MAPPING CLASS GROUPS OF 3-DIMENSIONAL HANDLEBODIES AND MERIDIAN DISKS

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1. INTRODUCTION

A genus $g$ handlebody, $H_g$, is an oriented 3-manifold, which is constructed from a 3-ball with attaching $g$ 1-handles. Let $\text{Diff}^+(H_g)$ (resp. $\text{Diff}^+(\partial H_g)$) be the group of orientation preserving diffeomorphisms on $H_g$ (resp. $\partial H_g$), $\mathcal{H}_g$ (resp. $\mathcal{M}_g$) be a group which consists of isotopy classes of $\text{Diff}^+(H_g)$ (resp. $\text{Diff}^+(\partial H_g)$). Generators of $\mathcal{H}_g$ are given in [11] and [7]. Wajnryb gave a presentation for $\mathcal{H}_g$ in [12]. In this note, we give a presentation for $\mathcal{H}_g$ with using other method. When $g \geq 3$, we use a simplicial action of $\mathcal{H}_g$ on simplicial complex (which is a subcomplex of a contractible complex defined by McCullough [10]) defined as follows: its vertices are isotopy classes of meridian disks in $H_g$ (essential 2-disks properly embedded in $H_g$), and its simplex is a system of isotopy classes of meridian disks which are represented by disks, which are disjoint and non-isotopic each other and whose complements is connected. This complex is $(g-2)$-connected, especially, if $g \geq 3$, it is simply connected. (When $g = 2$, unfortunately, this complex is not simply connected, hence we use a contractible complex defined in [10].) This is subcomplex of a complex $X$ defined by Harer in [5]. Since the orbit space of the former one by $\mathcal{H}_g$ is identical with the latter one by $\mathcal{M}_g$, our method can be applied to giving a presentation for $\mathcal{M}_g$ without using a complex defined by Hatcher and Thurston [4].

This note is a summary of a paper [6].

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2. A Presentation for $\mathcal{H}_g$

Before we state a presentation for $\mathcal{H}_g$, we set notations used there. Sometimes, we indicate an element of $\mathcal{H}_g$ by a figure. In Figure 1, the left hand side figure denotes an element given in the right hand side figure. The symbol $\Leftrightarrow$ means commute with. For example, if $L, M, N$ are any elements of $\mathcal{H}_g$, a relation $L \Leftrightarrow M, N$ means that $LM = ML$, $LN = NL$. The commutator of $A$ and $B$, $A^{-1}B^{-1}AB$, is denoted by $[A, B]$. In this paper, we consider that the group $\mathcal{H}_g$ acts on $H_g$ from the right: for any elements $\phi_1, \phi_2$ of $\mathcal{H}_g$, $\phi_1\phi_2$ means apply $\phi_1$ first, then apply $\phi_2$.

**Theorem 2.1.** Let $a_1, k_1, d_i$ ($2 \leq i \leq g$), $t(2)_{21}$, $r(2)_{21}$ be the elements of $\mathcal{H}_g$ indicated in Figure 2. The group $\mathcal{H}_g$ admits a presentation with generators $a_1, k_1, d_i$ ($2 \leq i \leq g$), $t(2)_{21}$, $r(2)_{21}$ and defining relations:

If $g \geq 4$,

$$d_m^{-1}a_md_m = a_{m-1}, \quad d_m^{-1}a_{m-1}d_m = a_m,$$

$$d_m^{-1}k_md_m = k_{m-1}, \quad d_m^{-1}k_{m-1}d_m = k_m,$$

where $2 \leq m \leq g$,
FIGURE 2
\(r_g = \begin{cases} \{ d_2d_3 \cdots d_g k_1^{-1}k_2^{-1} \cdots k_{g-1}^{-1}, & \text{g is odd,} \\ d_2d_3 \cdots d_g k_1^{-1}k_2^{-1} \cdots k_g^{-1}, & \text{g is even,} \end{cases}\)

\(d_1 = r_g^{-1}d_2r_g,\)

\(r(m)_{1j} = k_mr(m)_{2j}k_m^{-1}, \quad t(m)_{1j} = k_mt(m)_{2j}k_m^{-1},\)

\(r(m)_{ij} = d_jk_{j-1}^{-1}r(m)_{i,j-1}d_j^{-1}, \quad t(m)_{ij} = d_jk_{j-1}^{-1}t(m)_{i,j-1}d_j^{-1},\)

where \(m = 1, \cdots, g, \ i = 1, 2, \ j \neq m, m - 1, \) and index \(j\) is given modulo \(g,\)

