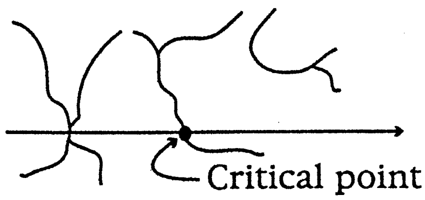


Figure 15:

This allows us to deduce exactly how the pleating variety $\mathcal{P}_{\gamma,\gamma'}$ meets Fuchsian space $\mathcal{F}$. On a fixed quakebend plane, l_γ has a fixed length which we denote by c. We denote this by writing $\mathcal{Q}^c_\gamma(\tau)$, and we denote by $p(\gamma,\gamma',c)$ the critical point t^0_γ. Here $l_{\gamma'}$ is minimal on the γ-earthquake path $\mathcal{E}_\gamma = \mathcal{E}^c_\gamma$. It follows from the antisymmetry of the derivative $dl_\mu/d\tau_\nu = -dl_\nu/d\tau_\mu$ that $p(\gamma,\gamma',c)$ is also the minimum of l_γ on the γ'-earthquake path through $p(\gamma,\gamma',c)$. (We are disguising in this some facts which are fairly easy to deduce from Kerckhoff's theorem 2.10, in particular that for a given γ, c there is a *unique* earthquake path on which $l_\gamma = c$. We shall come back to this in more detail in lecture 6, see also [KS98, §6].)

Theorem 2.11 (Bending Away Theorem Part 2). ([KS98, theorem 8.9]) *In $\mathcal{Q}^c_\gamma(\tau)$, $\mathcal{P}_{\gamma,\gamma'}$ meets $\mathcal{F}$* exactly *in the Kerckhoff critical point $p(\gamma,\gamma',c)$.*

Proof. Let δ' be a complementary generator to γ'. Since $\operatorname{Tr}\gamma' \in \mathbb{R}$ we can make the complex F-N construction *relative to* (γ',δ') and obtain a developed surface $\phi^{\gamma'}_{\tau'}(\mathbb{D})$ bent along γ' by an angle $\operatorname{Im}\tau'$. Since $\operatorname{Im}\tau' = 0$ on $\mathcal{F}$, and since τ' is a continuous function of $\tau = \tau_\gamma$, we see that near $\mathcal{F}$, $\operatorname{Im}\tau'$ is small and the same proof as before shows that $\phi^{\gamma'}_{\tau'}(\mathbb{D})$ is a component of $\partial\mathcal{C}$. □

Corollary 2.12. ([KS97, theorem 3.2]) *Suppose that $\gamma,\gamma' \in \mathcal{S}(\Sigma)$, $\gamma \neq \gamma'$. Then $\mathcal{P}_{\gamma,\gamma'} \neq \emptyset$.*

<u>Exercise</u> What is wrong with the above argument when $\gamma = \gamma'$?

Example 2.13. Suppose that A, B are generators of G (a quasifuchsian once punctured torus group) and we want to find groups such that $|p\ell^+| = \operatorname{Ax}(A)$, $|p\ell^-| = \operatorname{Ax}(B)$. In this special case there is an explicit formula which relates the traces and the twists:

$$\cosh(\tau_A/2) = \cosh(\lambda_B/2)\tanh(\lambda_A/2).$$

(This is proved in [PS95]; it can be checked by differentiating using Kerckhoff's formula $\frac{d\lambda_B}{d\lambda_A} = -\cosh(\delta(A,B))$.) So $\lambda_A, \tau_B \in \mathbb{R}$ implies that either $\operatorname{Im}\tau_A = 0$, in which case G is Fuchsian, or that $\operatorname{Re}\tau_A = 0$ in which case we have a *pure bend.* This is our special case example 2.1.

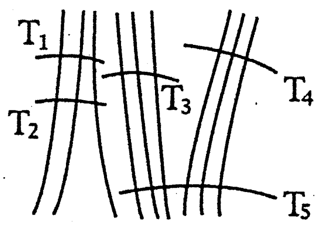
Lecture 3: Irrational measured laminations and Complex Length. Statements of the main technical results.

Irrational laminations can be viewed as a completion of the space $\mathcal{S}(\Sigma)$ of simple closed curves on a hyperbolic surface Σ. When Σ is a punctured torus, they can be thought of as families of lines of irrational slopes in the plane. In this lecture, we discuss how some key concepts extend to this case and then introduce the main technical results we shall need. Sections 3.1 to 3.6 apply to general hyperbolic surfaces Σ unless otherwise stated.

3.1 Geodesic Laminations

Good references for this section are [EM87] and [CEG87].

Definitions A *geodesic lamination* on a hyperbolic surface Σ is a closed set which is a union of pairwise disjoint simple (not necessarily closed) geodesics. A geodesic lamination is *measured* if it carries a transverse measure ν. This means there is a measure ν_T on each transversal T to ν which is invariant under the push-forward maps along leaves, as illustrated in figure 16.



$$\nu(T_1) = \nu(T_2)$$
$$\nu(T_5) = \nu(T_3) + \nu(T_4)$$

Figure 16:

Example Let $\gamma \in \mathcal{S}(\Sigma)$ be a simple closed loop. Then γ is a geodesic lamination which always carries the transverse measure $\nu = c\delta_\gamma$, $c > 0$ where $c\delta_\gamma(T) = ci(T,\gamma)$, the number of times γ intersects the transversal T.

Notation Here is some notation which we shall use:

$$\begin{aligned}\mathcal{GL}(\Sigma) &= \{\text{geodesic laminations on } \Sigma\}\\ \mathcal{ML}(\Sigma) &= \{\text{measured geodesic laminations on } \Sigma\}\\ \mathcal{ML}_{\mathbb{Q}}(\Sigma) &= \{c\delta_\gamma \mid \gamma \in \mathcal{S}(\Sigma)\}\end{aligned}$$

Thus $\mathcal{ML}_{\mathbb{Q}}(\Sigma)$ is the set of *rational* measured laminations on Σ. Notice that $\mathbb{R}^+$ acts on $\mathcal{ML}(\Sigma)$ by $t, \nu \mapsto t\nu$ where $(t\nu)(T) = t\nu(T)$, T a transversal. We define

$$\mathcal{PML}(\Sigma) = \{\textit{projective measured laminations on } \Sigma\} = \mathcal{ML}(\Sigma)/\mathbb{R}^+.$$

For $\nu \in \mathcal{ML}(\Sigma)$, we write $[\nu]$ for its projective class in $\mathcal{PML}$, and $|\nu|$ for its support in $\mathcal{GL}$.

WARNING Not all geodesic laminations are measured! A geodesic spiralling into a closed geodesic cannot support a transverse measure- the measure of transversal near the limit geodesic would be infinite. This is shown in figure 17.

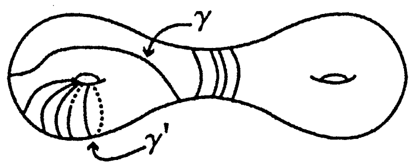


Figure 17:

3.2 Topologies on $\mathcal{GL}$ and $\mathcal{ML}$.

There are two topologies which are commonly used:

Geometric topology on $\mathcal{GL}$. In this topology, laminations $L_1, L_2 \in \mathcal{GL}$ are close if every point in L_1 is close to a point in L_2 and vice versa. Since geodesics diverge, this means tangent directions are close.

Measure topology on $\mathcal{ML}$. This is the weak topology of measures on transversals: $\nu_n \overset{\mathcal{ML}}{\to} \nu$ if $\nu_n(T) \to \nu(T)$ for all transversals T.

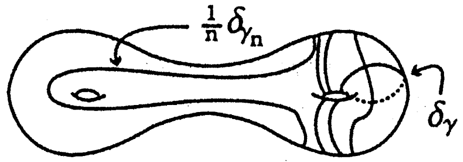


Figure 18:

WARNING These topologies are not the same!! For example, δ_γ and $100\delta_\gamma$ are close in $\mathcal{GL}$ but not in $\mathcal{ML}$.

A more subtle example is shown in figure 18. Take a sequence of closed geodesics (γ_n) of which some parts are far from γ, but which also spiral n times around γ. Then $\frac{1}{n}\delta_{\gamma_n} \overset{\mathcal{ML}}{\to} \delta_\gamma$ but $|\frac{1}{n}\delta_{\gamma_n}| = \gamma_n$ is far from γ in $\mathcal{GL}$.

A lamination L may carry several different projective measure classes, so we can have laminations equal in $\mathcal{GL}$ and but different in $\mathcal{PML}$. This does not happen on the punctured torus because of the property of *unique ergodicity*: a (measurable) lamination L is uniquely ergodic if it carries a unique projective measure class. In this case, up to a constant multiple, $\nu(T) = \lim_{n\to\infty} i(l_n, T)/n$ where l is any leaf of L and l_n is an arc of l of length n from some fixed initial point. On a general surface, the property of unique ergodicity is generic. However, it is special to the punctured torus (and four holed sphere) that it holds for every $L \in \mathcal{ML}$.

On the torus, the following lemma restricts the bad examples which occur.

Lemma 3.1 (Convergence Lemma). ([KS98, lemma 2.1]) *Let Σ be a once punctured torus and let $\nu_0 \in \mathcal{ML} - \mathcal{ML}_\mathbb{Q}$. If ν is close to ν_0 in $\mathcal{ML}$, then $|\nu|$ is close to $|\nu_0|$ in $\mathcal{GL}$.*

Remark We need the condition $\nu_0 \notin \mathcal{ML}_\mathbb{Q}$ because of the situation shown in figure 19, in which $\frac{1}{n}\delta_{A^nB} \overset{\mathcal{ML}}{\to} \delta_A$ but $|\delta_{A^nB}| \overset{\mathcal{GL}}{\not\to} |\delta_A|$.

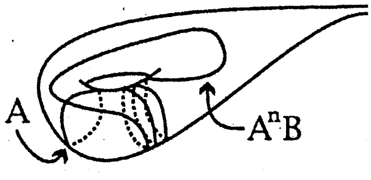


Figure 19:

In the case of a surface of higher genus the geodesic A of the above example can be replaced by an irrational lamination. This explains why in general the result of lemma 3.1 is false.

Proof. First, suppose that ν is close to ν_0 in $\mathcal{ML}$. The lamination $|\nu_0|$ can be covered by "flow boxes" as shown in figure 20. The "horizontal" sides are short and the "vertical" sides are long. Notice $|\nu_0|$ has no "horizontal" arcs. We claim that if ν is close enough to ν_0, the same is also true of $|\nu|$.

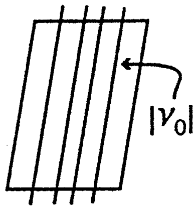


Figure 20:

The proof is by considering the measures of transversals: clearly, as shown in figure 21, $x+y \sim t$, $x+z+w \sim 0$, $w+v \sim t$, $y+z+v \sim 0$, $x, y, z, w, t \geq 0$ $\Rightarrow z = 0$. So $|\nu|$ has a "vertical leaf" close to $|\nu_0|$.

This part of the proof works in any genus.

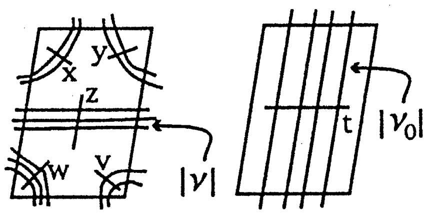


Figure 21:

Now for the converse. We need to show that there are leaves of $|\nu_0|$ near any long arc of $|\nu|$. If not, we can take a limit and find a leaf $l \notin |\nu_0|$ which is a limit of leaves of $|\nu_n|$ where $\nu_n \overset{\mathcal{ML}}{\to} \nu_0$. If $l \cap |\nu_0| = \emptyset$, then we get the pictures shown in figure 22. The picture on the left shows the punctured bigon obtained by cutting Σ along the two boundary leaves of $|\nu_0|$. (This is where we use the hypothesis $\nu_0 \notin \mathcal{ML}_{\mathbb{Q}}$.) The leaf l has no choice but to run from one ideal vertex of this bigon to itself.

Now cut the torus along any simple closed curve which meets l. The result is shown in on the right hand side of figure 22. Any lamination in $\mathcal{ML}$ would give equal weight to the inner and outer boundaries of the resulting punctured annulus $\mathcal{A}$. On the other hand, one sees from the figure that the intersection numbers of l with the two components of $\partial\mathcal{A}$ differ by 2. Thus it

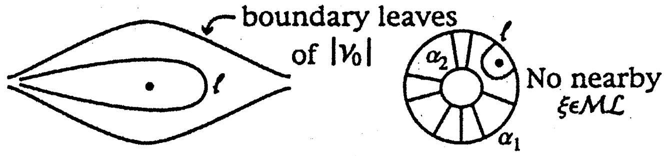


Figure 22:

is impossible to approximate any possible non zero weight on l by laminations in $\mathcal{ML}$. For more details, see [Th79, 9.5.2].

