A Stefan Problem with Memory and Nonlinear Boundary Condition

SERGIU AIZICOVICI
Department of Mathematics, Ohio University
321 Morton Hall, Athens, Ohio 45701-2979, USA
E-mail: aizicovic@bing.math.ohiou.edu

PIERLUIGI COLLI
Dipartimento di Matematica "F. Casorati"
Università di Pavia, Via Ferrata 1, I-27100 Pavia, Italy
E-mail: pier@dragon.ian.pv.cnr.it

MAURIZIO GRASSELLI
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9, I-20133 Milano, Italy
E-mail: maugra@mate.polimi.it

Abstract. This note is devoted to the study of a Stefan problem with memory that includes a third type boundary condition associated with a maximal monotone nonlinearity. The corresponding initial-boundary value problem can be formulated as a Cauchy problem for an abstract doubly nonlinear integrodifferential equation which belongs to a class already analyzed by the authors in a recent paper [2]. A slight variation of the abstract theory developed in [2] is then applied to deduce the existence of a solution to our Stefan problem.

1. Introduction

Let us consider a two-phase material which occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\Gamma$, at any time $t \in [0, T]$, $T > 0$ being fixed. This system is characterized by a pair of state variables, namely the (relative) temperature $\vartheta$ and the phase proportion $\chi$. We assume that the evolution of the pair $(\vartheta, \chi)$ is governed by the following energy balance equation (see [7, 8, 9] and references therein)

$$\partial_t (\vartheta + \chi + \varphi \ast \vartheta + \psi \ast \chi) - \Delta (\vartheta + k \ast \vartheta) = g$$

in $Q := \Omega \times (0, T)$ (1.1)

coupled with the condition

$$\chi \in H(\vartheta)$$

in $Q$ (1.2)

relating $\chi$ to $\vartheta$. Here, $\Delta$ is the usual Laplace operator acting on the space variables, $\partial_t = \partial/\partial t$, and $*$ denotes the convolution product with respect to time over $(0, t)$, that is, for instance,

$$(\varphi \ast \vartheta)(\cdot, t) = \int_0^t \varphi(t - s)\vartheta(\cdot, s)ds, \quad t \in [0, T].$$
In addition, $\mathcal{H}$ stands for the Heaviside graph $(\mathcal{H}(r) = 0$ if $r < 0$, $\mathcal{H}(0) = [0,1]$, $\mathcal{H}(r) = 1$ if $r > 0$) and the memory kernels $\varphi$, $\psi$, $k : (0,T) \to \mathbb{R}$ are given along with the function $g : Q \to \mathbb{R}$.

Initial and boundary value problems for the system (1.1)-(1.2) have been investigated in several papers (see [4, 6, 7, 9], cf. also [1, 5, 11] for related problems). Nevertheless, in all the mentioned literature, (1.1)-(1.2) is complemented with variational boundary conditions, that turn out to be linear with respect to $\vartheta$ and/or the outward normal derivative $\partial_n \vartheta$. On the contrary, in this note we prove the existence of solutions to an initial-boundary value problem for (1.1)-(1.2) characterized by a nonlinear boundary condition. To be more precise, we supply the system with

$$\partial_{\nu}(\vartheta + k * \vartheta) + \alpha(\vartheta) \ni h \quad \text{on} \quad \Sigma := \Gamma \times (0, T) \quad \text{(1.3)}$$

$$\vartheta(x, 0) = u_{0} \quad \text{in} \quad \Omega \quad \text{(1.4)}$$

where $\alpha : \mathbb{R} \to 2^{\mathbb{R}}$ denotes a maximal monotone graph, and the functions $h : \Sigma \to \mathbb{R}$ and $u_{0} : \Omega \to \mathbb{R}$ are known.

Problem (1.1)-(1.4) contains two monotone nonlinearities represented by the maximal monotone graphs $\mathcal{H}$ and $\alpha$. In Section 3, we consider an extended version of (1.1)-(1.4) in which the kernels $\varphi$ and $\psi$ are allowed to depend on the space variables too, and where the term $-k * \Delta \vartheta$ is replaced by a rather general second order linear convolution operator acting on $\vartheta$. Moreover, we let the right hand side $g$ of (1.1) incorporate an additional nonlinearity in order to represent not only a measurable function of $(x,t)$ but a Lipschitz continuous function of $\vartheta$ as well. Then we show that the resulting problem can be reformulated as a Cauchy problem for a doubly nonlinear integro-differential evolution equation.