\(r(m)_{ij} = d_mk_{m-1}^{-1}r(m-1)_{i,j}k_{m-1}d_m^{-1}, \quad t(m)_{ij} = d_mk_{m-1}^{-1}t(m-1)_{i,j}k_{m-1}d_m^{-1},\)

where \(m = 1, \cdots, g, \ i = 1, 2, \ j \neq m, m - 1, \) and index \(j\) is given modulo \(g,\)

\(c(m)_i[j, j+1, \cdots, j+k] = (\prod_{n=0}^{k}[t(m)_{i,j+n}, r(m)_{i,j+n}^{-1}])a_m^{-2(k+1)},\)

where \(i = 1, 2, \) and index \(j\) is given modulo \(g,\)

\(A_m = k_m^{-2}, \) where \(1 \leq m \leq g,\)

\(t_1 = t(2)21k_1^{-1}a_2,\)

\(d_i \sim d_j, \) where \(|i - j| \geq 2,\)

\(d_i^{-1}d_{i-1}^{-1}d_id_{i-1}d_i = d_{i-1}\) where \(2 \leq i \leq g,\)

\(k_1 \sim k_2, \quad a_1 \sim a_2,\)

\(a_1, k_1 \sim d_i, \) where \(3 \leq i \leq g,\)

\(a_g, k_g \sim d_j, \) where \(2 \leq j \leq g - 1,\)

