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MAPPING CLASS GROUPS OF 3-DIMENSIONAL HANDLEBODIES AND MERIDIAN DISKS

SUSUMU HIROSE

1. INTRODUCTION

A genus $g$ handlebody, $H_g$, is an oriented 3-manifold, which is constructed from a 3-ball with attaching $g$ 1-handles. Let $Diff^+(H_g)$ (resp. $Diff^+(\partial H_g)$) be the group of orientation preserving diffeomorphisms on $H_g$ (resp. $\partial H_g$), $\mathcal{H}_g$ (resp. $\mathcal{M}_g$) be a group which consists of isotopy classes of $Diff^+(H_g)$ (resp. $Diff^+(\partial H_g)$). Generators of $\mathcal{H}_g$ are given in [11] and [7]. Wajnryb gave a presentation for $\mathcal{H}_g$ in [12]. In this note, we give a presentation for $\mathcal{H}_g$ with using other method. When $g \geq 3$, we use a simplicial action of $\mathcal{H}_g$ on simplicial complex (which is a subcomplex of a contractible complex defined by McCullough [10]) defined as follows: its vertices are isotopy classes of meridian disks in $H_g$ (essential 2-disks properly embedded in $H_g$), and its simplex is a system of isotopy classes of meridian disks which are represented by disks, which are disjoint and non-isotopic each other and whose complements is connected. This complex is $(g-2)$-connected, especially, if $g \geq 3$, it is simply connected. (When $g = 2$, unfortunatelly, this complex is not simply connected, hence we use a contractible complex defined in [10].) This is subcomplex of a complex $X$ defined by Harer in [5]. Since the orbit space of the former one by $\mathcal{H}_g$ is identical with the latter one by $\mathcal{M}_g$, our method can be applied to giving a presentation for $\mathcal{M}_g$ without using a complex defined by Hatcher and Thurston [4].

This note is a summary of a paper [6].

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2. A presentation for $\mathcal{H}_g$

Before we state a presentation for $\mathcal{H}_g$, we set notations used there. Sometimes, we indicate an element of $\mathcal{H}_g$ by a figure. In Figure 1, the left hand side figure denotes an element given in the right hand side figure. The symbol $\rightarrow$ means commute with. For example, if $L, M, N$ are any elements of $\mathcal{H}_g$, a relation $L \rightarrow M, N$ means that $LM = ML, LN = NL$. The commutator of $A$ and $B$, $A^{-1}B^{-1}AB$, is denoted by $[A, B]$. In this paper, we consider that the group $\mathcal{H}_g$ acts on $H_g$ from the right: for any elements $\phi_1, \phi_2$ of $\mathcal{H}_g$, $\phi_1\phi_2$ means apply $\phi_1$ first, then apply $\phi_2$.

**Theorem 2.1.** Let $a_1, k_1, d_i (2 \leq i \leq g), t(2)_{21}, r(2)_{21}$ be the elements of $\mathcal{H}_g$ indicated in Figure 2. The group $\mathcal{H}_g$ admits a presentation with generators $a_1, k_1, d_i (2 \leq i \leq g), t(2)_{21}, r(2)_{21}$ and defining relations:

if $g \geq 4$,

\[
\begin{align*}
 d_m^{-1}a_m d_m &= a_{m-1}, & d_m^{-1}a_{m-1}d_m &= a_m, \\
 d_m^{-1}k_m d_m &= k_{m-1}, & d_m^{-1}k_{m-1}d_m &= k_m,
\end{align*}
\]

(A1)

where $2 \leq m \leq g$. 

Figure 2
\[ r_g = \begin{cases} 
  d_2d_3 \cdots d_gk_1^{-1}k_2^{-1} \cdots k_{g-1}^{-1}, & \text{g is odd}, \\
  d_2d_3 \cdots d_gk_1^{-1}k_2^{-1} \cdots k_g^{-1}, & \text{g is even}, 
\end{cases} \]

(A3) \[ d_1 = r_g^{-1} \, d_2 \, r_g, \]

(A4) \[ r(m)_{ij} = k_m r(m)_{2j}k_m^{-1}, \quad t(m)_{ij} = k_m t(m)_{2j}k_m^{-1}, \]

