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On the Well-posedness of a Linear Heat Equation with a Critical Singular Potential

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1 Introduction

This note is the joint work with Prof. M. Tsutsumi (Waseda Univ.).

Consider the initial-boundary value problem of a linear heat equation with a time-dependent singular potential $V = V(t,x)$:

$$\text{(IBVP)} \begin{cases} u_t - \Delta u = Vu & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and $T > 0$ is an arbitrary positive number. Here initial data u_0 is L^p-function on Ω, $p \geq 1$.

We are concerned with the well-posedness of IBVP on L^p if a potential V belongs to the class $L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$. Here, the class $L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$ may be regarded as a borderline case for the well-posedness. To see this situation, we shall briefly review the known results.

When a potential V belongs to $L^\infty(0,T;L^\sigma(\Omega))$ with $\sigma > N/2$, for every initial data $u_0 \in L^p(\Omega)$, $p \geq 1$ IBVP has a unique solution $u \in C([0,T];L^p(\Omega))$ which is acted on by the smoothing effect up to $u(t) \in L^\infty(\Omega)$ for $t \geq \varepsilon$ with $\varepsilon > 0$. More precisely, the following theorem is known (See Theorem A1 in [7] for instnance).

Theorem A. *Let $V \in L^\infty(0,T;L^\sigma(\Omega))$, $\sigma > N/2$. For every $u_0 \in L^p(\Omega)$, $p \geq 1$, there exists unique solution $u \in C([0,T];L^p(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,T;L^\infty(\Omega))$ of IBVP.*

On the other hand, if $V \in L^\infty(0,T;L^\sigma(\Omega))$ with $\sigma < N/2$, then such a class of the potential V is too singular for assuming the existence of a solution u of IBVP. In fact, Baras and Goldestein [3] proved the following ill-posedness result.

Theorem B. *Let $\Omega \ni 0$, and let V be a time-independent potential such that*

$$V(x) = \frac{C}{|x|^2}, \qquad \text{where} \quad C > \frac{(N-2)^2}{4}.$$

Then for every (smoothly) nontrivial nonnegative initial data $u_0 \in L^1(\Omega)$, there is no nonnegative solution $u \in C([0,T]; L^1(\Omega))$ of IBVP for any $T > 0$ in the following sense:

$$\begin{cases} u \geq 0 \quad \text{on } (0,T) \times \Omega, \quad Vu \in L^1_{\text{loc}}((0,T) \times \Omega), \\ u_t - \Delta u = Vu \quad \text{in } \mathcal{D}'((0,T) \times \Omega) \\ \lim_{t \downarrow 0} \int_\Omega u(t)\zeta = \int_\Omega u_0 \zeta \quad \text{for } \forall \zeta \in \mathcal{D}(\Omega). \end{cases}$$

Remark. (i) The above potential V is in $L^p(\Omega)$ for $p < N/2$ and does *not* belong to $L^{\frac{N}{2}}(\Omega)$.

(ii) $(N-2)^2/4$ is significant because the number is the optimal constant in Hardy inequality on a ball B or $\mathbb{R}^N$, that is,

$$\frac{(N-2)^2}{4} \int_B \frac{|\varphi|^2}{|x|^2} dx \leq \int_B |\nabla \varphi|^2 dx,$$

for all $\varphi \in H_0^1(B)$.

From Theorem A and Theorem B, we can say that the potential class $L^\infty(0,T; L^{\frac{N}{2}}(\Omega))$ is *critical* for the well-posedness of IBVP.

Our main results are, roughly speaking, as follows: If p is greater than one, then IBVP is well-posed on $L^p(\Omega)$. On the other hand, the well-posedness of IBVP breaks down on $L^1(\Omega)$. Precisely, the following theorems hold.

Theorem 1.1 *Let $V \in L^\infty(0,T; L^{\frac{N}{2}}(\Omega))$. Then for every $u_0 \in L^p(\Omega)$, $p > 1$, there exists a unique solution u satisfying the following (i) and (ii):*

(i) $u \in C([0,T]; L^p(\Omega)) \cap L^p(0,T; L^{\frac{Np}{N-2}}(\Omega)) \cap L^\infty_{\text{loc}}(0,T; L^q(\Omega))$ for any $q < +\infty$.