The abstract formulation we obtain essentially reduces to a particular case of a class of evolution equations studied in [2]. In that paper, two existence results are proved by means of a semi-implicit time discretization procedure. Here, in Section 2, we state a slight generalization of the main theorem of [2], whose proof can be achieved by performing simple changes in the original one. This result applies to the abstract equation

$$(M\vartheta)' + A\vartheta + B * \vartheta \ni f + \mathcal{F}(M\vartheta) + \mathcal{G}(\vartheta) \quad \text{in} \quad V', \quad \text{a.e. in} \quad (0,T) \quad \text{(1.5)}$$

where $V'$ is meant to be the dual space of $V = H^{1}(\Omega)$ in the framework of (1.1)-(1.4). We also point out that $M$ takes the place of $\mathcal{I} + \mathcal{H}$ ($\mathcal{I}$ being the identity mapping) and is maximal monotone from $H = L^{2}(\Omega)$ to the same space $H$ (identified with its dual space). The other maximal monotone operator is $A$ which works from $V$ to $V'$ and collects the contributions of $-\Delta \vartheta$ and $\alpha(\vartheta)$ from (1.1) and (1.3), while $B$ is a function from $[0,T]$ into the space of linear bounded operators from $V$ to $V'$. On the other hand, $f$ maps $(0,T)$ into $V'$ and $\mathcal{F}$, $\mathcal{G}$ are causal (cf. Section 2 for a precise definition) Lipschitz continuous operators on $L^{2}(0,T;H)$. In addition, $F$ is required to be linear and it is naturally applied to the same selection of $M(\vartheta)$ appearing on the left hand side of (1.5).

The existence of a solution to the Cauchy problem for (1.5) is established in the next section. Afterwards, the abstract result is used in Section 3 to deduce the existence of weak solutions to the above mentioned generalized version of (1.1)-(1.4).
2. Abstract result

On account of [2, Sect. 2], we introduce the hypotheses on the data of the Cauchy problem associated with (1.5).

(A1) Let $V$ and $W$ be reflexive real Banach spaces and let $H$ denote a real Hilbert space which is identified with its dual. We assume that

$$V \hookrightarrow W \hookrightarrow H \hookrightarrow W' \hookrightarrow V'$$

with dense and continuous injections, the first and the last embeddings being also compact.

(A2) $M$ is a maximal monotone operator from $H$ to $H$ that is linearly bounded, namely,

$$\exists C_1 > 0 : \|w\|_H \leq C_1 (1 + \|v\|_H) \quad \forall v \in H, \forall w \in M(v)$$

and $M^{-1}$ is Lipschitz continuous, i.e.,

$$\exists C_2 > 0 : C_2 \|v_1 - v_2\|_H^2 \leq \langle w_1 - w_2, v_1 - v_2 \rangle \quad \forall v_1, v_2 \in H, \forall w_1 \in M(v_1), \forall w_2 \in M(v_2)$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $H$.

(A3) $A$ is a maximal monotone and bounded operator from $V$ to $V'$ such that $A = A_1 + A_2$, where $A_i$ coincides with the subdifferential $\partial J_i$ of a convex and lower semicontinuous function $J_i : V \to \mathbb{R}$, for $i = 1, 2$. Furthermore, $A_1$ is linear, $A_2$ is bounded from $V$ to $W'$, and $J := J_1 + J_2$ satisfies

$$\frac{1}{2} \|v\|_H^2 + J(v) \geq C_3 \|v\|_V^p - C_4 \quad \forall v \in V$$

for some constants $p \geq 2$, $C_3 > 0$, $C_4 \geq 0$.

(A4) $B \in W^{1.1}(0, T; \mathcal{L}(V, V'))$, where $\mathcal{L}(V, V')$ stands for the Banach space of all the linear and continuous operators from $V$ to $V'$.

(A5) $F, G : L^2(0, T; H) \to L^2(0, T; H)$ are two Lipschitz continuous operators that are causal in the sense that

if $v_1, v_2 \in L^2(0, T; H)$, $t \in (0, T)$, and $v_1 = v_2$ a.e. in $(0, t)$, then $F(v_1) = F(v_2)$, $G(v_1) = G(v_2)$ a.e. in $(0, t)$.

Moreover, $F$ is linear.

(A6) $f \in L^2(0, T; H) + W^{1.1}(0, T; V')$.