We may therefore assume that $l \cap |\nu_0| \neq \emptyset$, which means we can find a flow box for $|\nu_0|$ in which l is a "horizontal" arc. This is also impossible by the first part of the proof. □

3.3 The Thurston Picture of $\mathcal{PML}$

In general, $\mathcal{PML}(\Sigma)$ is a sphere (of dimension $6g-7$ for a closed surface Σ of genus g). This sphere compactifies the Teichmüller space $\mathcal{T}(\Sigma)$, which is a ball of dimension $6g-6$. Roughly, (this is not the actual definition), $\xi_n \in \mathcal{T}(\Sigma) \to [\delta_\gamma] \in \mathcal{PML}$ if $\xi_n \to \partial\mathcal{T}$ and $l_\gamma(\xi_n)$ is bounded. In the torus case the picture was shown in figure 1. The set of irrational projective measured laminations is identified with $\mathbb{R} \cup \{\infty\} - \mathbb{Q}$, so that $\mathcal{PML} = S^1$. This fits with Wright's enumeration of $\mathcal{S}(\Sigma)$ as explained in lecture 1.

3.4 The Bending Measure on $\partial\mathcal{C}$ and the Continuity Theorems

Definition 3.2. ([EM87], [CEG87]) Let Σ be a hyperbolic surface, Γ a Fuchsian group with $\Sigma = \mathbb{H}/\Gamma$, G a Kleinian group. A *pleated surface* is a map $\sigma : \mathbb{H}^2 \to \mathbb{H}^3$ (or $\Sigma \to \mathbb{H}^3/G$) such that:

(a) σ is an isometry from $\mathbb{H}$ to its image with the path metric induced from $\mathbb{H}^3$.

(b) $\sigma_* : \pi_1(\Sigma) = \Gamma \to G$ is an injection.

(c) For each $x \in \Sigma$, there exists at least one geodesic γ containing x such that $\sigma|_\gamma$ is an isometry.

The *bending locus* is the set of geodesics in $\mathbb{H}$ through which there is only one direction as in (c). The bending locus is a geodesic lamination on Σ, denoted by $B(\sigma)$. In general, the image $\sigma(\mathbb{H})$ is neither convex nor embedded in $\mathbb{H}^3$.

Definition 3.3. *A lamination $L \in \mathcal{GL}(\Sigma)$ is* realized *in $\mathbb{H}^3/G$ if there is a pleated surface $\sigma : \Sigma \to \mathbb{H}^3/G$ with $L \subset B(\sigma)$.*

In this definition, the hyperbolic structure on Σ is left unspecified, to be determined by the map and the structure on $\mathbb{H}^3$.

Theorem 3.4. ([Th79], [CEG87]) *Let G be quasifuchsian and $L \in \mathcal{GL}(\Sigma)$. Then L is realized in $\mathbb{H}^3/G$.*

Proof. The idea is to make a direct (quite easy) construction if L has only finitely many leaves and then use "compactness of pleated surfaces" to take limits. It is explained in detail in [CEG87, chapter 5]. □

3.5 The Convex Hull Boundary

Theorem 3.5. (Thurston, [EM87] Chapter 1) *Let G be a finitely generated Kleinian group and let ∂C_0 be a component of ∂C, the convex hull boundary of $\mathbb{H}^3/G$. Then ∂C_0 carries an intrinsic hyperbolic metric induced from the metric $\mathbb{H}^3$, making it a pleated surface. The bending lamination carries a natural transverse measure, the* bending measure *$p\ell(\partial C_0) \in \mathcal{ML}(\partial C_0/G)$.*

Remark It is clear that in addition, ∂C_0 is convex (i.e. cuts off a convex half space) and embedded.

The idea for constructing the bending measure is illustrated in figure 23 A *support plane* is a half space touching ∂C with C entirely on one side. The

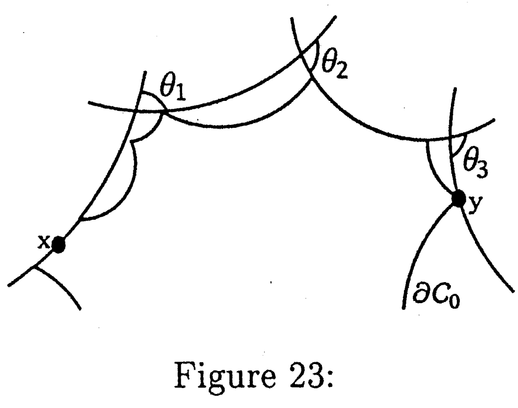


Figure 23:

figure 23 shows a collection of support planes forming a "roof" over C; adjacent planes meet at angles denoted in the figure by θ_i. The bending measure of a transversal T is defined by $p\ell(T) = \inf \sum \theta_i$,where the infimum is taken over all possible families of support planes sitting "over" the transversal T, which joins x to y in the figure. Distance on $\partial\mathcal{C}$ can be defined in a similar way. We call the induced hyperbolic metric on $\partial\mathcal{C}_0$, the *flat structure* $F(\partial\mathcal{C}_0)$ of $\partial\mathcal{C}_0$.

Theorem 3.6 (The continuity theorem). [KS95] *Let $\xi \mapsto G_\xi$ be a holomorphic family of Kleinian groups. Then for a fixed component $\partial\mathcal{C}_0$, the maps $\xi \mapsto p\ell(\xi) \in \mathcal{ML}$ and $\xi \mapsto F(\partial\mathcal{C}_0(\xi)) \in \mathcal{T}(\Sigma)$ are continuous.*

Proof. There is a retraction map $r : \Omega \to \partial\mathcal{C}$ which maps $z \in \Omega$ to the nearest point on $\partial\mathcal{C}$, seen by drawing expanding horoballs at z as in figure 24.

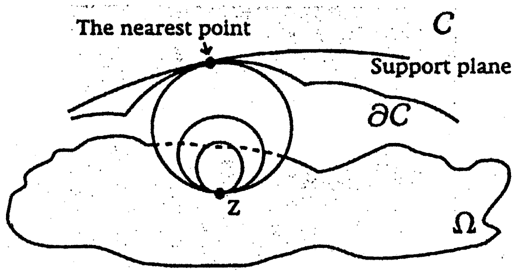


Figure 24:

This also defines a support plane at $r(z)$. We study the continuity of the map $\hat{r} : \Omega \times \mathrm{Def} \to Z(\partial\mathcal{C})$ where $Z(\partial\mathcal{C})$ is the set of support planes to $\partial\mathcal{C}$ with the obvious topology. One shows that $\hat{r}$ is uniformly continuous on compact subsets of $\Omega \times \mathrm{Def}$ so that any approximating roof for one group is close to a roof for a nearby group. It follows that the bending measure and flat metrics are also close. $\square$

3.6 Length and Complex Length

Length of a measured lamination on a hyperbolic surface Σ If $\gamma \in \mathcal{S}(\Sigma)$, then its hyperbolic length $l(\gamma)$ is given by $\mathrm{Tr}\,\gamma = 2\cosh(l(\gamma)/2)$. If $c\delta_\gamma \in \mathcal{ML}_{\mathbb{Q}}(\Sigma)$, set $l(c\delta_\gamma) = cl(\gamma)$. We want to extend this to $l : \mathcal{ML}(\Sigma) \to \mathbb{R}^+$. One way is to cover $|\nu|$, $\nu \in \mathcal{ML}$, by flow boxes B_i and integrate: $l(\nu) = \sum_i \int_{T_i} l(L_x \cap B_i) d\nu_{T_i}(x)$, where L_x is the leaf of $|\nu|$ through x in the transversal T_i.

Another way, following Kerckhoff, is to take limits:

Theorem 3.7. ([Ke83], [Ke85]) *If* $\nu_n \in \mathcal{ML}_{\mathbb{Q}}$, $\nu_n \overset{\mathcal{ML}}{\to} \nu$, $\xi \in \mathcal{T}(\Sigma)$, *then* $(l_{\nu_n}(\xi))$ *has a unique limit* $l_\nu(\xi)$. *The convergence is uniform on compact subsets of* $\mathcal{T}(\Sigma)$.

Complex length of a loxodromic. If $g \in \mathrm{SL}(2,\mathbb{C})$ then its complex length λ_g is given by $\operatorname{Tr} g = 2\cosh(\lambda_g/2)$. Here $\lambda_g = l_g + i\theta_g$ where l_g is the translation length along the axis and θ_g is the twist.
Note There is a major difficulty in extending this definition to $\mathcal{ML}_{\mathbb{Q}}$ since θ_g is only defined mod 2π. One possible solution is explained next.

Complex length of a measured lamination. We want to extend the length function l_ν from $\mathcal{T}(\Sigma)$ to $\mathcal{QF}(\Sigma)$. We have $\mathcal{F}(\Sigma) = \mathcal{T}(\Sigma) \hookrightarrow \mathcal{QF}(\Sigma)$. For $\gamma \in \mathcal{S}(\Sigma)$, choose the branch of $\lambda_g = \lambda_g(\xi)$, $g = g(\gamma)$, which is real valued on $\mathcal{F}$. Then define $\lambda_{c\delta_\gamma} = c\lambda_{g(\gamma)}$. Notice that this choice of a specific branch gets us round the difficulty of defining $c\theta_g$ mod 2π

Consider the family of functions $\lambda_{c\delta_\gamma} : \mathcal{QF} \to \mathbb{C}$. These are holomorphic and avoid the negative half plane $\mathbb{C}^- = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. Hence they are a *normal family*, see for example [Be91, theorem 3.3.5]. So if $\nu_n \overset{\mathcal{ML}}{\to} \nu$, $\nu_n \in \mathcal{ML}_{\mathbb{Q}}$, then (λ_{ν_n}) has a convergent subsequence. By Kerckhoff's theorem, (λ_{ν_n}) has a unique limit on $\mathcal{F}$ and hence (holomorphic functions!) on $\mathcal{QF}$. Moreover λ_ν is non-constant since it is non-constant on $\mathcal{F}$ by Kerckhoff. This defines complex length.

Note Equicontinuity means we can take diagonal limits: $\xi_n \to \xi \in \mathcal{QF}$ and $\nu_n \overset{\mathcal{ML}}{\to} \nu$ implies $\lambda_{\nu_n}(\xi_n) \to \lambda_\nu(\xi)$.

3.7 Statement of Main Technical Results

We can now state the main technical results we shall need. From now on, G is a quasifuchsian once punctured torus group, $p\ell^+$ and $p\ell^-$ the bending measures on $\partial\mathcal{C}^+/G$ and $\partial\mathcal{C}^-/G$. For $\mu, \nu \in \mathcal{ML}$, set

$$\mathcal{P}_{[\mu],[\nu]} := \{\xi \in \mathcal{QF} \mid [p\ell^+] = [\mu], [p\ell^-] = [\nu]\}.$$

(Often we shall be sloppy and write $\mathcal{P}_{\mu,\nu}$ for $\mathcal{P}_{[\mu],[\nu]}$.) Also we write

$$\mathcal{P}_\mu = \mathcal{P}_{[\mu]} = \{\xi \in \mathcal{QF} \mid [p\ell^+] = [\mu] \text{ or } [p\ell^-] = [\mu]\}.$$

The following result generalizes the key lemma 1.1.

Theorem 3.8 (Real Length Lemma). ([KS98a, theorem 6.5]) *Suppose that $\xi \in \mathcal{QF}, \xi \in \mathcal{P}_\mu$. Then $\lambda_\mu(\xi) \in \mathbb{R}$.*

The idea of the proof is obviously to take limits, but we need care to ensure that $\lambda_\mu(\xi) \in \mathbb{R}$ is impossible on open sets in $\mathcal{QF}$. The proof is given in lecture 5.

Theorem 3.9 (Local Pleating Theorem, Version 1). ([KS98, theorem 8.6]) *Suppose $\nu_0 \in \mathcal{ML} - \mathcal{ML}_\mathbb{Q}$, $\xi_0 \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$. Then there are neighbourhoods U of ξ_0 in $\mathcal{QF}$ and W of $[\nu_0] \in \mathcal{PML}$ such that $[\delta_\gamma] \in W \cap \mathcal{PML}_\mathbb{Q}$, $\xi \in U$, $\lambda_\gamma(\xi) \in \mathbb{R}$ implies $\xi \in \mathcal{P}_\gamma \cup \mathcal{F}$.*

Theorem 3.10 (Local Pleating Theorem, Version 2). ([KS98, theorem 8.1])

Suppose $\nu_0 \in \mathcal{ML}$, $\xi_0 \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$. Then there is a neighbourhood U of ξ_0 in $\mathcal{QF}$ such that $\xi \in U$, $\lambda_{\nu_0}(\xi) \in \mathbb{R}$ implies $\xi \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$.

Remarks on theorems 3.9 and 3.10. One should compare theorem 3.9 to the theory of local deformations for cone manifolds. (But notice it applies equally to irrational laminations and also that we do not need to assume there is a continuous path of deformations from ξ_0 to ξ.) It would be tempting to combine 3.9 and 3.10 and allow ν to vary in a neighbourhood of ν_0 in 3.10. However this result would be false in higher genus: take a surface of genus two and disjoint loops γ and γ'. Bending away from $\mathcal{F}$ in opposite directions along the two curves, we find $\frac{1}{n}\delta_{\gamma'} + \delta_\gamma \xrightarrow{\mathcal{ML}} \delta_\gamma$, $\lambda_{\gamma'}(\xi) \in \mathbb{R}$ but $\xi \notin \mathcal{P}_{\gamma'}$.