(A5) \[
\begin{align*}
  r(m)_{ij} &= d_jk_{j-1}^{-1}r(m)_{i,j-1}k_{j-1}d_j^{-1}, \\
  t(m)_{ij} &= d_jk_{j-1}^{-1}t(m)_{i,j-1}k_{j-1}d_j^{-1},
\end{align*}
\]
where \( m = 1, \ldots, g \), \( i = 1, 2 \), \( j \neq m, m - 1 \), and index \( j \) is given modulo \( g \),

(A6) \[
\begin{align*}
  r(m)_{ij} &= d_mk_{m-1}^{-1}r(m-1)_{i,j}k_{m-1}d_m^{-1}, \\
  t(m)_{ij} &= d_mk_{m-1}^{-1}t(m-1)_{i,j}k_{m-1}d_m^{-1},
\end{align*}
\]
where \( m = 1, \ldots, g \), \( i = 1, 2 \), \( j \neq m, m - 1 \), and index \( j \) is given modulo \( g \),

(A7) \[ c(m)_{i,j+1, \ldots, j+k} = \left( \prod_{n=0}^{k} t(m)_{i,j+n}, r(m)_{i,j+n}^{-1} \right) a_m^{-2k+1}, \]
where \( i = 1, 2 \), and index \( j \) is given modulo \( g \),

(A8) \[ A_m = k_m^{-2}, \quad \text{where } 1 \leq m \leq g, \]

(A9) \[ t_1 = t(2)_1k_1^{-1}a_2, \]

(B1) \[ d_i \Leftrightarrow d_j, \quad \text{where } |i - j| \geq 2, \]
\[ d_i^{-1}d_{i-1}^{-1}d_id_{i-1} = d_{i-1} \quad \text{where } 2 \leq i \leq g, \]

(B2) \[ k_1 \Leftrightarrow k_2, \]
\[ a_1 \Leftrightarrow a_2, \]

(B3) \[ a_1, k_1 \Leftrightarrow d_i, \quad \text{where } 3 \leq i \leq g, \]
\[ a_g, k_g \Leftrightarrow d_j, \quad \text{where } 2 \leq j \leq g - 1, \]

(B4) \[ k_1 \Leftrightarrow a_1, a_2, \]
\(d_3 \Leftarrow a_1,\)

\(t(3)_{11}, r(3)_{11} \Leftarrow t(3)_{22}, r(3)_{22},\)

\[t(2)_{21} A_2 t(2)_{11} A_2^{-1} t(2)_{21}^{-1} = t(2)_{11},\]

\[t(2)_{21} A_2 r(2)_{11} A_2^{-1} t(2)_{21}^{-1} = a_2^{-2} A_2 r(2)_{11}\]

\(r(2)_{21} A_2 t(2)_{11} r(2)_{21}^{-1} = a_2^{-2} t(2)_{11}\)

\[r(2)_{21} A_2 r(2)_{11} A_2^{-1} r(2)_{21}^{-1} = r(2)_{11}\]

\(r(2)_{21} r(2)_{21} = t(2)_{21}^{-1} r(2)_{21} t(2)_{21}^{-1} t(2)_{21},\)

\(k_1^{-1} t(2)_{21} k_1 = t(2)_{21}^{-1} r(2)_{21} t(2)_{21}^{-1} r(2)_{21}^{-1} t(2)_{21},\)

\(k_1^{-1} (r(2)_{21} k_1 = t(2)_{21}^{-1} r(2)_{21}^{-1} t(2)_{21},\)

\(a_1 \Leftarrow r(2)_{21},\)

\[a_1^{-1} t(2)_{21} a_1 = (r(2)_{21})^{-1} t(2)_{21}\]

\(a_2 \Leftarrow t(2)_{21}, r(2)_{21},\)

\(d_2 \Leftarrow r(2)_{21} (a_1)^{-1}\)

\(a_3, k_3 \Leftarrow t(2)_{21}, r(2)_{21},\)

\(t(2)_{21}, r(2)_{21} \Leftarrow t(4)_{2,3}, r(4)_{2,3},\)

\[t(2)_{2,1} t(3)_{2,1} t(2)_{2,1} = (t(3)_{2,1}^{-1} r(3)_{2,2})^{-1} t(3)_{2,1} t(3)_{2,1}^{-1} r(3)_{2,2} t(3)_{2,2},\]

\[t(2)_{2,1} t(3)_{2,1}^{-1} r(3)_{2,2} t(2)_{2,1} = a_3^{-1} t(3)_{2,2} t(3)_{2,1}^{-1} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\]

\[t(2)_{2,1} t(3)_{2,2} t(2)_{2,1} = a_3^{-1} t(3)_{2,2} t(3)_{2,1}^{-1} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\]