(A7) $u_0 \in H$, $\theta_0 := M^{-1}(u_0) \in V$, $J(\theta_0) < +\infty$. 


Here is the precise formulation of the Cauchy problem.

**Problem (P)** Find \( \vartheta \in L^\infty(0,T;V) \) and two auxiliary functions
\[
\begin{align*}
\vartheta &\in L^\infty(0,T;V), \\
u &\in W^{1,2}(0,T;V') \cap L^\infty(0,T;H), \\
\xi &\in L^\infty(0,T;V')
\end{align*}
\] such that
\[
\begin{align*}
\vartheta' + \xi + B \ast \vartheta &= f + F(u) + G(\vartheta) \quad \text{in } V', \text{ a.e. in } (0,T) \\
u(t) &= M(\vartheta(t)) \quad \text{for a.e. } t \in (0,T) \\
\xi(t) &= A(\vartheta(t)) \quad \text{for a.e. } t \in (0,T) \\
u(0) &= u_0 \quad \text{in } V'.
\end{align*}
\] (2.4)

The existence of a solution to (P) is ensured by

**Theorem 2.1** Let (A1)-(A7) hold. Then there exists at least one solution \((\vartheta, u, \xi)\) to Problem (P), with the additional property that \( \vartheta \in W^{1,2}(0,T;H) \).

A comparison between our Problem (P) and its counterpart in [2] shows that the term
\[
(B \ast \vartheta)(t) = \int_0^t B(t - s) \vartheta(s) ds, \quad t \in [0,T]
\] is now used in place of the original one, which is \( k \ast B \vartheta \) for a kernel \( k \) in \( W^{1,1}(0,T) \) and some operator \( B \in L(V,V') \) (in fact, \( k \ast B \) is a special case of \( B \ast \), cf. (A4)). However, a careful examination of the proof of Theorem 2.1 in [2] reveals that the procedure devised there also works in the present case. Basically, the main change concerns the proof of [2, Lemma 3.6], where one has to deduce [2, ineq. (3.19)]. This can be done by taking into account that [2, ineq. (3.25)] still follows from [2, ineq. (3.23)] in our current setup.

**Remark 2.2** Regarding (A3), we note that the subdifferential \( \partial J \) coincides with the sum \( \partial J_1 + \partial J_2 = A \) and that the functions \( J, J_1, \) and \( J_2 \) are all continuous from \( V \) to \( \mathbb{R} \) (cf. Remarks 2.3 and 2.4 in [2]).

### 3. Application

Here we consider a generalization of the Stefan problem (1.1)-(1.2) and provide a weak formulation of it in accordance with Problem (P). Then, the existence of solutions can be demonstrated by applying Theorem 2.1 (see [2, Sect. 5] for other possible applications of the abstract result).

Throughout this section, \( \Omega \) will denote a smooth bounded domain of \( \mathbb{R}^N \) \((N \geq 1)\) and the notation for \( \Gamma, Q, \Sigma \) is the same as in the Introduction. As usual, the variable in \( \Omega \cup \Gamma \) is indicated by \( x = (x_1, \ldots, x_N) \) and \( \partial x_j \) simply replaces \( \partial/\partial x_j, \) \( j = 1, \ldots, N. \)
We start by setting the (formal) Stefan problem for the unknowns \( \theta: Q \to \mathbb{R} \) and \( \chi: Q \to [0, 1] \) which have to satisfy

\[
\partial_t(\theta + \chi + \varphi(x, \cdot) * \theta + \psi(x, \cdot) * \chi) + A \theta + B * \theta = g(x, t, \theta) \quad \text{in } Q
\]

(3.1)

\[
\chi \in \mathcal{H}(\theta) \quad \text{in } Q
\]

(3.2)

\[
\partial_{\nu(A+B*)}\theta + \alpha(\theta) \ni h(x, t) \quad \text{on } \Sigma
\]

(3.3)

\[
(\theta + \chi)|_{t=0} = u_0 \quad \text{in } \Omega
\]

(3.4)

in a suitable sense, where \( \varphi, \psi: Q \to \mathbb{R} \) and \( g: Q \times \mathbb{R} \to \mathbb{R} \) are prescribed. Moreover, \( A \) is the linear second order differential operator

\[
(Av)(x) := -\sum_{j,m=1}^{N} \partial_{x_j}(a_{jm}(x) \partial_{x_m}v(x)), \quad x \in \Omega
\]