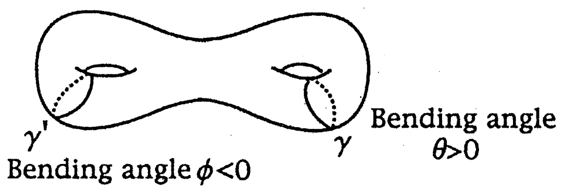


Figure 25:

For the special case of a torus, the result is true since there is a maximum of one bending angle (uniquely ergodicity). The proof is more complicated and not needed so omitted here.

Theorem 3.11 (Limit Pleating Theorem). ([KS98a, theorem 5.1]) *Suppose $\mu, \nu \in \mathcal{ML}$, $[\mu] \neq [\nu]$. (Equivalently, on the punctured torus, using unique ergodicity, $|\mu| \neq |\nu|$.) Suppose $\xi_n \in \mathcal{QF}$, $\xi_n \in \mathcal{P}_{\mu,\nu}$ and suppose that*

$l_\mu(\xi_n) \to c \geq 0$, $l_\nu(\xi_n) \to d \geq 0$. *Then* (ξ_n) *has a subsequence which converges algebraically to a group* $\xi_\infty = \langle A, B \mid [A,B] \text{ is parabolic}\rangle$. *Moreover,* $\xi_\infty \in \mathcal{QF} \iff c > 0$ *and* $d > 0$.

Remarks on 3.11 It follows from the continuity theorem 3.6 that if $\xi_\infty \in \mathcal{QF}$, then $c > 0$ and $d > 0$. If either $c = 0$ or $d = 0$ then ξ_∞ is either a cusp group (μ or ν rational) or a degenerate group with ending lamination $|\mu|$ or $|\nu|$ (or both). Notice that it is implicit in this statement that the convergence to a boundary group along a pleating variety is strong; that is, the geometric and algebraic limits agree. The gap in the original proof in [KS98] (see preface) revolved around the fact that differing limits had not been ruled out.

The proof in the case $\mu, \nu \in \mathcal{ML}_\mathbb{Q}$ can be done by relatively elementary means by studying the limit behaviour of F-peripheral circles in Λ. This is done in [KS93] and [KS94]. The proof for $\mu, \nu \notin \mathcal{ML}_\mathbb{Q}$ is much harder; it is rather similar to the first part of the proof of Thurston's double limit theorem (which states that if the complex structures $\omega^\pm(\xi_n)$ of $\Omega^\pm/G(\xi_n)$ converges to distinct points, then an algebraic limit exists). This theorem is crucial in allowing us to make analytic continuation along the subset of the real locus of $\lambda_\mu \times \lambda_\nu$ on which we know the groups is in $\mathcal{QF}$. The proof will be sketched at the end of next lecture.

Lecture 4: Pleating Coordinates in One Dimensional Examples

In this lecture we explain how the results stated at the end of the last lecture allow us to deduce a complete picture of the pleating varieties when the deformation space has one complex dimension. We concentrate on the Maskit embedding of the once punctured torus and also mention the Earle slice. We end the lecture with a proof of the limit pleating theorem 3.11.

Here is a list of our main technical tools. Theorem 3.11 is proved at the end of this lecture and the rest will be proved in lecture 5.

Complex length 3.6	Local pleating theorems 3.9, 3.10
Continuity theorems 3.6	Limit pleating theorem 3.11
Real length lemma 3.8	Bending away theorems 2.7 and 2.11.

From now on, we assume $\mathrm{Def}(G) \subset \mathbb{C}$. For the moment, we don't need to restrict to any particular slice. Here are some immediate consequences of the above results.

Corollary 4.1. *Let $\mu \in \mathcal{ML}(\Sigma)$. Then $\mathcal{P}_\mu = \{\xi \in \mathrm{Def}(G) \mid [p\ell(\partial\mathcal{C})] = [\mu]\}$ is a union of connected components of the real locus $\lambda_\mu^{-1}(\mathbb{R}) - \mathcal{F}$.*

Remarks

- If $\mu \in \mathcal{ML}_\mathbb{Q}$, then the length or trace function λ_γ is defined on all of $\mathbb{C}$ and so $\lambda_\mu^{-1}(\mathbb{R})$ is a subset of $\mathbb{C}$. (Of course, $|\operatorname{Tr}\gamma| > 2$ inside $\mathrm{Def}(G)$.) If $\mu \notin \mathcal{ML}_\mathbb{Q}$ then λ_μ is only defined on $\mathrm{Def}(G)$. This is important for computation.

- In the Maskit slice we need a bit more work to define λ_μ, see [KS93].

Corollary 4.2 (Density Theorem). ([KS98, theorem 4]) *The pleating varieties $\mathcal{P}_\mu$ for $\mu \in \mathcal{ML}_\mathbb{Q}$ are dense in* $\mathrm{Def}(G)$.

Proof. The length function λ_μ is non-constant on $\mathrm{Def}(G)$ for any $\mu \in \mathcal{ML}_\mathbb{Q}$, for otherwise by the local and limit pleating theorems 3.10, 3.11, if $\mathcal{P}_\mu \neq \emptyset$, then $\mathcal{P}_\mu = \mathrm{Def}(G)$ which is impossible (Why?). Pick $\xi \in \mathcal{P}_\mu$, and pick $\mu_n \in \mathcal{ML}_\mathbb{Q}$ with $\mu_n \overset{\mathcal{ML}}{\to} \mu$. By the real length lemma 3.8, $\lambda_\mu(\xi) \in \mathbb{R}$. Use Hurwitz's theorem (properties of normal families) to find $\xi_n \to \xi$ with $\lambda_{\mu_n}(\xi_n) = \lambda_\mu(\xi) \in \mathbb{R}$. By the local pleating theorem 3.9, $\xi_n \in \mathcal{P}_{\mu_n}$. □

Range of the length function and analytic continuation of pleating rays We now study the range of the length function $\lambda_\mu |_{\mathcal{P}_\mu}$. Clearly, $\lambda_\mu(\mathcal{P}_\mu) \subset \mathbb{R}^+$. By theorem 3.10, $\lambda_\mu(\mathcal{P}_\mu)$ is open. Hence by 3.11, we can continue along a pleating ray until either $\lambda_\mu \to 0$ (cusp or ending lamination), or $\lambda_\mu \to \infty$, or we reach a Fuchsian group where the bending measure is zero. In the Maskit embedding, the last case cannot occur.

Corollary 4.3 (Range of length function). [KS98, lemma 9.3] *Let K_μ be a connected component of a pleating variety $\mathcal{P}_\mu$. Then $\lambda_\mu(K_\mu) = (0, \infty)$ or $(0, d)$, where d is a critical value of λ_μ on $\mathcal{F}$. Furthermore (since the Kerckhoff critical point is unique), the degree of the map to $(0, d)$ is one.*

Corollary 4.4. *If K_μ contains a critical point, then $\lambda_\mu(K_\mu) = (0, \infty)$.*

Proof. Exercise. □

In order to get the full picture of pleating rays in $\mathrm{Def}(G)$, it remains to:

(a) Prove that $\mathcal{P}_\mu \neq \emptyset$.

(b) Determine the number of connected components of $\mathcal{P}_\mu$.

(c) Prove "non-singularity", that is, no component K_μ contains a critical point of λ_μ not on $\mathcal{F}$.

This will justify the assertions about pleating rays made in lecture 1.

4.1 The Maskit embedding

Now we specialise to the Maskit embedding as in lecture 1. Recall the special case example 2.1 in lecture 2, in which $[p\ell^+] = [\delta_B], [p\ell^-] = [\delta_A]$, where (A, B) are the generators of $G \in \mathcal{QF}$. We can extend the analysis to the case in which A is parabolic. Since such an element is always on the boundary of the convex core, $[p\ell^-] = [\delta_A]$. Thus all groups in $\mathcal{P}_{[B]} = \mathcal{P}_{[0/1]} = \{\xi \in \mathrm{Def}(G) \mid [p\ell^+] = [\delta_B], \mathrm{Tr}\, A = \pm 2\}$ are a limiting case of the special case examples $\mathrm{Tr}\, A, \mathrm{Tr}\, B \in \mathbb{R}$. Recall that in this case the axes of A and B are orthogonal and the complex F-N twist is a pure bend. (The parabolic A does not have an axis but it does have a translation direction, so the above makes sense.) The generator A is called an *accidental parabolic* since it does not represent a cusp in $\mathbb{H}^3/G$.

We can compute explicitly: from Appendix 1.3 in lecture 1 we have $\mathrm{Tr}\, B = -i\xi$. Hence $\lambda_B \in \mathbb{R}$ and $|\mathrm{Tr}\, B| > 2$ iff $\xi \in it$, $t \in \mathbb{R}$ and $|t| > 2$. In fact in this normalization we can take $\mathrm{Im}\, \xi > 0$, so $\xi = it$, $t \in (2, \infty)$ gives the pleating ray $\mathcal{P}_{[0/1]}$. The point $2i$ is the cusp group at which the loop corresponding to B is pinched.

More generally, $(A, A^{-n}B)$ form a generator pair and the same reasoning applies to $\delta_{[A^{-n}B]} = \delta_{[n/1]}$. We compute $\mathrm{Tr}(A^{-n}B) = -i(\xi - 2n)$. Thus $\mathcal{P}_n$ is the line $\mathrm{Re}\, \xi = 2n$, $\mathrm{Im}\, \xi > 2$, and $\xi = 2n + 2i$ is the cusp. We call these lines *integral pleating rays*; they are shown in figure 26.

Now we turn to non-integral p/q. We know from Appendix 1.3 that the region $\mathrm{Im}\, \xi > 2$ is contained in $\mathrm{Def}(G)$. It follows from easily from the Continuity theorem 3.6 and the above discussion about integral rays, that $\mathcal{P}_\mu \neq \emptyset$ for $\mu \in \mathbb{R} = \mathcal{PML} - \{\infty\}$. (Since $\infty = 1/0$ corresponds to the lamination $\delta_{[A]}$, this cannot be the pleating locus $[p\ell^+]$. Why?)

<u>Exercise</u> Show $\mathcal{P}_\mu \neq \emptyset$ for $\mu \in \mathbb{R}$.

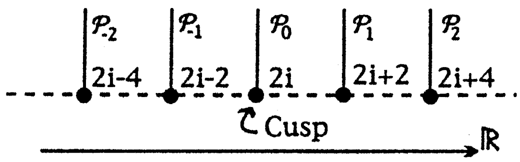


Figure 26:

We also have the formula $\operatorname{Tr} W_{p/q} = (-i)^q(\xi - 2p/q)^q + O(\xi^{q-2})$, from which it follows there is a unique branch of $\lambda^{-1}_{p/q}(\mathbb{R})$ on which $\lambda_{p/q}(\xi) \to \infty$ and which does not intersect $\mathcal{P}_n$ for any $n \in \mathbb{Z}$. (Here $q > 1$). It is now an easy exercise to conclude from corollary 4.3 that $\mathcal{P}_{p/q}$ has no critical points and only one connected component which is asymptotic to the line $\operatorname{Re}\xi = 2p/q$ as $|\xi| \to \infty$.

Challenge Make a limiting argument to show the same is true of the irrational rays $\mathcal{P}_\mu, \mu \notin \mathbb{Q}$. (The argument in [KS93] is wrong; see lecture 6.)

4.2 The Earle slice

This example is investigated in [KoS98]. Recall from lecture 1 that it is the slice of $\mathcal{QF}$ for which Ω^+ and Ω^- are conformally isomorphic under the involution induced by $A \to B$, $B \to A$. An argument using symmetry shows that, if $\mu, \nu \in \mathcal{ML}$, then $\mathcal{P}_{\mu,\nu} = \emptyset$ unless $\nu = 1/\mu$. In particular, by a variant of 2.11, we find that $\mathcal{P}_{\mu,1/\mu} \neq \emptyset$ unless $\mu = 1/\mu$, i.e. $\mu \neq \pm 1$, in which case clearly $\mathcal{P}_{\mu,1/\mu} = \emptyset$. In fact, one needs to show that the length function λ_μ has a unique critical point on the *Earle line* $\operatorname{Def}(G) \cap \mathcal{F}$; i.e. the set of Fuchsian groups which have rhombic symmetry. Existence is easy; uniqueness follows from an argument about boundaries of Riemann maps. We know we have a biholomorphic map $\psi : \mathbb{D} = \mathcal{T}(\Sigma) \to \operatorname{Def}(G)$, the Earle slice. We consider different branches of $\mathcal{P}_{p/q}$ going to the same cusp in $\operatorname{Def}(G)$ and show they define distinct prime ends of ψ. Thus the inverse images limit on *distinct* points on $\partial\mathbb{D}$, see figure 27. But on both branches the trace, and hence the extremal length, of the p/q-curve on Σ goes to zero. From what we know about extremal length in $\mathbb{D}$, this is impossible. This allows us to rule out critical points and multiple components in $\mathcal{P}_{p/q,q/p}$. Figure VIII is a drawing of the Earle slice.