(ii) For all $\varphi \in \mathcal{D}([0,T) \times \Omega)$ the above function u satisfies the following integral identity

$$\int_\Omega u_0 \varphi(0,x)dx + \int_0^T \int_\Omega [u\varphi_t + u\Delta\varphi + Vu\varphi] dxdt = 0. \tag{1.2}$$

Remark. We can not expect that $u(t)$ has L^∞-regularity for $t \geq \varepsilon$ with $\varepsilon > 0$. The reason is as follows: If $u \in L^\infty_{\text{loc}}(0,T; L^\infty(\Omega))$, then $Vu \in L^\infty_{\text{loc}}(0,T; L^{\frac{N}{2}}(\Omega))$. On the other hand, the maximal regularity result [10] gives that

$$u \in L^p_{\text{loc}}(0,T; W^{2,\frac{N}{2}}(\Omega) \cap W_0^{1,\frac{N}{2}}(\Omega)) \quad \text{for any } p < \infty.$$

But $W^{2,\frac{N}{2}}(\Omega) \not\subset L^\infty(\Omega)$.

Theorem 1.2 *Let $\Omega \ni 0$, and let Ω' be an arbitrary subdomain in Ω with $\Omega' \ni 0$ and $\overline{\Omega'} \subset \Omega$. Suppose that $V = V(x)$ is a nonnegative potential in $L^\infty(\Omega \backslash \Omega')$ having such a singularity as*

$$V(|x|) = \frac{C}{|x|^2}\left(\log \frac{1}{|x|^2}\right)^{-\alpha} \quad \text{near } x \approx 0, \tag{1.3}$$

where $\frac{2}{N} < \alpha \leq 1$ and $C > 0$. Then for any $C > 0$ there exists some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that there is no nonngative solution $u \in C([0,T]; L^1(\Omega))$ of IBVP for any $T > 0$ in the following sense:

$$\begin{cases} u \geq 0 \quad \text{on } (0,T) \times \Omega, \quad Vu \in L^1_{\text{loc}}((0,T) \times \Omega), \\ u_t - \Delta u = Vu \quad \text{in } \mathcal{D}'((0,T) \times \Omega), \\ \lim_{t \downarrow 0} \int_\Omega u(t)\zeta = \int_\Omega u_0 \zeta \quad \text{for } \forall \zeta \in \mathcal{D}(\Omega). \end{cases} \tag{1.4}$$

Remark. (i) Note that the above V is in $L^{\frac{N}{2}}(\Omega)$ if and only if $\alpha > 2/N$. In addition, V is not in *Kato class* $\mathcal{K}_N(\Omega)$ if and only if $\alpha \leq 1$. Recall that a measurable function V is in Kato class $\mathcal{K}_N(\Omega)$, if V satisfies

$$\lim_{r \downarrow 0}\left[\sup_{x \in \Omega} \int_{\{|x-y| \leq r\} \cap \Omega} \frac{|V(y)|}{|x-y|^{N-2}} dy\right] = 0.$$

If a potential V belongs to $\mathcal{K}_N(\Omega)$, then the Hamiltonian $H = -\Delta + V$ has several good properties (See B. Simon's survey [13], in which the related topics to Kato class $\mathcal{K}_N(\Omega)$ is discussed in detail, and see also [1]).

(ii) The assumption $Vu \in L^1_{\text{loc}}((0,T) \times \Omega)$ is by no means restrictive. In fact, Baras and Cohen [2] proved that if a nonnegative measurable function $F(t,x)$ is *not* in $L^1_{\text{loc}}((0,T) \times \Omega)$, then the solution u of $u_t = \Delta u + F(t,x)$ must have an instantaneous blow-up at $t = 0$ (see also [12] and [14]).

(iii) The ill-posedness result remains true if we replace the above V by any potential $\tilde{V}$, where $\tilde{V}(x) \geq V(x)$ in Ω.

Notation: Throughout this paper, we denote by $\mathcal{D}(\Omega)$ the space of all infinitely differentiable functions on Ω with compact supports, and $\mathcal{D}^+(\Omega) \equiv \{\varphi \in \mathcal{D}(\Omega) \; ; \; \varphi \geq 0\}$. By C we denote general positive constants, which may be different in each inequality.