\(r(2)_{2,1} t(3)_{2,2} r(2)_{2,1} = a_3^{-1} t(3)_{2,2} r(3)_{2,1}^{-1} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\)

\(r(2)_{2,1} r(3)_{2,2} r(2)_{2,1} = (t(3)_{2,2}^{-1} r(3)_{2,2})^{-1} r(3)_{2,2} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\)

\(r(2)_{2,1} t(3)_{2,2} r(2)_{2,1} = a_3^{-1} t(3)_{2,2} r(3)_{2,1}^{-1} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\)

\(r(2)_{2,1} r(3)_{2,2} r(2)_{2,1} = a_3^{-1} t(3)_{2,2} r(3)_{2,1}^{-1} t(3)_{2,2} r(3)_{2,2} t(3)_{2,2},\)

\(r(2)_{2,1} \Leftarrow r(3)_{2,2},\)

\(d_2^2 = a_2^{-4} \{(t(2)_{11})^{-1} r(2)_{11} t(2)_{11} r(2)_{11}^{-1} \} k_1^2 \{(t(2)_{21})^{-1} r(2)_{21} t(2)_{21} r(2)_{21}^{-1} \} k_1^2,"
\[ r(g)_{2,g-1}, t(g)_{2,g-1} \iff d_i \text{ where } 2 \leq i \leq g - 2, \]
\[ r(g)_{2,1}, t(g)_{2,1} \iff d_i \text{ where } 3 \leq i \leq g - 1, \]
\[ t(2)_{2,1}, t(2)_{2,1} \iff d_i \text{ where } 4 \leq i \leq g, \]

(B17)
\[ d_2^{-1}r(3)_{2,1}d_i = t(3)_{2,1}^{-1}r(3)_{2,1}t(3)_{2,1}^{-1}r(3)_{2,2}^{-1}t(3)_{2,2}^{-1}r(3)_{2,2}^{-1}t(3)_{2,2}^{-1}, \]
\[ d_2^{-1}t(3)_{2,1}d_2 = t(3)_{2,1}^{-1}r(3)_{2,1}t(3)_{2,1}^{-1}r(3)_{2,2}^{-1}t(3)_{2,2}^{-1}r(3)_{2,2}^{-1}t(3)_{2,2}^{-1}, \]

(B18)
\[ (d_2^{-1}t_1d_2t_1)^3 = d_2, \]

(B19)
\[ A_g = \prod_{i=1}^{g-1} [t(g)_{2,i}, r(g)_{2,i}^{-1}]a_g^{-2(g-1)}, \]

(B20)
\[ r(2)_{11} = r(1)_{12}a_2^{-1}, \]

(B21)
\[ t(2)_{11}t(2)_{21}A_2a_2^2 = \prod_{i=3}^{g} \{c(i)_2[2, \cdots , i-1]c(i)_{1}[2, \cdots , i-1]A_i^{-1}a_i^{-2}\}, \]

(B22)
\[ k_2^2A_2 = \prod_{i=3}^{g} \{c(i)_2[2, \cdots , i-1]c(i)_{1}[2, \cdots , i-1]A_i^{-1}a_i^{-2}\}, \]

and if \( g = 3 \), the above relations except (B13) and (B17) are satisfied and sufficient, and if \( g = 2 \), (A1), (A4), (A8), (A9), (B2), (B4), (B7), (B8), (B9), (B10), (B11), (B20) and

(B16')
\[ d_2^2 = 1, \]

(B19')
\[ (d_2^{-1}t_1d_2t_1)^3 = 1, \]

(B21')
\[ r(2)_{11}r(2)_{21}A_2a_2^2 = 1, \]

(B22')
\[ t(2)_{11}t(2)_{21}A_2a_2^2 = 1, \]

(B23')
\[ k_2^2A_2 = 1, \]
are satisfied and sufficient.

In this presentation, (A*)'s are the relations which define some generators from \( a_1, k_1, d_i (2 \leq i \leq g), t(2)_{2,1}, \) and \( r(2)_{2,1} \) (these are indicated in the sequel of this paper by Figures). (B*)'s are easily checked by drawing some figures. From here to the end of this paper, we will show sufficiency of these relations.