Variants of the same methods apply to the Riley slice [KS94] and Koebe

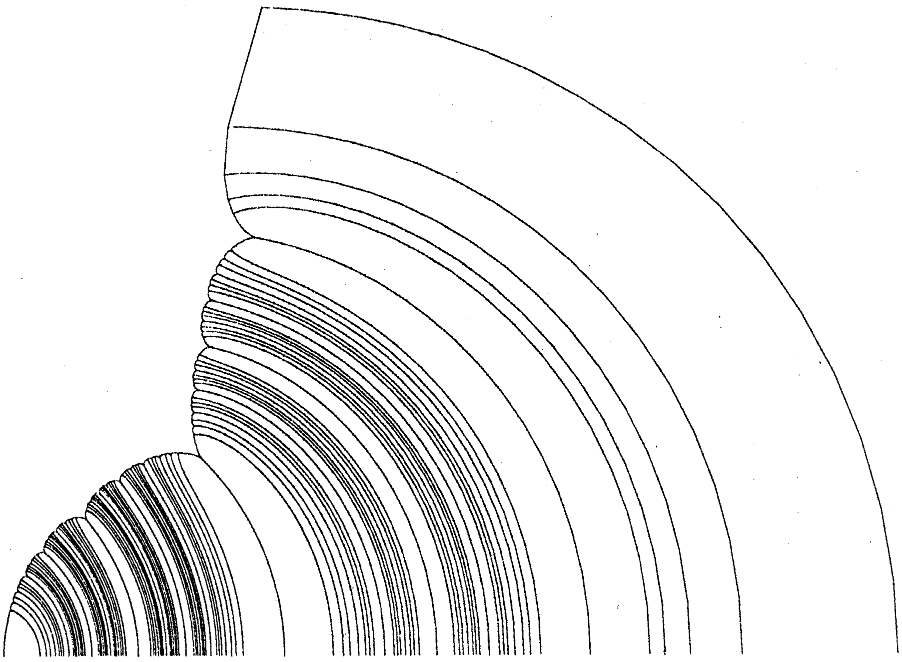


Figure VIII. The Earle Slice. *Graphics by Peter Liepa*

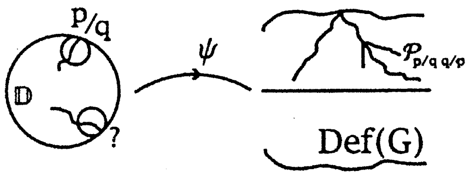


Figure 27:

slice [Pa99]. These examples differ from the cases we have studied, since $\mathbb{H}^3/G$ is a handlebody and $\pi_1(\partial\mathcal{C})$ is not injective. Further details appear in [KoS98a]

4.3 The Limit Pleating Theorem

For the remainder of this lecture, we discuss the proof of the limit pleating theorem, [KS98a, theorem 5.1].

As usual, Σ is a once punctured torus. Suppose $\mu, \nu \in \mathcal{ML}(\Sigma)$ with $[\mu] \neq [\nu]$. Suppose also $\xi_n \in \mathcal{P}_{\mu,\nu}$ and that $\lambda_\mu(\xi_n) \to c \geq 0$, $\lambda_\nu(\xi_n) \to d \geq 0$. We want to show that (up to a subsequence of ξ_n) the groups $G_n = G(\xi_n)$ have an algebraic limit G_∞ .

We use a fundamental estimate of Thurston about lengths of curves on pleated surfaces. This estimate is also crucial in the proof of the double limit theorem [Th86]. We state it in our special case only.

Theorem 4.5 (Efficiency of Pleated surfaces). ([Th86, theorem 3.2]) *Let L be the ideal triangulation of Σ whose leaves are the three geodesics from the cusp of Σ to itself in the homotopy classes of A, B, and AB respectively. Let $\sigma : \mathbb{D} \to \mathbb{H}^3$ be a pleated surface map realizing L and let $S = \sigma(\mathbb{D})/G$. Then there exists $C > 0$, depending only on L, such that $l_\mu(S) \leq l_\mu(M) + Ci(\mu, L)$ for all $\mu \in \mathcal{ML}$ and all 3-manifolds $M = \mathbb{H}^3/G$ with $G \in \mathcal{QF}$.*

Here $i(\mu, L)$ is the intersection number of μ with L and $l_\mu(S)$ is the length of μ on the pleated surface S. By definition, $l_\mu(M)$ is the length of μ *in the 3-manifold M*. If $\mu \in \mathcal{ML}_{\mathbb{Q}}$ the meaning is clear; for general μ, Thurston and Bonahon show how to extend this definition by continuity to $\mathcal{ML}$. In our case, we can take $l_\mu(M) = \operatorname{Re}\lambda_\mu(M)$, where λ_μ is the complex length of μ in M. The proof of this theorem is in [Th86]; a version for Schottky groups is given in [Ca93].

We deduce the existence of an algebraic limit as follows. We have a sequence of manifolds $M_n = M_n(\xi_n) = \mathbb{H}^3/G(\xi_n)$ with $\xi_n \in \mathcal{P}_{\mu,\nu}$. Let S_n be the pleated surface realizing L in M_n. We find

$$l_\mu(S_n) \leq l_\mu(M_n) + Ci(\mu, L) \text{ and } l_\nu(S_n) \leq l_\nu(M_n) + Ci(\nu, L).$$

Now, *since μ, ν are in $\partial\mathcal{C}$*, we have $l_\mu(M_n) = \lambda_\mu(M_n) \to c$ and $l_\nu(M_n) = \lambda_\nu(M_n) \to d$. Thus $(l_\mu(S_n))$ and $(l_\nu(S_n))$ are bounded sequences. By [Th86, corollary 2.3] we conclude that the surfaces S_n lie in a bounded region of $\mathcal{T}(\Sigma)$, and hence have a convergent subsequence. Along this subsequence, the curves α, β representing the generators A, B of Σ have definite bounded lengths $l_\alpha(S)$, $l_\beta(S)$. Now $l_\alpha(S) \geq l_A(M)$ and $l_\beta(S) \geq l_B(M)$ for any $M = \mathbb{H}^3/G$; hence $\operatorname{Re}\lambda_A(\xi_n)$ and $\operatorname{Re}\lambda_B(\xi_n)$ are bounded and so also $|\operatorname{Tr} A(\xi_n)|$ and $|\operatorname{Tr} B(\xi_n)|$. (Use $\operatorname{Tr} A = 2\cosh(\lambda_A/2)$!) Thus $\operatorname{Tr} A(\xi_n)$ and $\operatorname{Tr} B(\xi_n)$ have convergent subsequences. In a once punctured torus group these determine $\operatorname{Tr} AB(\xi_n)$ and hence ξ_{n_r} has an algebraic limit as required.

It remains only to prove the last statement of the theorem:

$$\xi_\infty \in \mathcal{QF} \iff c > 0 \text{ and } d > 0.$$

This is where the proof differs from that given in [KS98]. We need to use a strong result of Thurston's about joint continuity of the length function

$l_\mu(M)$. Let $AH(\Sigma)$ denote the set of Kleinian once punctured torus groups. If μ is not realised in a 3-manifold M, then set $l_\mu(M) = 0$.

Theorem 4.6 (Continuity of the Length Function). *The function $L : AH(\Sigma) \times \mathcal{ML} \to \mathbb{R}$, $L(H, \mu) = l_\mu(\mathbb{H}^3/H)$ is continuous.*

This result was asserted by Thurston in [Th86]; proofs have recently appeared in [Oh98, lemma 4.2] and [Br98, theorem 5.1].

It follows immediately from this theorem that the limits $\{l_\mu(\xi_n)\}, \{l_\nu(\xi_n)\}$ exist and converge to $\{l_\mu(G_\infty)\}, \{l_\nu(G_\infty)\}$. Clearly, if the limit group ξ_∞ is in $\mathcal{QF}$, then $c > 0$ and $d > 0$. By definition, if μ is not realised in M, then $l_\mu(M) = 0$. Thus it is enough to show that if μ and ν are both realised in M_∞, and if $c > 0$ and $d > 0$, then $\xi_\infty \in \mathcal{QF}$.

The main point is to show that if $c, d > 0$, then the pleated surfaces $\partial\mathcal{C}_n^\pm$ converge geometrically to pleated surfaces $\Pi^\pm$, each of whose quotients $\Pi^\pm/G_\infty$ is homeomorphic to Σ. Once we prove this, the remainder of the proof is as follows. From the geometric convergence, $\Pi^\pm$ are embedded and each bounds a convex half space. We deduce that $\Pi^\pm$ are components of $\partial\mathcal{C}(G_\infty)$ and therefore face simply connected G_∞ invariant components of $\Omega(G_\infty)$. It is well known that there can be at most two such components and we conclude that $G_\infty \in \mathcal{QF}$.

Geometric convergence is proved using compactness of pleated surfaces following [CEG87, §5.2]. There are two essential points to check: first, that the surfaces $\partial\mathcal{C}_n^\pm = \partial\mathcal{C}^\pm(\xi_n)$ all meet a fixed compact neighbourhood in $\mathbb{H}^3$ and second, that away from a neighbourhood of the cusp, the surfaces $\partial\mathcal{C}_n^\pm/G_n$ have bounded diameter. (The latter implies that G_∞ doesn't contain any accidental parabolics which is important for strong convergence.)

First let's show that the surfaces $\partial\mathcal{C}_n^\pm$ all meet a fixed compact neighbourhood in $\mathbb{H}^3$. With S_n as above, let $D_n^+ = \inf_{x\in\partial\mathcal{C}_n^+} \operatorname{dist}(x, S_n)$. We shall show that if $\{D_n^+\}$ is unbounded, then $l_\mu(\xi_n) = l_\mu(M_n) \to 0$.

If (D_n^+) is unbounded, then pick $\gamma_k \in \mathcal{S}$ with $[\gamma_k] \overset{\mathcal{PML}}{\to} [\mu]$ and normalize so that $c_k\gamma_k \overset{\mathcal{ML}}{\to} \mu$. Without loss of generality, assume $\mu \in \mathcal{ML} - \mathcal{ML}_\mathbb{Q}$. (The case $\mu \in \mathcal{ML}_\mathbb{Q}$ is easier.) Let $\gamma_k^+(n)$ be the representative of γ_k on $\partial\mathcal{C}_n^+$. By the convergence lemma 3.1, for fixed n, $\gamma_k^+(n) \overset{\mathcal{GL}}{\to} |\mu^+(n)|$ on $\partial\mathcal{C}_n^+$ and hence $\gamma_k^*(n) \to |\mu^*(n)|$ where γ_k^* and $|\mu^*|$ are the geodesics in $\mathbb{H}^3$ corresponding to $\gamma_k(n)$ and $|\mu(n)|$. Note $|\mu^*| = |\mu|$ since $\xi_n \in \mathcal{P}_\mu$. Now $l_{\gamma_k^+(n)}(\partial\mathcal{C}_n^+) \leq e^{-D_n^+} l_{\gamma_k}(S_n)$. By the above, γ_k^* and γ_k^+ are geometrically close for large n.

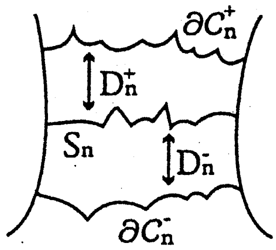


Figure 28:

Since orthogonal projection in $\mathbb{H}^3$ exponentially shrinks length, we deduce $l_\mu(\partial\mathcal{C}_n^+) \leq e^{-D_n^+} l_\mu(S_n)$ which forces $l_\mu(\partial\mathcal{C}_n^+) = l_\mu(M_n) \to 0$.

To prove the second point, let ϵ be the Margulis constant and let Σ_n^ϵ denote the surface $\partial\mathcal{C}_n^+/G_n$ with an ϵ-thin neighbourhood of the cusp removed. We shall show that the surfaces Σ_n^ϵ have uniformly bounded diameter by showing there is a uniform lower bound to the lengths of all curves on Σ_n^ϵ. This involves two applications of the fundamental length estimate which can be taken as the defining property of the Thurston boundary [Th86]: if $f_n \in \mathcal{F} \to [\lambda] \in \mathcal{PML}$, then there are laminations $\lambda_n \overset{\mathcal{ML}}{\to} \lambda$ and constants $c_n \to \infty$ such that

$$\frac{i(\eta, c_n\lambda_n)}{l_\eta(f_n)} \to 1,$$

for any lamination η with $i(\eta, \lambda) \neq 0$.