2 Proof of Theorem 1.1

We shall proceed by approximation. For any $n \in \mathbb{N}$, we truncate V by

$$V_n(t,x) = \begin{cases} -n & \text{if } V(t,x) \leq -n, \\ V(t,x) & \text{if } -n \leq V(t,x) \leq n, \\ n & \text{if } V(t,x) \leq n. \end{cases} \tag{2.1}$$

Then we have $V_n \in L^\infty((0,T)\times\Omega)$ and $V_n \to V$ strongly in $L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$ as $n \to \infty$.

Now we consider the sequence of approximate solutions $\{u_n\}_{n\in\mathrm{N}}$ which solves the following approximate problem:

$$\begin{cases} (u_n)_t - \Delta u_n = V_n u_n & \text{in } (0,T)\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

Then from Theorem A we can see that for every $u_0 \in L^p(\Omega)$ there exists a unique approximate solution $u_n \in C([0,T];L^p(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,T;L^\infty(\Omega))$.

(i) Existence. We can estanblish as a priori estimates of u_n Proposition 2.1 and Proposition 2.2 below, where the proofs are omitted.

Proposition 2.1 *There exists a constant $C > 0$ depending only on p, V, T and Ω such that*

$$||u_n(t)||_{L^p(\Omega)} \leq C||u_0||_{L^p(\Omega)}, \tag{2.3}$$

and

$$||\nabla|u_n|^{\frac{p}{2}}||_{L^2(0,T)\times\Omega)} \leq C||u_0||^{\frac{p}{2}}_{L^p(\Omega)}. \tag{2.4}$$

Moreover,

$$||u_n||_{L^p(0,T;L^{\frac{Np}{N-2}}(\Omega))} \leq C||u_0||_{L^p(\Omega)}. \tag{2.5}$$

Proposition 2.2 *Let $p_m = \left(\frac{N}{N-2}\right)^m p$ for any $m \in \mathbb{N}$. There exists a constant $C > 0$ such that*

$$||u_n(t)||_{L^{p_m}(\Omega)} \leq \frac{C}{t^{\frac{m}{p}}}||u_0||_{L^p(\Omega)}, \tag{2.6}$$

for $t \in (0,T)$.

From Proposition 2.1 and Proposition 2.2, there exists a limit function $u = \lim_{n\to\infty} u_n$ in the class $C(0,T;L^p(\Omega))\cap L^p(0,T;L^{\frac{Np}{N-2}}(\Omega))\cap L^\infty_{\mathrm{loc}}(0,T;L^q(\Omega))$ for any $q < \infty$.

(ii) Convergence. For all $\varphi \in \mathcal{D}([0,T)\times\Omega)$, the approximate solution u_n satisfies

$$\int_\Omega u_0\varphi(0,x)dx + \int_0^T\int_\Omega [u_n\varphi_t + u_n\Delta\varphi + V_n u_n\varphi]dxdt = 0.$$

We may only verify the convergence of the last term, since that of the remaining terms is obvious. Rewriting

$$\int_0^T\int_\Omega V_n u_n\varphi = \int_0^T\int_\Omega (V_n - V)u_n\varphi + \int_0^T\int_\Omega Vu_n\varphi,$$

then we estimate

$$\left|\int_0^T\int_\Omega (V_n - V)u_n\varphi\right|$$
$$\leq ||V_n - V||_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}||u_n||_{L^1(0,T;L^{\frac{N}{N-2}}(\Omega))}||\varphi||_{L^\infty((0,T)\times\Omega)}.$$

Letting $n \to \infty$, then we obtain

$$\int_0^T\int_\Omega V_n u_n\varphi \to \int_0^T\int_\Omega Vu\varphi. \tag{2.7}$$

(iii) Uniqueness. IBVP is the linear problem, so that we may only prove that if $u_0 \equiv 0$, then the soluton $u(t)$ is trivial. We give the proof of uniqueness by the duality method.