(3.5)

and \( B * \theta \) is defined by

\[
(B * v)(x, t) := -\sum_{j,m=1}^{N} \partial_{x_j} \int_{0}^{t} (b_{jm}(x, t-s) \partial_{x_m}v(x, s))ds, \quad (x, t) \in Q.
\]

(3.6)

Here the coefficients \( a_{jm} \) and \( b_{jm} \) are measurable functions from \( \Omega \) and \( Q \), respectively, to \( \mathbb{R} \). Note that both \( A \) and \( B \) are in divergence form. Besides, \( \partial_{\nu(A+B*)} \) denotes the conormal derivative related to the operator \( A + B* \) (see below for details), while \( h: \Sigma \to \mathbb{R} \) and \( u_0 : \Omega \to \mathbb{R} \) are given data.

Let us introduce now the assumptions that will enable us to reformulate (3.1)-(3.4) as (P).

(B1) \( \varphi, \psi \in W^{1,1}(0, T; L^\infty(\Omega)) \).

(B2) \( g \) is a Carathéodory function satisfying \( g(\cdot, \cdot, 0) \in L^2(Q) \) and

\[
|g(t, x, z_1) - g(t, x, z_2)| \leq c_1 |z_1 - z_2| \quad \text{for a.a. } (x, t) \in Q, \ \forall z_1, z_2 \in \mathbb{R}
\]

for some positive constant \( c_1 \).

(B3) \( a_{jm} = a_{mj} \in L^\infty(\Omega) \) and \( b_{jm} \in W^{1,1}(0, T; L^\infty(\Omega)) \) for \( j, m = 1, \ldots, N \). In addition, there exists a constant \( c_2 > 0 \) such that

\[
\sum_{j,m=1}^{N} a_{jm}(x) y_j y_m \geq c_2 |y|^2 \quad \forall y = (y_1, \ldots, y_N) \in \mathbb{R}^N, \ \text{for a.a. } x \in \Omega.
\]

(3.7)

Also, setting

\[
a(v, w) := \sum_{j,m=1}^{N} \int_{\Omega} a_{jm} v_{x_j} w_{x_m} \quad \forall v, w \in H^1(\Omega)
\]

and associating with any \( v \in L^2(0, T; H^1(\Omega)) \) the element \( \beta * v \in C^0([0, T]; H^1(\Omega)') \) specified by

\[
H^1(\Omega)'((\beta * v)(t), w)_{H^1(\Omega)} := \sum_{j,m=1}^{N} \int_{\Omega} (b_{jm} * v_{x_j}) (\cdot, t) w_{x_m}
\]

\[
\forall w \in H^1(\Omega), \ \forall t \in [0, T],
\]

(3.8)
we point out that the conormal derivative \( \partial_{\nu(A+B*)} \) is then defined for all \( v \in L^2(0,T;H^1(\Omega)) \) such that \((A+B*)v \in L^2(0,T;L^2(\Omega))\) by

\[
L^2(0,T;H^{-1/2}(\Gamma))(\partial_{\nu(A+B*)} v, w)_{L^2(0,T;H^{1/2}(\Gamma))} := \int_0^T \left( a(v(\cdot,t),w(\cdot,t)) + H^1(\Omega)^{(\beta \ast v)}(t),w(\cdot,t) \right)_{H^1(\Omega)} dt - \int_0^T \int_{\Omega} w(A+B*)v \forall w \in L^2(0,T;H^1(\Omega)). \tag{3.9}
\]

\((B4) \quad \alpha = \partial \phi \) where \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) is a convex potential satisfying \( \phi(z) \leq c_3(|z|^2 + 1) \forall z \in \mathbb{R} \) for some positive constant \( c_3 \).

\((B5) \quad h \in W^{1,1}(0,T;L^2(\Gamma)), u_0 \in L^2(\Omega), \) and \( \vartheta_0 = (\mathcal{I} + \mathcal{H})^{-1}(u_0) \in H^1(\Omega) \).

Therefore, on account of \((B1)-(B5)\), we can now state a weak formulation of the Stefan problem \((3.1)-(3.4)\). For the sake of convenience, in the sequel we denote by \( \cdot, \cdot \) the duality pairing between \( H^1(\Omega)' \) and \( H^1(\Omega) \).