3. Disk complices

Let \( H_g \) be a three dimensional handlebody of genus \( g \), \( E_1, \ldots, E_l \) be mutually disjoint 2-disks embedded in \( \partial H_g \). By a disc in \( (H_g, \{E_1, \ldots, E_l\}) \) we mean a properly imbedded 2-disc \( (D, \partial D) \subseteq (H_g, \partial H_g) \) which is disjoint from \( E_1 \cup \cdots \cup E_l \). The disc \( D \) is called meridian disc in \( (H_g, \{E_1, \ldots, E_l\}) \) when \( H_g - D \) is connected. Define the nonseparating disk complex of \( H_g \) to be the simplicial complex \( L'(H_g, \{E_1, \ldots, E_I\}) \) whose vertices(0-simplices) are the isotopy classes of meridian discs in \( (H_g, \{E_1, \ldots, E_I\}) \), and whose simplices are determined by the rule that a collection of \( n+1 \) distinct vertices spans an \( n \)-simplex if and only if it admits a collection of representative which are pairwise disjoint. Define the complex \( Y(H_g, \{E_1, \ldots, E_I\}) \) to be the subcomplex of \( L'(H_g, \{E_1, \ldots, E_I\}) \) whose \( n \)-simplex is determined by \( n+1 \) distinct vertices represented by pairwise disjoint discs \( D_0, D_1, \ldots, D_n \) such that \( H_g - D_0 \cup D_1 \cup \cdots \cup D_n \) is connected. If there is no distinguished discs \( \{E_1, \ldots, E_I\} \) on \( \partial H_g \), we denote these complices by the notation \( L'(H_g) \) and \( Y(H_g) \). We call a system of meridian discs the set of mutually disjoint and nonisotopic meridian discs in \( (H_g, \{E_1, \ldots, E_I\}) \). Each simplex of \( L'(H_g, \{E_1, \ldots, E_I\}) \) is represented by a system of meridian disks. The definition of \( L'(H_g, \{E_1, \ldots, E_I\}) \) is a modification of the disc complex defined in section 5 of [10]. The following theorem is proved by a slight modification of the proof for Theorem 5.2 of [10].

**Theorem 3.1.** \( L'(H_g, \{E_1, \ldots, E_I\}) \) is contractible. \( \square \)

We will show the following theorem.

**Theorem 3.2.** \( Y(H_g, \{E_1, \ldots, E_I\}) \) is \( (g-2) \)-connected.
This complex $Y(H_g, \{E_1, \ldots, E_l\})$ resembles complexes $X$ defined in [5] and $Y$ defined in [3]. We prove the above theorem as in the proof of Theorem 1.1 of [5] and the proof of Proposition of [3].

Proof. We prove this theorem by the induction of genus $g$.

At first, we prove $Y(H_2, \{E_1, \ldots, E_l\})$ is connected (0-connected). By 3.1, $L'(H_2, \{E_1, \ldots, E_l\})$ is connected. As is indicated in Figure 3, there is two types of edges in $L'(H_2, \{E_1, \ldots, E_l\})$. The first one is in $Y(H_2, \{E_1, \ldots, E_l\})$. The second one is not in $Y(H_2, \{E_1, \ldots, E_l\})$, but there is a bypass in $Y(H_2, \{E_1, \ldots, E_l\})$ given in Figure 4. Hence, $Y(H_2, \{E_1, \ldots, E_l\})$ is connected.