Since u belongs to $L^p(0,T;L^{\frac{Np}{N-2}})$, we have $Vu \in L^1(0,T;L^{q_0}(\Omega))$, with $\frac{1}{q_0} = \frac{N-2}{Np} + \frac{2}{N}$, $q_0 > 1$. Thus, we obtain that $u \in C([0,T];L^{q_0}(\Omega))$. On the other hand, let w_n be the solution of the backward (approximate) problem:

$$\begin{cases} -(w_n)_t - \Delta w_n = V_n w_n & \text{in } (-\infty, t_0)\times\Omega, \\ w_n = 0 & \text{on } (-\infty, t_0)\times\Omega, \\ w_n(t_0, x) = \zeta(x) & \text{in } \Omega, \end{cases} \tag{2.8}$$

where $\zeta \in \mathcal{D}(\Omega)$ and $t_0 \in (0,T)$ be arbitrary. Here we notice that $w_n \in C([0,t_0];L^q(\Omega)) \cap L^q(0,t_0;W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega))$ and $(w_n)_t \in L^q((0,t_0)\times\Omega)$ for every $q < \infty$ (See [11] or [5]).

Thanks to desirable regularities of u and w_n we can take $\varphi = w_n$ as a test function by the density argument and the cut-off procedure with respect to t at $t = t_0$. Therefore, we see that the following integral identity makes sense: For every $t_0 \in [0,T]$, solution $u \in C([0,T];L^{q_0}(\Omega))$ satisfies that

$$\int_\Omega u(t_0)\zeta = \int_0^{t_0}\int_\Omega [u(w_n)_t + u\Delta w_n + Vuw_n]$$
$$= \int_0^{t_0}\int_\Omega (V - V_n)uw_n.$$

Hence,

$$\left|\int_0^{t_0}\int_\Omega (V - V_n)uw_n\right|$$
$$\leq ||V - V_n||_{L^\infty(0,T;L^{\frac{N}{2}})}||u||_{L^1(0,T;L^{\frac{Np}{N-2}})}||w_n||_{L^\infty(0,T;L^q)}, \tag{2.9}$$

where $\frac{1}{q} = 1 - \frac{2}{N} - \frac{N-2}{Np}$, $q > 1$.

On the other hand, in the same manne as in the proof of Proposition 2.1 we obtain

$$||w_n||_{L^\infty(0,T;L^q(\Omega))} \leq C||\zeta||_{L^q(\Omega)}.$$

Letting $n \to \infty$ in (2.9), we have

$$\int_\Omega u(t_0)\zeta = 0.$$

The arbitariness of $t_0 \in (0,T]$ and of $\zeta \in \mathcal{D}(\Omega)$ yields that $u \equiv 0$.

Hence, we complete the proof of Theorem 1.1. ∎

Remark. If we use the parabolic version of Strichartz $L^p - L^q$ estimate in harmonic analysis (See [4] and [15]), we can give a more simple proof of Theorem 1.1 by the contraction mapping principle on the space-time function spaces.

An analogous poof of uniqueness in Theorem 1.1 gives that uniqueness of a solution of IBVP holds in the class $L^\infty(0,T;L^p(\Omega))$ provided $p > \frac{N}{N-2}$ as follows.

Theorem 2.3 *Let $V \in L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$. Suppose that $u \in L^\infty(0,T;L^p(\Omega))$ satisfies that*

$$\int_\Omega u_0\varphi(0,x)dx + \int_0^T \int_\Omega [u\varphi_t + u\Delta\varphi + Vu\varphi]dxdt = 0, \tag{2.10}$$

for all $\varphi \in \mathcal{D}([0,T) \times \Omega)$. If $p > \frac{N}{N-2}$, then uniqueness of u holds in the class.

Brezis and Cazenave [7] proved the same uniqueness result for $V \in C([0,T];L^{\frac{N}{2}}(\Omega))$. They suggested the question if one can replace the assumption $V \in C([0,T];L^{\frac{N}{2}}(\Omega))$ by $V \in L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$ (see Open problem 9 in [7]). Thus, we can conclude that the answer is "positive".

Remark. Uniqueness in Theorem 2.3 fails when $p = \frac{N}{N-2}$. In fact, we can construct that for some $V \in C([0,T];L^{\frac{N}{2}}(\Omega))$ there exists a nontrivial solution $u \in C([0,T];L^{\frac{N}{N-2}}(\Omega))$ for initial data $u_0 \equiv 0$ (see Remark A3 in [7]). Hence, this uniquenss result is optimal.

3 Proof of Theorem 1.2

The following lemma plays an essential role in proving Theorem 1.2.