**Problem (S)** Find \( \vartheta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \) and the auxiliary functions \( \chi \in L^\infty(Q), \eta \in L^\infty(0,T;L^2(\Gamma)) \) which satisfy

\[
\vartheta + \chi \in W^{1,1}(0,T;H^1(\Omega))' \tag{3.10}
\]

\[
< \partial_t(\vartheta + \chi + \varphi \ast \vartheta + \psi \ast \vartheta), v > + a(\vartheta, v) + < \beta \ast \vartheta, v > + \int_{\Gamma} \eta v = (g(\cdot,\cdot,\vartheta), v) + \int_{\Gamma} h v \forall v \in H^1(\Omega), \text{ a.e. in } (0,T) \tag{3.11}
\]

\[
\chi \in \mathcal{H}(\vartheta) \quad \text{a.e. in } Q \tag{3.12}
\]

\[
\eta \in \alpha(\vartheta) \quad \text{a.e. on } \Sigma \tag{3.13}
\]

\[
(\vartheta + \chi)(0) = u_0 \quad \text{in } H^1(\Omega)' \tag{3.14}
\]

Our main result is

**Theorem 3.1** Let \((B1)-(B5)\) hold. Then Problem \((S)\) admits a solution.

**Remark 3.2** It is worth noting that Theorem 3.1 can be viewed as a generalization of [10, Prop. 2.4]. Moreover, making a comparison between Problem \((S)\) and \((3.1)-(3.4)\), we observe that equation \((3.1)\) does not hold in \( L^2(Q) \) and, especially, the boundary condition \((3.3)\) cannot be recovered in the sense of traces in \( L^2(0,T;H^{-1/2}(\Gamma)) \) (contrary to the example developed in [2, Subsect. 5.1]). However, choosing \( v \in H^1_0(\Omega) \) as a test function in \((3.11)\), it is straightforward to deduce

\[
a(\vartheta, v) + < \beta \ast \vartheta, v > = - < \partial_t(\vartheta + \chi + \varphi \ast \vartheta + \psi \ast \vartheta) - g(\cdot,\cdot,\vartheta), v > \\
\forall v \in H^1_0(\Omega), \text{ a.e. in } (0,T).
\]
Then, integrating in time over \((0, t)\), \(t \in (0, T]\), and recalling \((B3), (3.5),\) and \((3.6),\) we obtain with the help of \((3.14)\)

\[
((A + B\ast)(1 \ast \vartheta))(t) = -(\vartheta + \chi + \varphi \ast \vartheta + \psi \ast \chi)(\cdot, t) + u_0 + \int_0^t g(\cdot, s, \vartheta(\cdot, s))ds
\]

in \(H^{-1}(\Omega)\), for a.a. \(t \in (0, T)\) \(3.15\)

where \((1 \ast \vartheta)(\cdot, t) = \int_0^t \vartheta(\cdot, s)ds\). Note that the right hand side of \((3.15)\) belongs to \(L^2(Q)\). Hence, we have that \((A + B\ast)(1 \ast \vartheta) \in L^2(Q)\) and, in view of \((3.9),\) the integrated boundary condition

\[
\partial_t((A + B\ast)(1 \ast \vartheta)) + 1 \ast \eta \ni 1 \ast h
\]

(cf. \((3.13)\) as well) holds in the sense of traces in \(L^2(0, T; H^{-1/2}(\Gamma))\). At this point, we could also argue that equation \((3.1)\) makes sense, e.g., in \(W^{-1,2}(0, T; H^{-1}(\Omega))\).

**Proof of Theorem 3.1.** It suffices to show that Problem \((S)\) can be put in the abstract framework of \((P)\). Then, the existence will follow from Theorem 2.1. Hence, let \(V = H^1(\Omega), H = L^2(\Omega),\) and introduce the new variable

\[
u = \vartheta + \chi.
\]

Note that, owing to \((B1),\) the relations \((3.11)-(3.12)\) can be rewritten in the form

\[
< \partial_t u, v > + a(\vartheta, v) + \int_{\Gamma} \eta v + < \beta \ast \vartheta, v > = < f, v > + (F(u) + G(\vartheta), v)
\]

\(\forall v \in V',\ a.e.\ in\ (0, T)\)

\(u \in (I + H)(\vartheta)\ a.e.\ in\ Q\)

where

\[
< f(t), v > = \int_{\Gamma} h(\cdot, t)v
\]

for any \(v \in V\) and almost any \(t \in [0, T]\). Here, we have set

\[
F(u)(x, t) = -\psi(x, 0)u(x, t) - (\partial_t \psi \ast u)(x, t)
\]

\[
G(\vartheta)(x, t) = g(x, t, \vartheta(x, t)) + (\psi - \varphi)(x, 0)\vartheta(x, t) + (\partial_t(\psi - \varphi) \ast \vartheta)(x, t)
\]

for almost all \((x, t) \in Q\). Using \((B1)-(B2)\) and Young's inequality for convolution products, it is not difficult to check that \(F\) and \(G\) are Lipschitz continuous and causal operators from \(L^2(0, T; H)\) to itself, whence \((A5)\) is fulfilled.