\(k_1 \sim a_1, a_2,\)
\[(B5) \quad d_3 \Leftarrow a_1,\]
\[(B6) \quad t(3)_{11}, \ r(3)_{11} \Leftarrow t(3)_{22}, \ r(3)_{22},\]
\[t(2)_{21} A_2 t(2)_{11} A_2^{-1} t(2)_{21}^{-1} = t(2)_{11},\]
\[t(2)_{21} A_2 r(2)_{11} A_2^{-1} t(2)_{21}^{-1} = a_2^{-2} A_2 r(2)_{11}\]
\[(B7) \quad r(2)_{21} A_2 t(2)_{11} r(2)_{21}^{-1} = a_2^{-2} t(2)_{11}\]
\[r(2)_{21} A_2 r(2)_{11} A_2^{-1} r(2)_{21}^{-1} = r(2)_{11}\]
\[(B8) \quad k_1^{-1} t(2)_{21} k_1 = t(2)_{21}^{-1} r(2)_{21} t(2)_{21}^{-1} r(2)_{21}^{-1} t(2)_{21},\]
\[k_1^{-1} r(2)_{21} k_1 = t(2)_{21}^{-1} r(2)_{21}^{-1} t(2)_{21},\]
\[(B9) \quad a_1 \Leftarrow r(2)_{21},\]
\[a_1^{-1} t(2)_{21} a_1 = (r(2)_{21})^{-1} t(2)_{21}\]
\[(B10) \quad a_2 \Leftarrow t(2)_{21}, \ r(2)_{21},\]
\[(B11) \quad d_2 \Leftarrow r(2)_{21} (a_1)^{-1}\]
\[(B12) \quad a_3, k_3 \Leftarrow t(2)_{21}, \ r(2)_{21},\]
\[(B13) \quad t(2)_{2,1}, \ r(2)_{2,1} \Leftarrow t(4)_{2,3}, \ r(4)_{2,3},\]
\[(B14) \quad t(2)_{2,1}^{-1} t(3)_{2,1} t(2)_{2,1} = (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2})^{-1} t(3)_{2,1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2}),\]
\[t(2)_{2,1}^{-1} t(3)_{2,1} t(2)_{2,1} = a_3^{-1} t(3)_{2,1}^{-1} r(3)_{2,1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2}),\]
\[t(2)_{2,1}^{-1} t(3)_{2,2} t(2)_{2,1} = a_3^{-1} t(3)_{2,2} t(3)_{2,1}^{-1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2}),\]
\[t(3)_{2,1} \Leftarrow r(3)_{2,2},\]
\[r(2)_{2,1}^{-1} t(3)_{2,1} r(2)_{2,1} = a_3^{-1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2})^{-1} t(3)_{2,1},\]
\[r(2)_{2,1}^{-1} r(3)_{2,1} r(2)_{2,1} = (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2})^{-1} r(3)_{2,1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2}),\]
\[r(2)_{2,1}^{-1} t(3)_{2,2} r(2)_{2,1} = a_3^{-1} t(3)_{2,2} r(3)_{2,1}^{-1} (t(3)_{2,2}^{-1} r(3)_{2,2} t(3)_{2,2}),\]
\[r(2)_{2,1} \Leftarrow r(3)_{2,2},\]
\[(B15) \quad d_2^2 = a_2^{-4} \{(t(2)_{11})^{-1} r(2)_{11} t(2)_{11} (r(2)_{11})^{-1}\} k_1^2 \{(t(2)_{21})^{-1} r(2)_{21} t(2)_{21} (r(2)_{21})^{-1}\} k_1^2,\]
\[ r(g)_{2,g-1}, t(g)_{2,g-1} \Rightarrow d_i \text{ where } 2 \leq i \leq g-2, \]
\[ r(g)_{2,1}, t(g)_{2,1} \Rightarrow d_i \text{ where } 3 \leq i \leq g-1, \]
\[ t(2)_{2,1}, t(2)_{2,1} \Rightarrow d_i \text{ where } 4 \leq i \leq g, \]
\[ d_2^{-1} r(3)_{2,1} d_i \]
\[ = t(3)_{2,1} r(3)_{2,1} t(3)_{2,1} r(3)_{2,1} t(3)_{2,2} x t(3)_{2,1} r(3)_{2,2} t(3)_{2,2} \]
\[ (B17) \]
\[ d_2^{-1} t(3)_{2,1} d_2 \]
\[ = t(3)_{2,1} r(3)_{2,1} t(3)_{2,1} r(3)_{2,1} t(3)_{2,2} x t(3)_{2,1} r(3)_{2,2} t(3)_{2,2} \]
\[ (B18) \]
\[ (d_2^{-1} t_1 d_2 t_1)^3 = d_2^2, \]
\[ (B19) \]
\[ A_g = \prod_{i=1}^{g-1} [t(g)_{2,i}, r(g)_{2,i}^{-1}] a_{g-2(g-1)}, \]
\[ (B20) \]
\[ r(2)_{11} = r(1)_{12} a_1 a_2^{-1} \]
\[ (B21) \]
\[ t(2)_{11} t(2)_{21} A_2 a_2^2 = \prod_{i=3}^{g} \{ t(i)_{2,1} t(i)_{1,1} c(i)_{2,1} [2, \cdots, i-1] c(i)_{1,1} [2, \cdots, i-1] A_i^{-1} a_i^{-4} \}, \]
\[ (B22) \]
\[ k_1^2 A_2 = \prod_{i=3}^{g} \{ c(i)_{2,1} [2, \cdots, i-1] c(i)_{1,1} [2, \cdots, i-1] A_i^{-1} a_i^{-2} \}, \]
\[ (B23) \]

and if \( g = 3 \), the above relations except \( (B13) \) and \( (B17) \) are satisfied and sufficient, and if \( g = 2 \), \( (A1), (A4), (A8), (A9), (B2), (B4), (B7), (B8), (B9), (B10), (B11), (B20) \) and \( (B16') \)
\[ d_2^2 = 1, \]
\[ (B16') \]
\[ (d_2^{-1} t_1 d_2 t_1)^3 = 1, \]
\[ (B19') \]
\[ r(2)_{11} r(2)_{21} A_2 a_2^2 = 1, \]
\[ (B21') \]
\[ t(2)_{11} t(2)_{21} A_2 a_2^2 = 1, \]
\[ (B22') \]
\[ k_1^2 A_2 = 1, \]
\[ (B23') \]
are satisfied and sufficient.

In this presentation, (A*)'s are the relations which define some generators from $a_1$, $k_1$, $d_i$ ($2 \leq i \leq g$), $t(2)_{2,1}$, and $r(2)_{2,1}$ (these are indicated in the sequel of this paper by Figures). (B*)'s are easily checked by drawing some figures. From here to the end of this paper, we will show sufficiency of these relations.