Lemma 3.1 *Assume $\Omega \ni 0$. Let $v \in C([0,\infty);L^1(\Omega))$ be the solution of the heat equation:*

$$(\text{HE}) \begin{cases} v_t = \Delta v & \text{in } (0,\infty) \times \Omega, \\ v = 0 & \text{on } (0,\infty) \times \Omega, \\ v(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Then there exists some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ *such that*

$$\int_0^1 \int_{\Omega'} V v dx dt = +\infty, \tag{3.2}$$

where V *is the potential in Theorem 1.2.*

Remark. Of course, $v \geq 0$ by the maximum principle.

Proof. Without loss of generality, we may assume that $\Omega = B(1)$ and $\Omega' = B(1/2)$, where $B(R) \equiv \{x \in \mathbb{R}^N \ ; \ |x| < R\}$. Moreover, we may assume that

$$V(|x|) = \begin{cases} \dfrac{1}{|x|^2}\left(\log \dfrac{1}{|x|^2}\right)^{-\alpha} & \text{on } B(1/2), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We shall give the proof by contradiction. Suppose that for every $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, the solution v of (HE) satisfies

$$\int_0^1 \int_{B(1/2)} V(|x|) v dx dt < +\infty. \tag{3.4}$$

Applying the closed graph theorem to the linear mapping

$$u_0 \mapsto v|_{(0,1)\times B(1/2)},$$

then there exists a constant $C > 0$ such that

$$\int_0^1 \int_{B(1/2)} V(|x|) v dx dt \leq C ||u_0||_{L^1(B(1/2))}, \tag{3.5}$$

for every $u_0 \in L^1(B(1/2))$. We consider a sequence $\{u_0^n\} \subset \mathcal{D}(B(1/2))$ such that

$$||u_0||_{L^1(B(1/2))} \leq 1 \quad \text{and} \quad u_0^n \to \delta \quad \text{weakly in } \mathcal{M}(B(1/2)),$$

where δ is the Dirac measure at 0 and $\mathcal{M}(B(1/2))$ is the space of signed Radon measures on $B(1/2)$. Let $G(t,x)$ be the corresponding Green function determined by (HE), then by letting $n \to \infty$,

$$v_n \to v = G * \delta = G(t,x).$$

Applying to u_0^n in (HE) and using Fatou's lemma, then we have

$$\int_0^1 \int_{B(1/2)} V(|x|) G(t,x) dx dt \leq C.$$

On the other hand, we know that

$$G(t,x) \approx E(t,x) \quad \text{on } (0,1) \times B(1/2), \tag{3.6}$$

where E is the fundamental solution of (HE) in $\Omega = \mathbb{R}^N$. Thus, we can estimate that

$$
\begin{aligned}
&\int_\varepsilon^1 \int_{B(1/2)} V(|x|)E(t,x)dxdt \\
&\geq \begin{cases} \dfrac{\omega_N}{(4\pi)^{\frac{N}{2}}} \displaystyle\int_0^{\frac{1}{2}} \frac{r^{N-3}e^{-\frac{r^2}{4}}}{t} \left[\frac{1}{1-\alpha}\left(\log \frac{1}{tr^2}\right)^{1-\alpha}\right]_{t=1}^{t=\varepsilon} dr & \text{if } \frac{2}{N} < \alpha < 1, \\ \dfrac{\omega_N}{(4\pi)^{\frac{N}{2}}} \displaystyle\int_0^{\frac{1}{2}} \frac{r^{N-3}e^{-\frac{r^2}{4}}}{t} \left[\log\log \frac{1}{tr^2}\right]_{t=1}^{t=\varepsilon} dr & \text{if } \alpha = 1, \end{cases}
\end{aligned}
$$

where ω_N is the measure of the unit $(N-1)$-dimensional sphere. By using elementary inequalities: for any $a,\ b > 0$

$$
\begin{aligned}
&(a+b)^\alpha \geq \frac{1}{2^{1-\alpha}}(a^\alpha + b^\alpha) \quad (0 < \alpha < 1), \\
\text{and} \quad &\log(a+b) \geq \frac{1}{2}(\log a + \log b),
\end{aligned}
$$