Let F_n^+ be the flat structure of $\partial\mathcal{C}_n^+/G_n$. If the F_n^+ lie in a compact set in Teichmüller space then they lie in a compact set in moduli space and the result follows from Mumford's lemma [CEG87]. If not, $F_n^+ \to [\lambda]$ in the Thurston boundary $\mathcal{PML}$. If $\lambda \neq \mu$ then by the fundamental estimate there are laminations $\lambda_n \to \lambda$ and constants $c_n \to \infty$ such that

$$\frac{i(\mu, c_n\lambda_n)}{l_\mu(\partial\mathcal{C}^+(\xi_n)/G_n)} \to 1,$$

forcing $l_\mu(\partial\mathcal{C}_n^+/G_n) = l_\mu(M_n) \to \infty$. Likewise if there is no uniform lower bound on the lengths of closed geodesics on Σ_n^ϵ, we find a sequence $\gamma_n \in \mathcal{S}$ with $l_{\gamma_n}(\Sigma_n^\epsilon) \to 0$ and $\gamma_n \to \eta$ in $\mathcal{PML}$. As before, we conclude $\eta = \mu$.

Now there is always a sequence $d_n \to 0$ such that $d_n\delta_{\gamma_n} \to \mu$ in $\mathcal{ML}$. (Why?) Using $l_{\gamma_n}(\xi_n) \leq l_{\gamma_n}(\Sigma_n^\epsilon)$, we have $l_{d_n\gamma_n}(\xi_n) \to 0$ which, together with continuity of the length function would force $c = 0$.

This shows that the non-cuspidal parts of the pleated surfaces $\partial\mathcal{C}_n^+/G_n$ meet a uniformly bounded neighbourhood of S_n and have uniformly bounded diameter. To apply compactness of pleated surfaces, we also need to know that there is a uniform lower bound on the injectivity radius of $\mathbb{H}^3/G_n$ in some compact neighbourhood of the non-cuspidal part of S_n. If not, there is a short geodesic in $\mathbb{H}^3/G_n$ meeting this neighbourhood. This geodesic is contained in a very large Margulis tube, which therefore entirely contains Σ_n^ϵ. In particular, loops corresponding to two distinct non-commuting elements of G_n lie inside this Margulis tube, which is impossible.

Since $S_n \to S_\infty$ in the geometric topology, we can now apply the compactness of pleated surfaces as in [CEG87, 5.2.2, 5.2.11].

Lecture 5: The Local Pleating Theorem

We shall prove the real length lemma 3.8 and the local pleating theorems 3.9, 3.10. From now on, Σ and $\mathcal{QF}$ always refer to a once punctured torus.

We begin with the real length lemma.

Proof of 3.8.

Proof. [1] The idea of the proof is the following. Suppose that we could show that arbitrarily near ξ_0 there are points $\xi' \in \mathcal{QF}$ with $\xi' \notin \mathcal{P}_\mu^\pm$. Let $t \mapsto \sigma(t)$ be a path in $\mathcal{QF}$ from ξ' to ξ_0. Using continuity of $\xi \mapsto [p\ell^+(\xi)]$, and the fact that $\mathcal{PML}$ is one dimensional, we conclude there are points $\xi_n \in \mathcal{QF}$ with $\xi_n \to \xi_0$ and $\xi_n \in \mathcal{P}_{\mu_n}^\pm$, $\mu_n \in \mathcal{ML}_\mathbb{Q}$. Thus by the key lemma 1.1, $\lambda_{\mu_n}(\xi_n) \in \mathbb{R}$ and by continuity of $p\ell$ again and equicontinuity of λ_{μ_n}, $\lambda_{\mu_n}(\xi_n) \to \lambda_\mu(\xi_0)$.

Thus we need only show that there are points ξ' arbitrarily near ξ_0 with $\xi' \notin \mathcal{P}_\mu^+$. Now the map $\mathcal{QF} - \mathcal{F} \to \mathcal{ML} \times \mathcal{F}$ that takes $q \in \mathcal{QF} - \mathcal{F}$ to $(p\ell^+, F^+)$, where F^+ is the flat structure of $\partial\mathcal{C}^+/G$, is continuous by theorem 3.6. The map is also injective because the hyperbolic structure F^+ together with the bending data $p\ell^+$ completely determine the group G. (See the discussion about irrational quakebends in lecture 6 below.) If $[p\ell^+]$ were constant on an open neighbourhood of ξ_0, then a four dimensional

[1] The original version of these notes contained a more complicated proof. The original proof, which may well still be of interest, may be found at the end of this lecture. The same longer proof appears in the original preprint [KS98, theorem 6.9].

neighbourhood would have a three dimensional image, violating invariance of domain. □

Most of the rest of this lecture is about the proof of the local pleating theorem version 1, 3.9. Recall:

Local Pleating Theorem Version 1 *Suppose $\nu_0 \in \mathcal{ML} - \mathcal{ML}_{\mathbb{Q}}$, $\xi_0 \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$. Then there are neighbourhoods $U \ni \xi_0$ and $W \ni [\nu_0]$ such that $[\delta_\gamma] \in W \cap \mathcal{PML}_{\mathbb{Q}}$, $\xi \in U$, and $\lambda_\gamma(\xi) \in \mathbb{R}$ implies $\xi \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$.*

We begin by showing that version 2, 3.10, is an easy corollary.

Proof of 3.10 from 3.9 Suppose $\nu_0 \in \mathcal{ML}$, $\xi_0 \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$. We have to show that near ξ_0 in $\mathcal{QF}$, the condition $\lambda_{\nu_0}(\xi) \in \mathbb{R}$ implies $\xi \in \mathcal{P}_{\nu_0} \cup \mathcal{F}$. We may as well assume that $\nu_0 \notin \mathcal{ML}_{\mathbb{Q}}$, otherwise this can be proved like the bending away theorem 2.7, see especially Remark (b) following that proof.

Pick $\nu_n \in \mathcal{ML}_{\mathbb{Q}}$, $\nu_n \to \nu_0$. Then $\lambda_{\nu_n} \to \lambda_{\nu_0}$ (uniformly on compact sets in $\mathcal{QF}$), so by Hurwitz's theorem, we can find $\xi_n \in \mathcal{QF}$ with $\xi_n \to \xi$ and $\lambda_{\nu_n}(\xi_n) \in \mathbb{R}$ (in fact with $\lambda_{\nu_n}(\xi_n) = \lambda_{\nu_0}(\xi)$). Since $\xi_0 \in \mathcal{P}_{\nu_0}$, we have $\xi_n \in \mathcal{P}_{\nu_n}$ by the local pleating theorem 3.9, and hence $p\ell(\xi) = \lim p\ell(\xi_n) = \lim(\nu_n) = \nu_0$ as required. □

5.1 The condition $\lambda_\mu(\xi) \in \mathbb{R}$.

Before we begin the proof of theorem 3.9, it is worth taking some time to get a better understanding of the significance of the condition $\lambda_\gamma(\xi) \in \mathbb{R}$. We begin with rational laminations.

Recall from lecture 2 the construction of a developed surface $\phi^{\gamma}_{\tau}(\mathbb{D})$ from the complex Fenchel-Nielsen coordinates $(\lambda_\gamma, \tau_\gamma)$. Clearly, $\phi^{\gamma}_{\tau} : \mathbb{D} \to \mathbb{H}^3$ is a pleated surface as explained in 3.4. It is also clear that its bending locus $B(\phi^{\gamma}_{\tau})$ contains γ, so that ϕ^{γ}_{τ} *realizes* γ (c.f. theorem 3.4). In fact, it is easy to see that in this case, $B(\phi^{\gamma}_{\tau}) = \gamma$, that is, *the bending lamination of the pleated surface realizing γ is exactly γ.* This is very special situation, and is equivalent to the condition $\lambda_\gamma(\xi) \in \mathbb{R}$. To understand this, let us consider $\hat{\gamma} \in \mathcal{GL}$, a general geometric lamination containing γ. It is not hard to see that $\hat{\gamma}$ can contain:

(a) two leaves from the cusp spiralling into γ from opposite sides; these two leaves may each spiral in either direction.

(b) a leaf from the cusp to itself in the homotopy class (on the torus with puncture filled in) of γ.

It is not possible to add any further leaves. Figure 29 (i) shows the geodesic

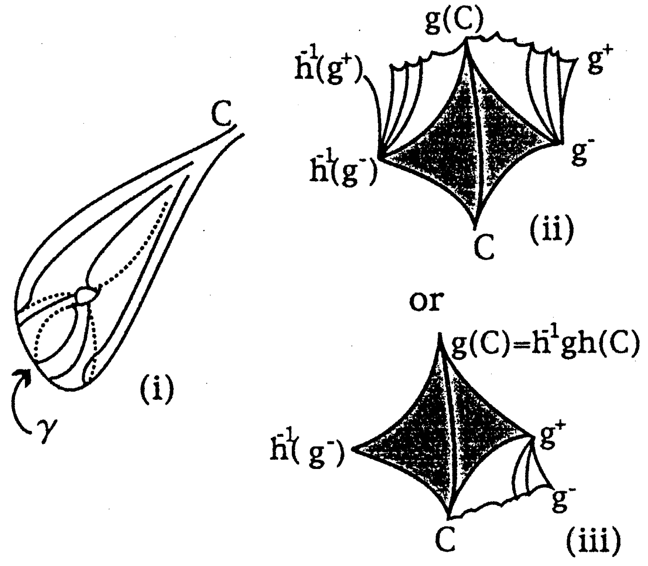


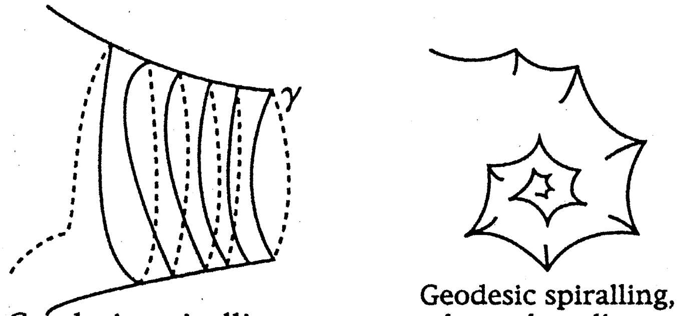
Figure 29:

γ and the two leaves spiralling into γ from each side. Figure 29 (ii) and (iii) shows the lifts of (i) to $\mathbb{H}^3$, after Σ is cut along γ and these two leaves. In these pictures, $g = g(\gamma)$ is the covering transformation whose axis projects to γ and $g^{\pm}$ are its fixed points. The points C and $g(C)$ project to the cusp on Σ and the element $h \in G$ corresponds to a curve which cuts γ once. The shaded areas in (ii) and (iii) project to $\Sigma - \gamma$ and the fan of ideal quadrilaterals with one vertex at g^-, respectively g^+, are its images under g. One can see the geodesics joining $g(C)$ to g^- and C to g^+ spiralling into $\mathrm{Ax}(g)$.

Now notice (c.f. lecture 2, complex F-N coordinates):
$B(\phi^{\gamma}_{\tau}) = \gamma$ iff all of the triangles in (ii) and (iii) are coplanar, which happens iff the points C, g^-, $h^{-1}(g^-)$, $g(C)$ are concyclic, which happens in turn iff $\lambda_g = \lambda_\gamma \in \mathbb{R}$.

The difference between the two cases is shown in figure 30: if $\lambda_\gamma \in \mathbb{R}$ then the developed surface in $\mathbb{H}^3/G$ is flat near γ while if $\lambda_\gamma \notin \mathbb{R}$, then along the lines of the spiral we see a bend.

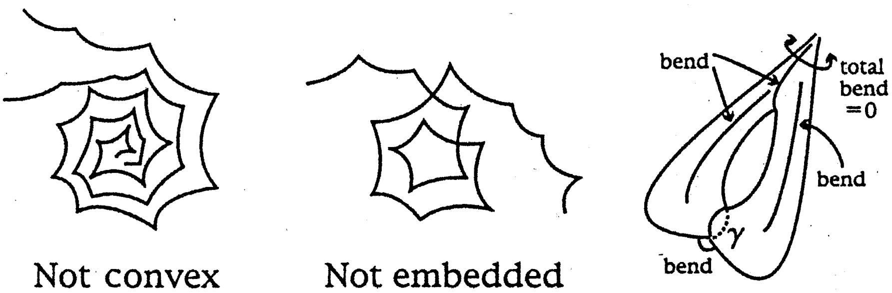
In figure 31 we see the spiralling from both sides. It is clear from this picture that unless $\lambda_\gamma \in \mathbb{R}$, the surface $\phi^{\gamma}_{\tau}(\mathbb{D})$ is never convex and probably not embedded either, so cannot possibly be a component of $\partial\mathcal{C}$.



Geodesic spiralling into γ, Flat case

Geodesic spiralling, bent along line from g(C) to g±

Figure 30:



Not convex

Not embedded

Figure 31:

The last part of figure 31 shows that in general, there is also a bend along the geodesics which run from the cusp to itself (from C to $g(C)$ in figure 29). (But note the total bend round the cusp is 0, so the bending must be in opposite directions along the two lines.)

5.2 Ensuring $\phi^{\gamma}_{\tau}(\mathbb{D})$ is embedded and convex

The main point in proving the local pleating theorem 3.9 is to show that the surface $\phi^{\gamma}_{\tau}(\mathbb{D})$ is *embedded* and *convex*. (Convexity here means that it cuts off a convex half space in $\mathbb{H}^3$.) In this situation, it is not hard to show that ϕ^{γ}_{τ} extends to a continous map $\partial\mathbb{D} \to \partial\mathbb{H}^3$ whose image is exactly the limit set $\Lambda(G)$, and to deduce that $\phi^{\gamma}_{\tau}(\mathbb{D})$ is a component of $\partial\mathcal{C}$. For details, see [KS98, props. 6.2 and 7.5].