3. DISK COMPLICES

Let $H_g$ be a three dimensional handlebody of genus $g$, $E_1, \ldots , E_l$ be mutually disjoint 2-disks embedded in $\partial H_g$. By a disc in $(H_g, \{E_1, \ldots , E_l\})$ we mean a properly imbedded 2-disc $(D, \partial D) \subseteq (H_g, \partial H_g)$ which is disjoint from $E_1 \cup \cdots \cup E_l$. The disc $D$ is called meridian disc in $(H_g, \{E_1, \ldots , E_l\})$ when $H_g - D$ is connected. Define the nonseparating disc complex of $H_g$ to be the simplicial complex $L'(H_g, \{E_1, \ldots , E_l\})$ whose vertices(0-simplices) are the isotopy classes of meridian discs in $(H_g, \{E_1, \ldots , E_l\})$, and whose simplices are determined by the rule that a collection of $n+1$ distinct vertices spans an $n$-simplex if and only if it admits a collection of representative which are pairwise disjoint. Define the complex $Y(H_g, \{E_1, \ldots , E_l\})$ to be the subcomplex of $L'(H_g, \{E_1, \ldots , E_l\})$ whose $n$-simplex is determined by $n+1$ distinct vertices represented by pairwise disjoint discs $D_0, D_1, \ldots , D_n$ such that $H_g - D_0 \cup D_1 \cup \cdots \cup D_n$ is connected. If there is no distinguished discs $\{E_1, \ldots , E_l\}$ on $\partial H_g$, we denote these complexes by the notation $L'(H_g)$ and $Y(H_g)$. We call a system of meridian discs the set of mutually disjoint and nonisotopic meridian discs in $(H_g, \{E_1, \ldots , E_l\})$. Each simplex of $L'(H_g, \{E_1, \ldots , E_l\})$ is represented by a system of meridian disks. The definition of $L'(H_g, \{E_1, \ldots , E_l\})$ is a modification of the disc complex defined in section 5 of [10]. The following theorem is proved by a slight modification of the proof for Theorem 5.2 of [10].

Theorem 3.1. $L'(H_g, \{E_1, \ldots , E_l\})$ is contractible.  

We will show the following theorem.

Theorem 3.2. $Y(H_g, \{E_1, \ldots , E_l\})$ is $(g-2)$-connected.
This complex $Y(H_g, \{E_1, \ldots, E_l\})$ resembles complices $X$ defined in [5] and $Y$ defined in [3]. We prove the above theorem as in the proof of Theorem 1.1 of [5] and the proof of Proposition of [3].

\textbf{Proof.} We prove this theorem by the induction of genus $g$.

At first, we prove $Y(H_2, \{E_1, \ldots, E_l\})$ is connected (0-connected). By 3.1, $L'(H_2, \{E_1, \ldots, E_l\})$ is connected. As is indicated in Figure 3, there is two types of edges in $L'(H_2, \{E_1, \ldots, E_l\})$. The first one is in $Y(H_2, \{E_1, \ldots, E_l\})$. The second one is not in $Y(H_2, \{E_1, \ldots, E_l\})$, but there is a bypass in $Y(H_2, \{E_1, \ldots, E_l\})$ given in Figure 4. Hence, $Y(H_2, \{E_1, \ldots, E_l\})$ is connected.

We assume, for any integer $k$ smaller than $g$, $Y(H_k, \{E_1, \ldots, E_l'\})$ is $(k-2)$-connected. Let $i$ be an integer smaller than or equal to $g-2$, and $f : S^i \rightarrow Y(H_g, \{E_1, \ldots, E_l\})$ be a continuous map. Since $L'(H_g, \{E_1, \ldots, E_l\})$ is contractible, there is a continuous map $\tilde{f} : D^{i+1} \rightarrow L'(H_g, \{E_1, \ldots, E_l\})$ such that $\tilde{f}|_{\partial D^{i+1}} = f$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Figure 4}
\end{figure}
We may assume \( \tilde{f} \) is piecewise linear with respect to some triangulation of \( D^{i+1} \). If \( \tilde{f}(D^{i+1}) \) is contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \), this means that \( f \) is null-homotpic. We assume \( \tilde{f}(D^{i+1}) \) is not contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \). Then, there is a simplex \( \sigma \) of \( D^{i+1} \) such that \( \tilde{f}(\sigma) \) is not contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \), and this means that there is a representative \( \{D_0 \cup \cdots \cup D_j\} \) of \( \tilde{f}(\sigma) \) with disconnected complement. Let \( G_\sigma \) be a graph defined as follows. Each vertex corresponds to components of \( H_g - D_0 \cup \cdots \cup D_j \). Each edge corresponds to one of \( D_0, \ldots, D_j \) and connecting vertices corresponding to the components containing this disc. Since \( D_0 \cup \cdots \cup D_j \) has disconnected complement, \( G_\sigma \) has at least two components. There is a nonempty subgraph of \( G_\sigma \) whose edges connects distinct vertices. Let \( \mathcal{D} \) be the subsystem of \( \{D_0, \ldots, D_j\} \) which corresponds to the edges of this graph. This system \( \mathcal{D} \) satisfies the following property:

\[
(*) \text{ Let } \mathcal{D} = \{D'_0, \ldots, D'_m\}. \text{ Each } D'_a \in \mathcal{D} \text{ separates } H_g - \bigcup_{n \neq a} D'_n.
\]

A system of meridian discs which satisfies property\((*)\) is called purely separating system of meridian discs, and a simplex of \( L'(H_g, \{E_1, \ldots, E_l\}) \) is called purely separating if it is represented by a purely separating system of meridian discs. \( \mathcal{D} \) represent a face of \( \tilde{f}\sigma \), so there is a simplex \( \tau \) of \( D^{i+1} \) such that \( \tilde{f}(\tau) \) is purely separating. In general, we have shown the following lemma.

**Lemma 3.3.** Each simplex in \( L'(H_g, \{E_1, \ldots, E_l\}) \) has a face which is purely separating.

Let \( \sigma \) be a simplex of \( D^{i+1} \) of maximal dimension \( p \) such that \( \tilde{f}(\sigma) \) is purely separating. This simplex \( \sigma \) is not contained in \( \partial D^{i+1} \), hence, \( \text{Link } \sigma \) is homeomorphic to \( S^{i-p} \). Let \( \{D_0, \ldots, D_p\} \) be a system of meridian discs which represents \( \tilde{f}(\sigma) \), and let \( M = H_g - D_0 \cup \cdots \cup D_p \). We define the complex \( Y(M, \ast) \) for \( (M, \{E_1, \ldots, E_l\}, \text{discs which are emerged as a cut end along } D_0 \cup \cdots \cup D_p) \) in the same manner as \( Y(H_g, \{E_1, \ldots, E_l\}) \). Then, each disc which represents each vertex of \( \text{Link } \sigma \) does not separate \( M \). In fact, if some vertex \( v \) of \( \text{Link } \sigma \) separates \( M \), then
$	ilde{f}(v * \sigma)$ is purely separating, and this fact contradicts the maximality assumption of $p$. Hence, each vertex of $\text{Link} \ \sigma$ is mapped into $Y(M, *)$ by $\tilde{f}$. If there is a simplex $\rho$ of $\text{Link} \ \sigma$ such that $\tilde{f}(\rho)$ separates $M$, then, by 3.3 there is a face $\tau$ of $\rho$ such that $\tilde{f}(\tau)$ is purely separating. Then $\tilde{f}(\tau * \sigma)$ is purely separating, and $\tau * \sigma \subset \text{Link} \ \sigma \subset D^{i+1}$. This fact contradicts the maximality assumption of $p$. Hence, $\tilde{f}(\text{Link} \ \sigma) \subset Y(M, *)$.

By the way, $M$ is a disjoint union of handle bodies, and the total number $m$ of these genera is at least $g - p$. $Y(M)$ is a join of $Y's$ of each components of $M$. Hence, by the hypothesis for the induction, $Y(M)$ is $(m - 2)$-connected. Here, we remember that $i \leq g - 2$, then we can see $i - p \leq g - 2 - p \leq m - 2$. This shows that $\tilde{f}|_{\text{Link} \ \sigma}$ is null homotopic in $Y(M)$, hence, in $Y(H_g, \{E_1, \ldots, E_l\})$. Therefore, we can homotope $\tilde{f}$ such that $\tilde{f}(\text{Link} \ \sigma) \subset Y(H_g, \{E_1, \ldots, E_l\})$. We do the same way for other simplices whose images are purely separating, then $\tilde{f}$ is homotoped to a continuous map whose image is in $Y(H_g, \{E_1, \ldots, E_l\})$. $\square$