We assume, for any integer $k$ smaller than $g$, $Y(H_k, \{E_1, \ldots, E_l\})$ is $(k - 2)$-connected. Let $i$ be an integer smaller than or equal to $g - 2$, and $f : S^i \rightarrow Y(H_g, \{E_1, \ldots, E_l\})$ be a continuous map. Since $L'(H_g, \{E_1, \ldots, E_l\})$ is contractible, there is a continuous map $\tilde{f} : D^{i+1} \rightarrow L'(H_g, \{E_1, \ldots, E_l\})$ such that $\tilde{f}|_{\partial D^{i+1}} = f$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Figure 4}
\end{figure}
We may assume \( \tilde{f} \) is piecewise linear with respect to some triangulation of \( D^{i+1} \). If \( \tilde{f}(D^{i+1}) \) is contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \), this means that \( f \) is null-homotopic. We assume \( \tilde{f}(D^{i+1}) \) is not contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \). Then, there is a simplex \( \sigma \) of \( D^{i+1} \) such that \( \tilde{f}(\sigma) \) is not contained in \( Y(H_g, \{E_1, \ldots, E_l\}) \), and this means that there is a representative \( \{D_0 \cup \cdots \cup D_j\} \) of \( \tilde{f}(\sigma) \) with disconnected complement. Let \( G_\sigma \) be a graph defined as follows. Each vertex corresponds to components of \( H_g - D_0 \cup \cdots \cup D_j \). Each edge corresponds to one of \( D_0, \ldots, D_j \) and connecting vertices corresponding to the components containing this disc. Since \( D_0 \cup \cdots \cup D_j \) has disconnected complement, \( G_\sigma \) has at least two components. There is a nonempty subgraph of \( G_\sigma \) whose edges connects distinct vertices. Let \( \mathcal{D} \) be the subsystem of \( \{D_0, \ldots, D_j\} \) which corresponds to the edges of this graph. This system \( \mathcal{D} \) satisfies the following property:

\[(*) \text{ Let } \mathcal{D} = \{D'_0, \ldots, D'_m\}. \text{ Each } D'_a \in \mathcal{D} \text{ separates } H_g - \bigcup_{n \neq a} D'_n.\]

A system of meridian discs which satisfies property\((*)\) is called purely separating system of meridian discs, and a simplex of \( L'(H_g, \{E_1, \ldots, E_l\}) \) is called purely separating if it is represented by a purely separating system of meridian discs. \( \mathcal{D} \) represent a face of \( \tilde{f} \sigma \), so there is a simplex \( \tau \) of \( D^{i+1} \) such that \( \tilde{f}(\tau) \) is purely separating. In general, we have shown the following lemma.

**Lemma 3.3.** Each simplex in \( L'(H_g, \{E_1, \ldots, E_l\}) \) has a face which is purely separating.

Let \( \sigma \) be a simplex of \( D^{i+1} \) of maximal dimension \( p \) such that \( \tilde{f}(\sigma) \) is purely separating. This simplex \( \sigma \) is not contained in \( \partial D^{i+1} \), hence, \( \text{Link} \ \sigma \) is homeomorphic to \( S^{i-p} \). Let \( \{D_0, \ldots, D_p\} \) be a system of meridian discs which represents \( \tilde{f}(\sigma) \), and let \( M = H_g - D_0 \cup \cdots \cup D_p \). We define the complex \( Y(M, \ast) \) for \((M, \{E_1, \ldots, E_l\}, \text{discs which are emerged as a cut end along } D_0 \cup \cdots \cup D_p) \) in the same manner as \( Y(H_g, \{E_1, \ldots, E_l\}) \). Then, each disc which represents each vertex of \( \text{Link} \ \sigma \) does not separate \( M \). In fact, if some vertex \( v \) of \( \text{Link} \ \sigma \) separates \( M \), then
\[\tilde{f}(v * \sigma)\] is purely separating, and this fact contradicts the maximality assumption of \(p\). Hence, each vertex of \(\text{Link} \sigma\) is mapped into \(Y(M, \ast)\) by \(\tilde{f}\). If there is a simplex \(\rho\) of \(\text{Link} \sigma\) such that \(\tilde{f}(\rho)\) separates \(M\), then, by 3.3 there is a face \(\tau\) of \(\rho\) such that \(\tilde{f}(\tau)\) is purely separating. Then \(\tilde{f}(\tau * \sigma)\) is purely separating, and \(\tau * \sigma \subset \text{Link} \sigma \subset D^{i+1}\). This fact contradicts the maximality assumption of \(p\). Hence, \(\tilde{f}(\text{Link} \sigma) \subset Y(M, \ast)\).