On the other hand, the maximal monotone operator \(M\) defined by

\[
Mv = (I + H)(v), v \in H
\]

clearly satisfies \((A2)\) and, in particular, \((2.1)-(2.2).\) Next, let us take \(W = H^{3/4}(\Omega)\), so that \((A1)\) holds, and specify the functions

\[
J_1(v) = \frac{1}{2} a(v, v), J_2(v) = \int_{\Gamma} \phi(v), v \in V.
\]

3.21
In view of (B3), the quadratic form \( a \) is continuous and symmetric. Therefore \( A_1 = \partial J_1 \) is a linear and bounded operator from \( V \) to \( V' \) which is given by

\[
< A_1(v), w > = a(v, w) \quad \forall v, w \in V.
\]  

(3.22)

As far as \( A_2 = \partial J_2 \) is concerned, we can invoke, for instance, [2, Lemmas 5.1 and 5.2] and verify that

\[
w \in A_2(z) \quad \text{if and only if} \quad \langle w, v \rangle = \int_{\Gamma} \omega v \quad \forall v \in V,
\]

for some \( \omega \in L^2(\Gamma) \) such that \( \omega \in \partial \phi(z) \) a.e. in \( \Gamma \).

(3.23)

In addition, from (B4) it follows that (see, e.g., [2, Lemma 5.2]) there exists a positive constant \( C_b \), depending only on \( c_3 \) and the surface measure of \( \Gamma \), such that

\[
|< w, v >| \leq C_b \left( 1 + \| z \|_{L^2(\Gamma)} \right) \| v \|_{L^2(\Gamma)} \quad \forall z, v \in V, \quad \forall w \in A_2(z).
\]  

(3.24)

Since the trace operator \( v \mapsto v|_{\Gamma} \) is continuous from \( W \) to \( L^2(\Gamma) \), by (3.24) we deduce that \( A_2 = \partial J_2 \) maps bounded sets of \( V \) into bounded sets of the dual space of \( W \). Then, in order to conclude the verification of (A3), it remains to check (2.3). Note, however, that (2.3) is a direct consequence of (3.21), (3.7), and the fact that \( \phi \) is bounded from below by an affine function (see, e.g., [3, Prop. 2.1, p. 51]). Hence, by recalling that \( A = A_1 + A_2 \), it turns out that assumption (A3) is completely satisfied.

Next, we introduce the operator

\[
< B(t)v, w > = \sum_{j,m=1}^{N} \int_{\Omega} b_{jm}(\cdot, t)v_{x_j}w_{x_m} \quad \forall v, w \in V, \quad \forall t \in [0, T].
\]  

(3.25)

and use (B3) to infer that \( B \) fulfills (A4). Moreover, on account of (3.8), it is clear that the image of \( v \in L^2(0, T; V) \) under \((B*)\) is \( \beta * v \in L^2(0, T; V') \).

Finally, we observe that (B4), (B5), (3.17), (3.20), and (3.21) entail the validity of (A6) and (A7).

In conclusion, thanks to (3.16)-(3.23) and (3.25), we deduce that Problem (S) can be equivalently set as Problem (P). Indeed, the solution component \( \xi \) in (P) satisfies \( \xi = A_1 \eta + \xi_2 \) for some \( \xi_2 \in A_2(\theta) \) almost everywhere in \((0, T)\), and \( \eta \) in (S) is exactly the boundary function corresponding to \( \xi_2 \) in (3.23). Thus, the \( L^\infty(0, T; L^2(\Gamma)) \) regularity of \( \eta \) follows from (2.4) and (3.24). Note also that \( \chi \in L^\infty(Q) \) comes directly from (3.12), which actually implies that \( 0 \leq \chi \leq 1 \) almost everywhere in \( Q \). Then, Theorem 2.1 enables us to conclude the proof. \( \square \)

References