From the bad pictures in figure 31, it should be clear that a necessary condition is $\lambda_{\gamma}(\xi) \in \mathbb{R}$. In the torus case, this condition ensures that there are no other bending lines besides the axis of γ itself. Convexity of the half space it cuts off follows automatically. (In general one needs the added condition of convexity: this is one of the difficulties in extending to higher genus.)

Thus $\phi^{\gamma}_{\tau}(\mathbb{D})$ will be a component of $\partial\mathcal{C}$ exactly when it is embedded. It will fail to be embedded if the bending angle is too large compared to the distance between bending lines. (A bent Euclidian surface of infinite extent can *never* be embedded.) The possibility of having an embedded pleated surface exists only because of the property of hyperbolic space expressed in the cone lemma 2.9.

The remaining idea in proving theorem 3.9, is that being embedded is obviously a local property; if $\phi^{\gamma}_{\tau}(\mathbb{D})$ is embedded, so is $\phi^{\gamma'}_{\tau'}(\mathbb{D})$ for nearby τ' and γ', see 2.8 and especially remark (b). Our strategy is to show that under the hypotheses of the theorem, the developed surface $\phi^{\gamma}_{\tau(\xi)}(\mathbb{D})$ is, in a suitable sense, near to the convex hull boundary $\partial\mathcal{C}^+(\xi_0)$.

In general, a component, say $\partial\mathcal{C}^+$, of $\partial\mathcal{C}$ is determined by:

- its intrinsic hyperbolic structure, the *flat structure* $F^+(\xi)$ and
- the bending measure $p\ell^+(\xi)$.

(In the general case the *quakebend construction*, see lecture 6, defines a pleated surface map from $(\mathbb{D}, F^+(\xi)) \to \mathbb{H}^3$ with image $\partial\mathcal{C}^+(\xi)$. This is done by quakebending the flat structure $F^+(\xi)$ along the lamination $|p\ell^+(\xi)|$ by an amount $ip\ell^+(\xi)$.)

The developed surface $\phi^\gamma_\tau(\mathbb{D})$ also carries naturally

- an intrinsic hyperbolic measure
- a "bending measure" $b = b^\gamma_\tau$.

To define the bending measure, let T be a transversal to γ and let $b(T) = i(\gamma, T)\theta$, where $i(\gamma, T)$ is the intersection number of T with γ and θ is the bending angle $\operatorname{Im}\tau$ along γ. The flat pieces in $\phi^\gamma_\tau(\mathbb{D}) - \gamma$ define "pseudo-support planes" to $\phi^\gamma_\tau(\mathbb{D})$ in an obvious way. The developed surface $\phi^\gamma_\tau(\mathbb{D})(\xi)$ will be near $\partial\mathcal{C}^+(\xi')$, $\xi, \xi' \in \mathcal{QF}$, iff the support planes of $\partial\mathcal{C}^+(\xi')$ and the "pseudo-support planes" of $\phi^\gamma_\tau(\mathbb{D})(\xi)$ are close in the geometric topology. This will be enough to ensure $\phi^\gamma_\tau(\mathbb{D})(\xi)$ is embedded and hence equal to $\partial\mathcal{C}^+(\xi)$. More precisely:

Theorem 5.1 (Local Variation 1). ([KS98] prop. 8.3) *Suppose that $\nu_0 \in \mathcal{ML} - \mathcal{ML}_{\mathbb{Q}}$ and $\xi_0 \in \mathcal{P}^+_{\nu_0} \cup \mathcal{F}$. Let $\nu \in \mathcal{ML}_{\mathbb{Q}}$, ν near ν_0 in $\mathcal{ML}$ and suppose $\lambda_\nu(\xi) \in \mathbb{R}$, ξ near ξ_0 in $\mathcal{QF}$. Then $\phi^{|\nu|}_{\tau(\xi)}(\mathbb{D})$ has intrinsic hyperbolic structure and "bending measure" near those of $\partial\mathcal{C}^+(\xi_0)$.*

Here $\phi^{|\nu|}_{\tau(\xi)}(\mathbb{D})$ denotes the developed surface obtained from the complex F-N construction made for the group $(\lambda_{|\nu|}(\xi), \tau_{|\nu|}(\xi))$ where $\lambda_{|\nu|}(\xi)$ is the length of $|\nu|$ at ξ and $\tau_{|\nu|}(\xi)$ the twist along $|\nu|$.

Proof. By the convergence lemma 3.1, $|\nu|$ is close to $|\nu_0|$ in the geometric topology on $\mathcal{GL}$. (This is where we use $\nu_0 \notin \mathcal{ML}_{\mathbb{Q}}$.) This forces the support planes of $\partial\mathcal{C}^+(\xi_0)$ and the pseudo-support planes of $\phi^{|\nu|}_\tau(\xi)$ to be close. The result now follows by a construction similar to the proof of the continuity theorem 3.6, see [KS98, appendix 12.2]. □

Theorem 5.2 (Local Variation 2). ([KS98] theorem 8.6) *Suppose that $\nu \in \mathcal{ML} - \mathcal{ML}_{\mathbb{Q}}$ and $\xi_0 \in \mathcal{P}^+_\nu$. Suppose that $\phi^\gamma_{\tau(\xi)}(\mathbb{D})$ is a developed surface such that*

(i) *γ is close to $|\nu|$ in $\mathcal{GL}$*

(ii) *the flat structures of $\partial\mathcal{C}^+(\xi_0)$ and $\phi^\gamma_{\tau(\xi)}(\mathbb{D})$ are close in $\mathcal{F}$*

(iii) *the bending measures of $\partial\mathcal{C}^+(\xi_0)$ and $\phi^\gamma_{\tau(\xi)}(\mathbb{D})$ are close in $\mathcal{ML}$.*

Then $\phi^{\gamma}_{\tau(\xi)}(\mathbb{D})$ *is convex and embedded and hence equal to* $\partial\mathcal{C}^+(\xi)$.

Proof. If γ is close to $|\nu|$ in $\mathcal{GL}$ then the bending lines and hence the support planes of $\phi^{\gamma}_{\tau(\xi)}(\mathbb{D})$ and $\partial\mathcal{C}^+(\xi_0)$ are close. This requires two points: first, $|\gamma|$ is close to $|\nu|$ in $\mathcal{GL}$, and second, the end points of geodesics in $\Lambda(\xi)$ are close to those in $\Lambda(\xi_0)$. The second assertion follows from the λ-lemma from holomorphic dynamics, see 5.3 below. Now a surface cut out by "pseudo-support planes" which are geometrically close to one which is embedded is still embedded, which completes the proof. Notice that in the torus case convexity of the half space cut out is automatic; in higher genus this would follow from the above proof. $\square$

The proof of theorem 3.9 is now an easy corollary of 5.1 and 5.2.

5.3 Appendix: Alternative proof of 3.8.

The original proof of the real length lemma involved introducing another function $\chi_\mu : \mathcal{QF} \to \mathbb{C}$. This function was the cross-ratio of a certain four points in $\Lambda(G(\xi))$, chosen so as to be concyclic whenever $\xi \in \mathcal{P}^+_\mu$. Thus $\chi_\mu(\xi') \notin \mathbb{R} \Rightarrow \xi' \notin \mathcal{P}^+_\mu$. The function χ_μ is holomorphic and non-constant on $\mathcal{QF}$, which, together with the first paragraph of the proof given above, proves theorem 3.8.

The four points are chosen as follows. Let Σ be a hyperbolic once punctured torus, and suppose $\mu \in \mathcal{ML} - \mathcal{ML}_{\mathbb{Q}}$. (If $\mu \in \mathbb{Q}$ we have nothing to prove.) Such a lamination $|\mu|$ has two boundary leaves which together bound a punctured bigon ([Th79]; recall that no leaf of *measured* geodesic lamination can approach the cusp).

Cutting the bigon along a geodesic which runs from the cusp into one vertex of the bigon, we obtain an ideal quadrilateral. The point is, that in the special case in which $\xi \in \mathcal{P}^+_\mu$ and $\Sigma = \partial\mathcal{C}^+/G(\xi)$, the punctured bigon lies in one hyperbolic plane so that the vertices of the ideal quadrilateral are concyclic. Using the λ-lemma, it is possible to choose the limit points $z_1, \ldots, z_4$ as shown in the diagram so as to vary holomorphically with ξ. (Let $F(\xi) \subset \Lambda(\xi)$ denote the fixed points of hyperbolic elements of $G(\xi)$, $\xi \in \mathcal{QF}$. Let ξ_0 be a fixed group. We have a map

$$F(\xi_0) \times \mathcal{QF} \to \mathbb{C}, \quad (f, \xi) \to \text{the fixed point of } f \text{ at } \xi$$

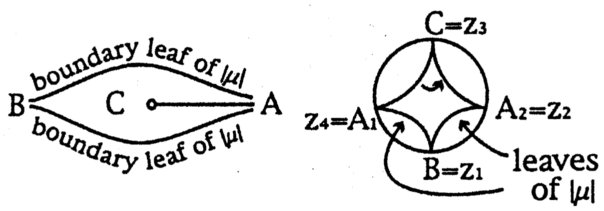


Figure 32:

which is holomorphic in ξ for fixed f and injective for fixed ξ. It therefore extends to a holomorphic map $\Lambda(\xi_0) \times \mathcal{QF} \to \mathbb{C}$.) We set $\chi_\mu(\xi) = [z_1(\xi), \dots, z_4(\xi)]$.)

It remains only to show that χ_μ is non-constant. One way to do this is to use a *shearing deformation* as defined in [Bo96]. To do this, join C to B in figure 32 by a geodesic and shear the right hand triangle CA_2B relative to the left one CA_1B which is held fixed. This is exactly equivalent to moving the point A_2 while fixing C, A_1, B. Such a deformation exists as long as we make sure the holonomy round the puncture remains parabolic. This can be done by making a compensating shear along CA_2.

Lecture 6: QuasiFuchsian space for Once Punctured Tori

In this final lecture we discuss the complete description of $\mathcal{QF}$.

Recall our notation: $G \subset \mathrm{PSL}(2,\mathbb{C})$ is a quasifuchsian punctured torus group, $\mathbb{H}^3/G = \Sigma \times (0,1)$, where Σ is a once punctured torus, and $\partial\mathcal{C}^{\pm}$ are the two components of the convex hull boundary facing the two components $\Omega^{\pm}$ of the regular set $\Omega(G)$. We denote by $p\ell^{\pm}(G) \in \mathcal{ML}(\Sigma)$ the bending measures of $\partial\mathcal{C}^{\pm}(G)$ and set $\mathcal{P}_{\mu,\nu} = \{\xi \in \mathcal{QF} \mid [p\ell^+] = [\mu], [p\ell^-] = [\nu]\}$, $\mu, \nu \in \mathcal{ML}$.

Recall also the Kerckhoff-Thurston picture of $\mathcal{T}(\Sigma)$, naturally identified with Fuchsian space $\mathcal{F} = \mathbb{H}$, [KS98, §3]. As in figure 33, $\mathcal{F}$ is a disc with boundary S^1 which should be thought of as the Thurston compactification $\mathcal{PML}(\Sigma)$, see lecture 3. We also denote the compactification by $\partial_{Th}(\mathcal{F})$. For each $c > 0$, there is a unique path $\mathcal{E}^c_\mu(t)$ of earthquakes along the lamination μ whose length is fixed, $l_\mu \equiv c$, on $\mathcal{E}^c_\mu$. This path converges as $t \to \pm\infty$ to $[\mu] \in \partial_{Th}(\mathcal{F})$. If $[\mu] \neq [\nu]$, then paths $\mathcal{E}^c_\mu(t_\mu)$ and $\mathcal{E}^d_\nu(t_\nu)$ are tangent at the

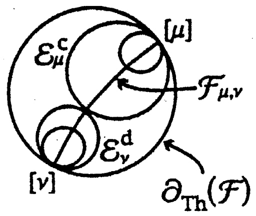


Figure 33:

point $p(\mu,\nu,c) = p(\nu,\mu,d)$ where $\frac{dl_\nu}{d\tau_\mu} = \frac{dl_\mu}{d\tau_\nu} = 0$. (Recall the antisymmetry of the derivative: $\frac{dl_\nu}{d\tau_\mu} = -\frac{dl_\mu}{d\tau_\nu}$. This point is the unique minimum of l_ν on $\mathcal{E}^c_\mu$, and of l_μ on $\mathcal{E}^d_\nu$.) We set $f(\mu,\nu,c) = l_\nu(p(\mu,\nu,c)$, the length of l_ν at the minimum point on $\mathcal{E}^c_\mu$. These critical points all lie on the *critical line* $\mathcal{F}_{\mu,\nu}$ which is a real analytic curve from $[\mu]$ to $[\nu]$. The lengths l_μ, l_ν are monotonic on $\mathcal{F}_{\mu,\nu}$, with $l_\mu \to \infty$ as $\xi \to [\nu]$ and $l_\mu \to 0$ as $\xi \to [\mu]$. The functions $p(\mu,\nu,c)$ and $f(\mu,\nu,c)$ are continuous in all their variables. All these facts are easily deduced from [Ke85] and [Ke83], see [KS98, §3]. (The critical lines are what Kerckhoff has called *lines of minima*, for a discussion see the revised version [KS98a, §3].)