then we obtain

$$
\int_\varepsilon^1 \int_{B(1/2)} V(|x|)E(t,x)dxdt \geq \begin{cases} C_1\left(\log \dfrac{1}{\varepsilon}\right)^{1-\alpha} - C_2 & \text{if } \frac{2}{N} < \alpha < 1, \\ C_3 \log\log \dfrac{1}{\varepsilon} - C_4 & \text{if } \alpha = 1, \end{cases} \tag{3.7}
$$

where C_i $(i = 1,2,3,4)$ is positive constant. Hence, letting $\varepsilon \downarrow 0$ in (3.7), we find that

$$
V(|x|)E(t,x) \notin L^1((0,1) \times B(1/2)).
$$

It follows from (3.6) that

$$
V(|x|)G(t,x) \notin L^1((0,1) \times B(1/2)), \tag{3.8}
$$

which contradicts the assumption (3.4). Therefore, we complete the proof of Lemma 3.1. ∎

Proof of Theorem 1.2. We argue by contradiction. Suppose that for any $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, there exists some $T > 0$ and a nonnegative solution u of IBVP in the sense of (1.4).

By the standard argument, we can see that the solution u of IBVP in $C([0,T]; L^1(\Omega))$ satisfies

$$
\begin{aligned}
&\int_\Omega u(T-\varepsilon)\zeta dx - \int_\Omega u(\varepsilon)\zeta dx + \int_\varepsilon^{T-\varepsilon} \int_\Omega u(-\Delta\zeta)dx \\
&= \int_\varepsilon^{T-\varepsilon} \int_\Omega V u \zeta dxdt,
\end{aligned}
$$

for any $\zeta \in \mathcal{D}(\Omega)$ and small $\varepsilon > 0$. Since $u \in C([0,T]; L^1(\Omega))$, by letting $\varepsilon \downarrow 0$, we see that each term in the left hand side converges as follows,

$$\int_\Omega u(T-\varepsilon)\zeta dx \to \int_\Omega u(T)\zeta dx,$$
$$\int_\Omega u(\varepsilon)\zeta dx \to \int_\Omega u_0\zeta dx,$$
$$\int_\varepsilon^{T-\varepsilon}\int_\Omega u(-\Delta\zeta)dx \to \int_0^T\int_\Omega u(-\Delta\zeta)dx.$$

The above convergence implies that

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^{T-\varepsilon}\int_\Omega Vu\zeta dxdt = \int_0^T\int_\Omega Vu\zeta dxdt < \infty.$$

Taking $\zeta \in \mathcal{D}^+(\Omega)$ such that $\zeta \geq 1$ on Ω', we deduce that

$$\int_0^T\int_{\Omega'} Vudxdt < \infty, \tag{3.9}$$

i.e., $Vu \in L^1((0,T)\times\Omega')$ (note that $V \geq 0$ and $u \geq 0$).

On the other hand, we have the following maximum principle:

Proposition 3.2 *Assume $F \in L^1((0,T)\times\Omega)$. Let $w \in C([0,T]; L^1(\Omega))$ be a supersolution defined by*

$$\begin{cases} w_t \geq \Delta w + F(t,x) & \text{in } \mathcal{D}'((0,T)\times\Omega), \\ w \geq 0 & \text{on } (0,T)\times\partial\Omega, \\ w(0,x) = w_0(x) \geq 0 & \text{in } \Omega. \end{cases} \tag{3.10}$$

If $F \geq 0$, then $w \geq 0$ on $[0,T]\times\Omega$.

Let v be the solution of the heat equation such that

$$(\text{HE}')\begin{cases} v_t = \Delta v & \text{in } (0,\infty)\times\Omega', \\ v = 0 & \text{on } (0,\infty)\times\partial\Omega', \\ v(0,x) = u_0(x)|_{\Omega'} & \text{in } \Omega', \end{cases}$$

then it follows from Proposition 3.2 that u is a supersolution of (HE′), and hence,

$$u(t) \geq v(t) \geq 0 \quad \text{on } [0,T]\times\Omega'.$$

In particular, taking $u_0 \in L^1(\Omega)$ as in Lemma 3.1, then the nonnegative solution u of IBVP must satisfy

$$\int_0^1\int_{\Omega'} Vudxdt = +\infty, \tag{3.11}$$

which contradicts (3.9). Hence, we complete the proof of Theorem 1.2. ∎

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