By the way, \(M\) is a disjoint union of handle bodies, and the total number \(m\) of these genera is at least \(g - p\). \(Y(M)\) is a join of \(Y\)'s of each components of \(M\). Hence, by the hypothesis for the induction, \(Y(M)\) is \((m - 2)\)-connected. Here, we remember that \(i \leq g - 2\), then we can see \(i - p \leq g - 2 - p \leq m - 2\). This shows that \(\tilde{f}|_{\text{Link} \sigma}\) is null homotopic in \(Y(M)\), hence, in \(Y(H_g, \{E_1, \ldots, E_t\})\). Therefore, we can homotope \(\tilde{f}\) such that \(\tilde{f}(\text{Link} \sigma) \subset Y(H_g, \{E_1, \ldots, E_t\})\). We do the same way for other simplices whose images are purely separating, then \(\tilde{f}\) is homotoped to a continuous map whose image is in \(Y(H_g, \{E_1, \ldots, E_t\})\). \(\Box\)

4. Obtaining a Presentation from the Action of \(\mathcal{H}_g\) on \(Y(H_g)\)

For each element \(\phi\) of \(\mathcal{H}_g\) and simplex \(([D_0], \ldots, [D_n])\) of \(Y(H_g)\) (resp.\(\Delta'(H_g)\)), \(([\phi(D_0)], \ldots, [\phi(D_n)])\) is also a simplex of \(Y(H_g)\) (resp.\(\Delta'(H_g)\)). Hence, we
can define a right action of $\mathcal{H}_g$ on $Y(H_g)$ (resp. $\Delta'(H_g)$) by $([D_0], \ldots, [D_n])\phi = ([\phi(D_0)], \ldots, [\phi(D_n)])$. We can see that, if $g = 2$, each of the $2$-simplices of $\Delta'(H_2)/\mathcal{H}_2$, $\{1$-simplices of $\Delta'(H_2)/\mathcal{H}_2$ and $\{vertices of $\Delta'(H_2)/\mathcal{H}_2$ consists of one element, each of which is represented by $([D_0], [D_1], [D_2]), ([D_0], [D_1], [D_0]),$ where $D_0, D_1, D_2$ are indicated in Figure 5, and if $g \geq 3$, each of the $2$-simplices of $Y(H_g)/\mathcal{H}_g$, $\{1$-simplices of $Y(H_g)/\mathcal{H}_g$ and $\{vertices of $Y(H_g)/\mathcal{H}_g$ consists of one element, each of which is represented by $([D_0], [D_1], [D_2]), ([D_0], [D_1], [D_0]),$ where $D_0, D_1, D_2$ are indicated in Figure 6. If stabilizer of each vertex is finitely presented, and if that of each 1-simplex is finitely generated, we can obtain a presentation for $\mathcal{H}_g$ as in the way of [9], [12]. Here, we will mention these method. The action of $\mathcal{H}_2$ on $\Delta'(H_2)$ is similar to the action of $\mathcal{H}_g$ on 2-skelton of $Y(H_g)$ when $g \geq 3$. Hence, we mention just on the case of $g \geq 3$.

We fix a vertex $v_0$ of $Y(H_g)$, fix an edge (= a 1-simplex with orientation) $e_0$ of $Y(H_g)$ which emanates from $v_0$ and fix a 2-simplex $f_0$ of $Y(H_g)$ which contains $v_0$. Let $D_0, D_1$ and $D_2$ be meridian disks indicated in Figure 6, we set $v_0 = [D_0], e_0 = ([D_0], [D_1])$ and $f_0 = ([D_0], [D_1], [D_2])$. We choose an element $r_0$ of $\mathcal{H}_g$ which switches the vertices of $e_0$, in our situation, we set $r_0 = d_g$. By this notation, we see $e_0 = (v_0, (v_0)d_g)$ We denote the stabilizer of $v_0$ by $(\mathcal{H}_g)_{v_0}$, that of $e_0$ by $(\mathcal{H}_g)_{e_0}$ and an infinite cyclic group generated by $d_g$ by $<d_g>$. The free product $(\mathcal{H}_g)_{v_0} * <d_g>$ with the following three types of relation define a presentation for $\mathcal{H}_g$.

(Y1) $d_g^2 = a$ a presentation of $d_g^2$ as an element of $(\mathcal{H}_g)_{v_0}$.