By corollary 2.11 to the bending away theorem 2.7, and its extension to irrational laminations (see the discussion in 6.1 below), we know that the *pleating variety* $\mathcal{P}_{\mu,\nu}$ meets $\mathcal{F}$ along $\mathcal{F}_{\mu,\nu}$. In fact we have

Theorem 6.1 (Non-Singularity Theorem). ([KS98, theorem 2]) *Let* $\mu,\nu \in \mathcal{ML}$, $[\mu] \neq [\nu]$. *Then the map* $\lambda_\mu \times \lambda_\nu \colon \mathcal{P}_{\mu,\nu} \to \mathbb{R}^+ \times \mathbb{R}^+$, $\lambda_\mu \times \lambda_\nu(\xi) = (\lambda_\mu(\xi), \lambda_\nu(\xi))$, *is a diffeomorphism onto the region bounded by the two positive axes in* $\mathbb{R}^+ \times \mathbb{R}^+$ *and the graph of the function* $t \mapsto f(\mu,\nu,t)$.

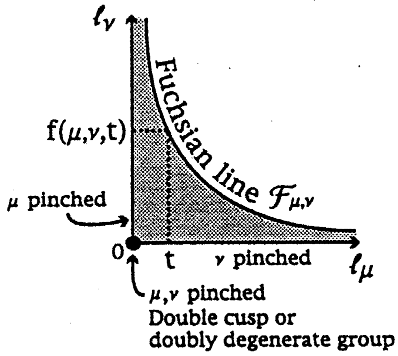


Figure 34:

Figure 34 shows the image of $\mathcal{P}_{\mu,\nu}$ under this map. Notice:

(a) λ_μ, $\lambda_\nu\ |_{\mathcal{P}_{\mu,\nu}}$ are real valued by the real length lemma 3.8.

(b) $f(\mu,\nu,t)$ is the length of ν when $l_\mu = t$. Thus the graph of this function represents the *critical line* $\mathcal{F}_{\mu,\nu}$.

(c) The pleating variety $\mathcal{P}_{\nu,\mu}$, in which $[p\ell^+] = [\nu]$, $[p\ell^-] = [\mu]$, has an identical description. With $\mathcal{P}_{\mu,\nu}$, it forms a continuous sheet which meets $\mathcal{F}$ orthogonally along $\mathcal{F}_{\mu,\nu}$.

(d) Non-singularity implies immediately:

Theorem 6.2 (Pleating Invariant Theorem). ([KS98, theorem 1]) *Fix a choice of $\mu \in [p\ell^+]$ and $\nu \in [p\ell^-]$. Then $\xi \in \mathcal{QF}$ is uniquely determined by $[p\ell^+]$, $[p\ell^-]$, and the values of λ_μ and λ_ν.*

(e) The above shows that not only is $\mathcal{P}_{\mu,\nu}$ diffeomorphic to a real 2-dimensional submanifold in $\mathbb{C}^2 \sim \mathbb{R}^4$, but that it is totally real, i.e. holomorphically equivalent to $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$

(f) The boundary of $\mathcal{P}_{\mu,\nu}$ consists of the Fuchsian line $\mathcal{F}_{\mu,\nu}$ and groups for which either λ_μ or λ_ν vanish. These groups either represent cusps ($\mu,\nu \in \mathcal{ML}_\mathbb{Q}$) or degenerate groups with ending laminations $|\mu|$ or $|\nu|$ (μ or $\nu \notin \mathcal{ML}_\mathbb{Q}$).

<u>Exercise</u> In the special case in which $[\mu] = [\delta_A]$, $[\nu] = [\delta_B]$, i.e. $|p\ell^+|$ and $|p\ell^-|$ are the generators of $G = G(\xi)$, find $\mathcal{P}_{A,B}$ explicitly from the formulae of lecture 2 (special case example 2.1 and 2.13) and verify directly it has the correct shape.

<u>Harder Exercise</u> Do the same thing for $\mathcal{P}_{AB,AB^{-1}}$. (See [KS98, §11.1])

VERY IMPORTANT REMARK It is theorem 6.1, together with the density of rational pleating planes 6.8 below, which enables us to compute arbitrarily accurate pictures of $\mathcal{QF}$ with respect to any given holomorphic parameters. This works as follows. For $\mu,\nu \in \mathcal{ML}_\mathbb{Q}$, then we have only to find $\mathcal{P}_{\gamma,\gamma'}$ where $\gamma = |\mu|$, $\gamma' = |\nu|$. (We remark that, in general, $\mathcal{P}_{\mu,\nu}$ only depends on the projective classes $[\mu]$, $[\nu]$; the graph $f(\mu,\nu,t)$ is scaled by an affine map: $f(c\mu,\nu,t) = f(\mu,\nu,t/c)$ while $f(\mu,c\nu,t) = cf(\mu,\nu,t)$,

$c > 0$.) Now λ_γ, $\lambda_{\gamma'}$ are essentially the (computable) trace functions $\operatorname{Tr} g(\gamma)$, $\operatorname{Tr} g(\gamma')$. Thus we can find the critical line $\mathcal{F}_{\gamma,\gamma'}$ and follow the branch of $\operatorname{Tr} g(\gamma) \times \operatorname{Tr} g(\gamma') \in \mathbb{R} \times \mathbb{R}$ out of $\mathcal{F}$ (where all traces are real). Theorem 6.1 asserts that there will be no other critical points and that we can follow the branch to the boundary $\operatorname{Tr} g(\gamma) = \pm 2$ or $\operatorname{Tr} g(\gamma') = \pm 2$. All of this branch automatically lies in $\mathcal{QF}$: nothing else bad can happen.

Such branches are dense in $\mathcal{QF}$ (and obviously pairwise disjoint). The enumeration of pairs γ, γ' can be carried out by a recursive scheme as explained in lecture 1- they are $\{(p/q, r/s) \in \hat{\mathbb{Q}} \times \hat{\mathbb{Q}} - \text{diagonal}\}$.

It would be very nice to have a computer programme to implement this in practice!

The idea of the proof of the Non-singularity theorem 6.1 is to work in *quakebend planes* which extend the earthquake lines $\mathcal{E}^c_\mu$ holomorphically into $\mathcal{QF} \subset \mathbb{C}^2$. This gives us a family of one complex dimensional holomorphic slices of $\mathcal{QF}$ in which we can apply one dimensional techniques similar to those discussed in lecture 4. First, we briefly discuss *quakebends* when $\mu \notin \mathcal{ML}_{\mathbb{Q}}$.

6.1 Irrational Quakebends

References for this section are [EM87],[KS97].

A *quakebend* on a rational lamination $c\delta_\gamma$, $c > 0$, $\gamma \in \mathcal{S}(\Sigma)$ is just a complex Fenchel Nielsen twist in the case in which the length λ_γ is real: $\mathcal{Q}_{c\delta_\gamma}(\tau) : (\lambda_\gamma, \tau_\gamma) \mapsto (\lambda_\gamma, \tau_\gamma + c\tau)$. Here $(\lambda_\gamma, \tau_\gamma)$ are the complex Fenchel Nielsen coordinates of the starting group. Notice that if the rational lamination is $c\delta_\gamma$ then we twist by $c\tau$. Thus quakebends depend on $\mu \in \mathcal{ML}$ not just $[\mu] \in \mathcal{PML}$.

Now the generalization of the (real) Fenchel-Nielsen twist to irrational laminations μ is an *earthquake*, see for example [Ke83],[Ke85]. A time t earthquake shifts complementary components of the lamination $|\mu|$ on the surface Σ a distance $t_\mu(\tau)$ relative to one another, where $\mu(T)$ is the μ-measure of a transversal T joining the two components. (This is independent of the choice of transversal: Why?) A *quakebend* by $\tau \in \mathbb{C}$ along μ is defined in exactly the same way, where now we *shift* by $(\operatorname{Re}\tau)\mu(T)$ and *bend* by $(\operatorname{Im}\tau)\mu(T)$.

Theorem 6.3. ([EM87, chapter 3]) *Starting from a Fuchsian group $\xi_0 \in \mathcal{F}$, a quakebend induces a well-defined deformation of G. One obtains a*

Figure 35:

representation $\rho_\tau : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{C})$ for which the matrix coefficients of $\rho_\tau(A)$ and $\rho_\tau(B)$ are holomorphic in τ.

The idea of the construction in [EM87] is to approximate the required quakebend by a sequence of finite quakebends, that is, deformations which are products of shifts and twists about a finite number of disjoint lines. This defines a bent surface in $\mathbb{H}^3$ which converges to a developed surface associated to $\mathcal{Q}_\mu(\tau)$.

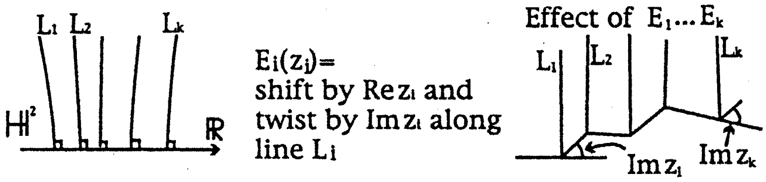


Figure 36:

We replace the (in general infinitely many) lines of $|\mu|$ intersecting a transversal T by lines $L_i, 1 = 1 \ldots, k$; note that these lines do not have to be in the lamination $|\mu|$. The measure of T is divided equally amoung these lines by setting $a = \mu(T)/k$. We then approximate $\mathcal{Q}_\mu(\tau)$ by a product of twists $E_1(\tau a) \ldots E_k(\tau a)$, where $E_i(z)$ is a shift by $\mathrm{Re}\, z$ and a twist by $\mathrm{Im}\, z$ along the axis L_i. This product should be viewed as a kind of approximating Riemann sum.

Remark One can also consider $\mathcal{Q}_\mu(\tau)$ as a limit of complex Fenchel-Nielsen twists $\mathcal{Q}_{\mu_n}(\tau)$ for $\mu_n \overset{\mathcal{ML}}{\to} \mu$, $\mu_n \in \mathcal{ML}_\mathbb{Q}$, but this seems to be technically tricky to handle. (As $c_n\delta_{\gamma_n} \to \mu$ then $c_n \to 0$ and $\mathcal{Q}_{\delta_{c_n\gamma_n}}(\tau) = \mathcal{Q}_{\delta_{\gamma_n}}(\tau/c_n)$ so $\tau_{\gamma_n}/c_n \to \tau_\mu$.)

As in lecture 2, this construction can also be made starting from a point $\xi \in \mathcal{P}_\mu$. The *quakebend plane* $\mathcal{Q}^\xi_\mu$ through $\xi \in \mathcal{P}_\mu \cup \mathcal{F}$ is the set of groups $\mathcal{Q}^\xi_\mu(\tau)$, $\tau \in \mathbb{C}$. (Note that unlike the rational case, it is not immediately obvious whether or not $\mathcal{Q}^\xi_\mu(\tau) = \mathcal{Q}^{\xi'}_\mu(\tau)$ for $\xi \neq \xi'$, $\xi, \xi' \in \mathcal{P}_\mu \cup \mathcal{F}$.) We need the bending away theorem 2.7 for $\mu \notin \mathcal{ML}_{\mathbb{Q}}$. This depended on the cone lemma 2.9. Recall this required a definite distance $d = d(\theta)$ between bending lines, depending on the bending angle θ, (and the cone angle α).

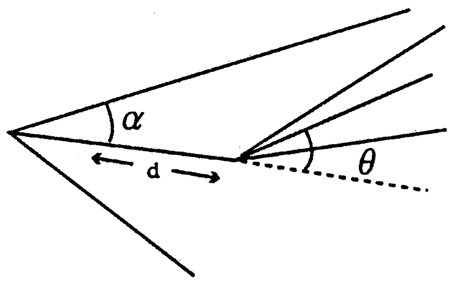


Figure 37:

This now fails, as the distance between bending lines $|\mu|$ is arbitrarily small. However the bending angle along very short transversals is also very small (equal $\operatorname{Im}\tau \cdot \mu(T)$ where $\mu(T) \to 0$). Thus it is possible to prove a suitable variant of the cone lemma by approximation. The details are very technical and are given in [KS97, §8].

6.2 Quakebend Planes

We now study quakebend planes $\mathcal{Q}^{\xi}_{\mu}(\tau)$, for $\xi \in \mathcal{P}_\mu \cup \mathcal{F}$. Our plan is first to consider rational planes for which $\mu = c\delta_\gamma \in \mathcal{ML}_{\mathbb{Q}}$ and then to extend our results to general μ. We shall use the same methods as in lecture 4 to study pleating rays $\mathcal{P}_{\mu,\nu}$ in $\mathcal{Q}_\mu$. Before we begin, we collect some useful results which apply to general $\mathcal{Q}_\mu$.