4. Obtaining a presentation from the action of $\mathcal{H}_g$ on $Y(H_g)$

For each element $\phi$ of $\mathcal{H}_g$ and simplex $([D_0], \ldots, [D_n])$ of $Y(H_g)$ (resp.$\Delta'(H_g)$ ), $([\phi(D_0)], \ldots, [\phi(D_n)])$ is also a simplex of $Y(H_g)$ (resp.$\Delta'(H_g)$ ). Hence, we
can define a right action of $\mathcal{H}_g$ on $Y(H_g)$ (resp. $\Delta'(H_g)$) by $\phi([D_0], \ldots, [D_n]) = ([\phi(D_0)], \ldots, [\phi(D_n)])$. We can see that, if $g = 2$, each of the 2-simplices of $\Delta'(H_2)/\mathcal{H}_2$, the 1-simplices of $\Delta'(H_2)/\mathcal{H}_2$ and the vertices of $\Delta'(H_2)/\mathcal{H}_2$ consists of one element, each of which is represented by $([D_0], [D_1], [D_2])$, $([D_0], [D_1])$, and $([D_0])$, where $D_0, D_1, D_2$ are indicated in Figure 5, and if $g \geq 3$, each of the 2-simplices of $Y(H_g)/\mathcal{H}_g$, the 1-simplices of $Y(H_g)/\mathcal{H}_g$ and the vertices of $Y(H_g)/\mathcal{H}_g$ consists of one element, each of which is represented by $([D_0], [D_1], [D_2])$, $([D_0], [D_1])$, and $([D_0])$, where $D_0, D_1, D_2$ are indicated in Figure 6. If stabilizer of each vertex is finitely presented, and is that of each 1-simplex is finitely generated, we can obtain a presentation for $\mathcal{H}_g$ as in the way of [9], [12]. Here, we will mention these method. The action of $\mathcal{H}_2$ on $\Delta'(H_2)$ is similar to the action of $\mathcal{H}_g$ on 2-skelon of $Y(H_g)$ when $g \geq 3$. Hence, we mention just on the case of $g \geq 3$.

We fix a vertex $v_0$ of $Y(H_g)$, fix an edge (= a 1-simplex with orientation) $e_0$ of $Y(H_g)$ which emanates from $v_0$ and fix a 2-simplex $f_0$ of $Y(H_g)$ which contains $v_0$. Let $D_0, D_1$ and $D_2$ be meridian disks indicated in Figure 6, we set $v_0 = [D_0], e_0 = ([D_0], [D_1])$ and $f_0 = ([D_0], [D_1], [D_2])$. We choose an element $r_0$ of $\mathcal{H}_g$ which switches the vertices of $e_0$, in our situation, we set $r_0 = d_g$. By this notation, we see $e_0 = (v_0, (v_0)d_g)$ We denote the stabilizer of $v_0$ by $(\mathcal{H}_g)_{v_0}$, that of $e_0$ by $(\mathcal{H}_g)_{e_0}$ and an infinite cyclic group generated by $d_g$ by $< d_g >$. The free product $(\mathcal{H}_g)_{v_0} * < d_g >$ with the following three types of relation define a presentation for $\mathcal{H}_g$.

(Y1) $d_g^2 = \text{a presentation of } d_g^2$ as an element of $(\mathcal{H}_g)_{v_0}$.

(Y2) For each element $t$ of $(\mathcal{H}_g)_{e_1}$, $(d_g)^{-1}( \text{a presentation of } t \text{ as an element of } (\mathcal{H}_g)_{v_0})d_g$

$= \text{a presentation of } (d_g)^{-1}td_g \text{ as an element of } (\mathcal{H}_g)_{v_0}.$

(Y3) For the loop $\partial f_0$ in $Y(H_g)$, we define an element $W_{f_0}$ in the follwing manner. The loop $\partial f_0$ consists of three vertices $v_0, v_1, v_2$ and three edges $e_1, e_2, e_3$ such that $e_1 = (v_0, v_1), e_2 = (v_1, v_2), e_3 = (v_2, v_0)$. There is an element $h_1$ of $(\mathcal{H}_g)_{v_0}$ such that $e_0h_1 = e_1$ i.e. $e_1 = (v_0, (v_0)d_gh_1)$, then $e_2(d_gh_1)^{-1}$ is an edge emanat-
ing from $v_0$. Hence, there is an element $h_2$ of $(\mathcal{H}_g)_{v_0}$ such that $e_0 h_2 = e_2 (d_g h_1)^{-1}$ i.e. $e_2 = ((v_0) d_g h_1, (v_0) d_g h_2 d_g h_1)$, then $e_3 (d_g h_2 d_g h_1)^{-1}$ is an edge emanating from $v_0$. So, there is an element $h_3$ of $(\mathcal{H}_g)_{v_0}$ such that $e_0 h_3 = e_2 (d_g h_3 d_g h_2 d_g h_1)$. We define $W_{f_0} = d_g h_3 d_g h_2 d_g h_1$. This element $W_{f_0}$ fixes $v_0$, so the following is a relation for $\mathcal{H}_g$. $W_{f_0}$ is a presentation of $W_{f_0}$ as an element of $(\mathcal{H}_g)_{v_0}$. If $g \geq 3$, then this type of relation is $d_g d_{g-1} d_g = d_{g-1} d_g d_{g-1}$, and if $g = 2$, then this type of relation is $(d_2^{-1} t_1 d_2 t_1)^3 = 1$, where $t_1 = t(2)^2 k_1^{-1} a_2$.