(Y2) For each element $t$ of $(\mathcal{H}_g)_{e_1}$,

\[
(d_g)^{-1}(\text{a presentation of } t \text{ as an element of } (\mathcal{H}_g)_{v_0})d_g
\]

\[
= \text{a presentation of } (d_g)^{-1}td_g \text{ as an element of } (\mathcal{H}_g)_{v_0}.
\]

(Y3) For the loop $\partial f_0$ in $Y(H_g)$, we define an element $Wf_0$ in the following manner. The loop $\partial f_0$ consists of three vertices $v_0, v_1, v_2$ and three edges $e_1, e_2, e_3$ such that $e_1 = (v_0, v_1), e_2 = (v_1, v_2), e_3 = (v_2, v_0)$. There is an element $h_1$ of $(\mathcal{H}_g)_{v_0}$ such that $e_0h_1 = e_1$ i.e. $e_1 = (v_0, (v_0)d_gh_1)$, then $e_2(d_gh_1)^{-1}$ is an edge emanat-
ing from \(v_0\). Hence, there is an element \(h_2\) of \((\mathcal{H}_g)_{v_0}\) such that \(e_0h_2 = e_2(d_gh_1)^{-1}\) i.e. \(e_2 = ((v_0)d_gh_1, (v_0)d_gh_2d_gh_1)\), then \(e_3(d_gh_2d_gh_1)^{-1}\) is an edge emanating from \(v_0\). So, there is an element \(h_2\) of \((\mathcal{H}_g)_{v_0}\) such that \(e_0h_2 = e_2(d_gh_2d_gh_1)^{-1}\) i.e. \(e_3 = ((v_0)d_gh_2d_gh_1, (v_0)d_gh_3d_gh_2d_gh_1)\). We define \(W_{f_0} = d_gh_3dh_2d_gh_1\). This element \(W_{f_0}\) fixes \(v_0\), so the following is a relation for \(\mathcal{H}_g\). \(W_{f_0}\) is an \(n\)-string pure braid group \(P_n(\Sigma_g)\) to be the fundamental group of \(F_n\Sigma_g\). Let \(p_1\) and \(p_2\) be points on \(\partial H_{g-1}\), and \(p_{0,1}, p_{0,2}, p_{1,1}\), and \(p_{1,2}\) be points on \(\partial H_{g-2}\). We can get a presentation for \((\mathcal{H}_g)_{v_0}\) by investigating the following three exact sequences:

\[
\begin{align*}
(1) & \quad P_2\Sigma_{g-1} \xrightarrow{\beta} \pi_0(Diff^+(H_g, p_1, p_2)) \xrightarrow{\alpha#} \mathcal{H}_{g-1}(\pi_0(Diff^+(H_g-1))) \rightarrow 0, \\
(2) & \quad 0 \rightarrow \pi_0(Diff^+(H_g-1, p_1, p_2)) \xrightarrow{\delta} \pi_0(Diff^+(H_{g-1}, \{p_1, p_2\})) \xrightarrow{\gamma} \mathbb{Z}_2 \rightarrow 0, \\
(3) & \quad 0 \rightarrow \mathbb{Z} \rightarrow \pi_0(Diff^+(H_{g-2}, \{p_0,1, p_{0,2}, p_{1,1}, p_{1,2}\})) \xrightarrow{\lambda} \pi_0(Diff^+(H_{g-2}, \{p_{0,1}, p_{0,2}\})) \rightarrow 0.
\end{align*}
\]

We can get a set of generators of \((\mathcal{H}_g)_{v_0}\) by investigating the following three exact sequences:

\[
\begin{align*}
(4) & \quad P_4\Sigma_{g-2} \xrightarrow{\beta} \pi_0(Diff^+(H_{g-2}, p_{0,1}, p_{0,2}, p_{1,1}, p_{1,2})) \xrightarrow{\alpha#} \mathcal{H}_{g-2}(\pi_0(Diff^+(H_g-2))) \rightarrow 0, \\
(5) & \quad 0 \rightarrow \pi_0(Diff^+(H_{g-2}, \{p_{0,1}, p_{0,2}, \{p_{1,1}, p_{1,2}\})) \xrightarrow{\delta} \pi_0(Diff^+(H_{g-2}, \{p_{0,1}, p_{0,2}\}, \{p_{1,1}, p_{1,2}\})) \xrightarrow{\gamma} \mathbb{Z}_2 \rightarrow 0.
\end{align*}
\]
0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow \pi_0(\text{Diff}^+(H_g, D_0, D_1))
\xrightarrow{\lambda} \pi_0(\text{Diff}^+(H_{g-2}, \{p_{0,1}, p_{0,2}\}, \{p_{1,1}, p_{1,2}\})) \longrightarrow 0.

We can give relations of type (Y1) and (Y2) by drawing some figures.
For details please see [6].

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