Lemma 6.4 (Bound on Bend). ([KS98, prop.8.10]) *Suppose that $\mu \in \mathcal{ML}$, $\xi \in \mathcal{P}_\mu \cup \mathcal{F}$. Then given $K > 0$, there exists $B > 0$ such that $|\operatorname{Re}\tau| < K$ and $|\operatorname{Im}\tau| > B$ implies $\mathcal{Q}^{\xi}_{\mu}(\tau) \notin \mathcal{P}_\mu$. (Here $G^{\xi}_{\mu}(\tau)$ denotes the group obtained from $G(\xi)$ by quakebending by τ along the lamination μ.)*

Proof. The idea is similar to that in the discussion of the generalized cone lemma and bending away theorem above. As long as we know that a definite amount of μ-measure is concentrated in some transversal T to $|\mu|$ of bounded length $l(T)$, then by increasing $\operatorname{Im}\tau$ sufficiently we can cause the developed surface in $\mathbb{H}^3$ not to be embedded.

We need the condition $|\operatorname{Re}\tau| < K$ to make the conclusion because otherwise we may have $l(T) \to \infty$. The condition $|\operatorname{Re}\tau| < K$ ensures that the flat structure (obtained by earthquaking G^{ξ}_{μ} by $\operatorname{Re}\mu$ along μ) stays in a bounded set in $\mathcal{F}$. □

Lemma 6.5 (Bound on Twist). ([KS98, lemma 9.4]) *Suppose that $\xi \in \mathcal{P}_{\mu,\nu}$ and let $\mathcal{Q}^{\xi}_{\mu}$ be the quakebend plane through ξ with parameter τ. Then $\tau_n \in \mathcal{Q}^{\xi}_{\mu} \cap \mathcal{P}_{\mu,\nu}$, $|\lambda_\nu(\tau_n)| \to \infty$ implies $|\operatorname{Re}\tau_n| \to \infty$.*

Proof. Since $q_n \in \mathcal{P}_{\mu,\nu}$ we know $\lambda_\nu(q)$ is real. Moreover$\lambda_\nu(q)$ is bounded above by the length of ν on the flat structure of $\partial\mathcal{C}^+/G(q_n)$. This flat structure is determined by the length of μ, which is fixed, and the earthquake parameter $\operatorname{Re}(\tau_n)$. Thus if $|\operatorname{Re}(\tau_n)|$ is bounded, so is $\lambda_\nu(q_n)$. □

Corollary 6.6. *Suppose that $\xi \in \mathcal{P}_{\mu,\nu}$ and let $\mathcal{Q}^{\xi}_{\mu}$ be the quakebend plane through ξ. Then the length function λ_ν is non-constant on $\mathcal{Q}^{\xi}_{\mu} \cap \mathcal{QF}$.*

Proof. If not, then by the local and limit pleating theorems, then $\mathcal{P}_{\mu,\nu} = \mathcal{Q}^{\xi}_{\mu}$, which is impossible by lemma 6.4. □

Corollary 6.7. *Let $\xi \in \mathcal{P}_{\mu,\nu} \cup \mathcal{F}$. In $\mathcal{Q}^{\xi}_{\mu}$, $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}^{\xi}_{\mu}$ is a union of connected components of $\lambda_\nu^{-1}(\mathbb{R}^+)$ and the range of $\lambda_\nu \mid_K$, where $K \subset \mathcal{P}_{\mu,\nu} \cap \mathcal{Q}^{\xi}_{\mu}$ is any connected component of $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}^{\xi}_{\mu}$, is either $(0,\infty)$ or $(0,d)$ where $d = f(\mu,\nu,l_\mu(\xi))$. Moreover there is only one component whose image is $(0,d)$ and the degree of the map on this component is one.*

Proof. Same as 4.1 and 4.3 in lecture 4. □

Corollary 6.8 (Density Theorems). ([KS98, theorems 4 and 10.1])

(a) *The pleating varieties $\mathcal{P}_{\mu,\gamma'}$ for $\gamma' \in \mathcal{ML}_{\mathbb{Q}}$ are dense in $\mathcal{Q}^{\xi}_{\mu}$, for $\xi \in \mathcal{P}_\mu$.*

(b) *The rational pleating varieties $\mathcal{P}_{\gamma,\gamma'}$, $\gamma,\gamma' \in \mathcal{ML}_{\mathbb{Q}}$ are dense in $\mathcal{QF}$.*

Proof. Exercise, see 4.2 in lecture 4. The proof of (b) is an elaboration of the same argument. □

6.3 Rational Quakebend Planes

Now we proceed to study rational quakebend planes $\mathcal{Q}^{\xi}_{\gamma}(\tau)$. Notice that, in terms of complex Fenchel-Nielsen coordinates, $\mathcal{Q}^{\xi}_{\gamma}(\tau) = \{(c,\tau) \in \mathbb{R} \times \mathbb{C} \mid c = \lambda_\gamma(\xi)\}$. Thus $\mathcal{Q}^{\xi}_{\gamma}(\tau)$ is just the set on which the length λ_γ is fixed at its value $\lambda_\gamma(\xi)$ at ξ. Clearly, $\mathcal{Q}^{\xi}_{\gamma}$ meets Fuchsian space $\mathcal{F}$ along the earthquake line $\mathcal{E}^c_\gamma(t)$, where $c = \lambda_\gamma(\xi)$. In this case, the bending away theorem (or its improved version for irrational μ) gives that $\mathcal{P}_{\gamma,\mu} \cap \mathcal{Q}^c_\gamma \neq \emptyset$ for all $\mu \notin [\delta_\gamma]$.

We proceed much as in the case of the Maskit embedding as in lecture 4. Let δ be a complementary generator to γ, i.e., $i(\gamma, \delta) = 1$. We study first the *integral rays* $\mathcal{P}_{\gamma,\gamma^n\delta}$. Recall the formula from 2.13:

$$\cosh(\tau_\gamma) = \cosh(\lambda_\gamma/2)\tanh(\lambda_\gamma/2)$$

or more generally (exercise!)

$$\cosh(\tau_\gamma - n\lambda_\gamma) = \cosh(\lambda_{\gamma^n\delta}/2)\tanh(\lambda_\gamma/2)\ .$$

Using this, it is easy to show that $\mathcal{P}_{\gamma,\gamma^n\delta} \cap \mathcal{Q}^c_\gamma$ is exactly the line segment $\operatorname{Re}\tau_\gamma = nc$, $0 < \operatorname{Im}\tau_\gamma < 2\operatorname{arccosh}(\tanh(c/2))$.

We also prove directly that if $|\operatorname{Im}\tau_\gamma|$ is outside this range, then $G(\tau_\gamma) \notin \mathcal{QF}$. (Exercise, see [PS95, theorem 6.3].) (If $\operatorname{Im}\tau_\gamma < 0$ this just means we have bent in the "opposite" direction, so we are in $\mathcal{P}_{\cdot,\gamma}$ instead of $\mathcal{P}_{\gamma,\cdot}$.)

We can now argue on the same lines as in lecture 4 to show that $\mathcal{P}_{\gamma,\nu}$ for arbitrary ν has a unique connected component with no critical points which meets $\mathcal{Q}^c_\gamma \cap \mathcal{F} = \mathcal{E}^c_\gamma$ in the critical point $p(\gamma, \nu, c)$. We need to use the bound on twist, lemma 6.5, to control the behaviour of λ_ν as $|\tau| \to \infty$.

6.4 Irrational Quakebend Planes

Finally we pass to the case of a general quakebend plane $\mathcal{Q}^\xi_\mu$ with $\mu \notin \mathcal{ML}_\mathbb{Q}$. Notice that we can no longer assume that $\mathcal{Q}^\xi_\mu$ contains $\mathcal{E}^{\lambda_\mu(\xi)}_\mu$; this is a consequence of our proof. The problem is that there are no integral rays to get us started; these were used in the rational case to rule out $\lambda_\nu(\tau) \to \infty$. This is the only problem in extending the proof explained above.

We can rule out components $K \subset \mathcal{P}_{\mu,\nu} \cap \mathcal{Q}^\xi_\mu$ with $\lambda_\nu(K) = (0, \infty)$ by a limiting argument as follows. In the case $\nu = \delta_\gamma \in \mathcal{ML}_\mathbb{Q}$, we have already shown that $\lambda_\gamma(\mathcal{P}_{\mu,\gamma}) = (0, l_\gamma(p(\mu, \delta_\gamma, c)))$, where $c = \mu(\xi)$ and $p(\mu, \delta_\gamma, c)$ is the Kerckhoff critical point of l_γ on $\mathcal{E}^\xi_\mu$. More generally, $\lambda_\nu(\mathcal{P}_{\mu,\nu}) = (0, f(\mu, \nu, c))$ for $\nu \in \mathcal{ML}_\mathbb{Q}$ where $f(\mu, \nu, c) = \lambda_\nu(p(\mu, \nu, c))$. Now let $\nu \in \mathcal{ML}$ and $\nu_n \xrightarrow{\mathcal{ML}} \nu$, $\nu_n \in \mathcal{ML}_\mathbb{Q}$. Let $\xi^* \in \mathcal{P}_{\mu,\nu} \cap \mathcal{Q}^\xi_\mu$. By Hurwitz theorem in $\mathcal{Q}^\xi_\mu$, we can find $\xi_n \to \xi^*$ with $\lambda_{\nu_n}(\xi_n) = \lambda_\nu(\xi^*) \in \mathbb{R}$. By the local pleating theorem, $\xi_n \in \mathcal{P}_{\mu,\nu_n}$ so $\lambda_{\nu_n}(\xi_n) \leq f(\mu, \nu_n, c)$. Now use continuity of f to conclude $\lambda_\nu(\xi^*) \leq f(\mu, \nu, c)$. (Actually this needs a bit of work; we need to prove the bound on $\lambda_\nu(\xi^*)$

with the roles of μ, ν reversed, see [KS98a] theorem 6.) By corollary 4.3 we conclude that $\lambda_\nu \;|_{\mathcal{P}_{\mu,\nu}}$ has degree 1 and hence $\mathcal{P}_{\mu,\nu}$ has a unique connected component with no critical points (except at $\overline{\mathcal{P}_{\mu,\nu}} \cap \mathcal{F} = \{p(\mu, \nu, c)\}$.)

Proof of the non-singularity theorem 6.1. This is now an easy exercise: the quakebend planes intersect $\mathcal{P}_{\mu,\nu}$ in horizontal and vertical segments and we use the local and limit pleating theorems.

We can also prove the following result about the structure of quakebend planes.

Theorem 6.9. ([KS98, §10]) *For any $\mu \in \mathcal{ML}$ and $c > 0$, there is a unique quakebend plane $\mathcal{Q}_\mu = \mathcal{Q}_\mu^c$ which meets $\mathcal{F}$ in $\mathcal{E}_\mu^c$. (i.e., $\mathcal{Q}_\mu^\xi = \mathcal{Q}_\mu^{\xi'}$ for any $\xi, \xi' \in \mathcal{E}_\mu^c$, see the discussion above.) This plane is simply connected and retracts onto $\mathcal{E}_\mu^c$.*

Our final picture of $\mathcal{QF}$ is shown in figure 38. We have shown only half

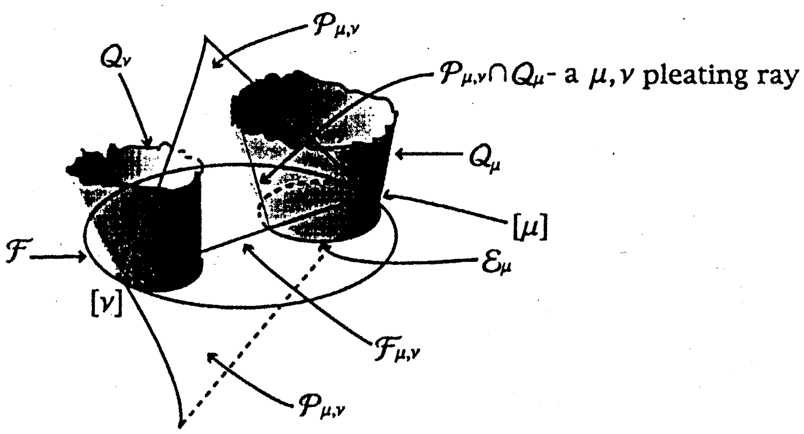


Figure 38:

of each of the quakebend planes $\mathcal{Q}_\mu, \mathcal{Q}_\nu$. Remember: $\mathcal{QF} \subset \mathbb{C}^2$ and $\mathcal{F}$ is embedded as $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ not $\mathbb{C} \subset \mathbb{C}^2$, since the Bers map is $\omega \mapsto (\omega, \overline{\omega})$.

Further Directions Here are some further directions for study.

(a) Higher genus: analytic (the non-singularity theorem) and combinational (how best to do the Farey recursion).

(b) Relationship between lengths and bending angle; connections with deformations of cone manifolds.

(c) Precise shape of boundaries of 1-dimensional slices.

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