For subsets $A_1, \ldots, A_n$ of $H_g$, we define $Diff^+(H_g, A_1, \ldots, A_n) = \{\phi \in Diff^+(H_g)| \phi(A_1) = A_1, \cdots, \phi(A_n) = A_n\}$. We can see $(\mathcal{H}_g)_{v_0} = \pi_0(Diff^+(H_g, D_0))$, and $(\mathcal{H}_g)_{e_0} = \pi_0(Diff^+(H_g, D_0, D_1))$ Let $\Sigma_g = \partial H_g$ and let $F_n\Sigma_g$ be the space of all ordered $n$-tuples of distinct points of $\Sigma_g$: $F_n\Sigma_g = \{(p_1, \cdots, p_n)|$ each $p_i \in \Sigma_g$, and $p_i \neq p_j$ if $i \neq j\}$. We define the $n$-string pure braid group $P_n(\Sigma_g)$ to be the fundamental group of $F_n \Sigma_g$. Let $p_1$ and $p_2$ be points on $\partial H_{g-1}$, and $p_{0,1}, p_{0,2}, p_{1,1},$ and $p_{1,2}$ be points on $\partial H_{g-2}$.

We can get a presentation for $(\mathcal{H}_g)_{v_0}$ by investigating the following three exact sequences:

\begin{enumerate}
  \item $P_2 \Sigma_{g-1} \xrightarrow{\beta} \pi_0(Diff^+(H_{g-1}, p_1, p_2)) \xrightarrow{\alpha^#} \mathcal{H}_{g-1} (= \pi_0(Diff^+(H_{g-1}))) \longrightarrow 0,$
  \item $0 \longrightarrow \pi_0(Diff^+(H_{g-1}, p_1, p_2)) \xrightarrow{\delta} \pi_0(Diff^+(H_{g-1}, \{p_1, p_2\})) \xrightarrow{\gamma} \mathbb{Z}_2 \longrightarrow 0,$
  \item $0 \longrightarrow \mathbb{Z} \longrightarrow \pi_0(Diff^+(H_g, D_0)) \xrightarrow{\lambda} \pi_0(Diff^+(H_{g-1}, \{p_1, p_2\})) \longrightarrow 0.$
\end{enumerate}

We can get a set of generators of $(\mathcal{H}_g)_{e_0}$ by investigating the following three exact sequences:

\begin{enumerate}
  \item $P_4 \Sigma_{g-2} \xrightarrow{\beta} \pi_0(Diff^+(H_{g-2}, p_{0,1}, p_{0,2}, p_{1,1}, p_{1,2})) \xrightarrow{\alpha^#} \mathcal{H}_{g-2} (= \pi_0(Diff^+(H_{g-2}))) \longrightarrow 0,$
  \item $0 \longrightarrow \pi_0(Diff^+(H_{g-2}, p_{0,1}, p_{0,2}, p_{1,1}, p_{1,2})) \xrightarrow{\delta} \pi_0(Diff^+(H_{g-2}, \{p_{0,1}, p_{0,2}\}, \{p_{1,1}, p_{1,2}\})) \xrightarrow{\gamma} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 0.$
\end{enumerate}
$0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow \pi_0(Diff^+(H_g, D_0, D_1))$
\[\lambda \pi_0(Diff^+(H_{g-2}, \{p_0,1, p_0,2\}, \{p_1,1, p_1,2\})) \longrightarrow 0.\]

We can give relations of type (Y1) and (Y2) by drawing some figures.
For details please see [6].

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REFERENCES


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