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Structure of unbounded viscosity solutions to semilinear degenerate elliptic equations

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1. Introduction.

We consider the Dirichlet problem for a semilinear degenerate elliptic equation (DP):

\[(1) \quad -g(|x|) \Delta u + f(|x|, u) = 0 \quad \text{in} \quad \mathbb{R}^N\]

\[(2) \quad \lim_{|x| \to \infty} \frac{u(x)}{h(|x|)} = 1,\]

where \(N \geq 2,\)

\[g(|x|) = ||x| - a_1|^\lambda_1 ||x| - a_2|^\lambda_2 \cdots ||x| - a_k|^\lambda_k (|x| + 1)^{-\lambda^*},\]

\(0 < a_1 < a_2 < \cdots < a_k, \quad 0 < \lambda_i \quad (i = 1, 2, \cdots, k) \quad \text{and} \quad \lambda^* \geq 0,\)

\(\Delta\) is the Laplacian, and \(h(|x|) \in C(|x| > a_k)\) will be determined later.

We discuss the problem (DP) under the following assumptions:

(A.1) \(f(t, y) \in C([0, \infty) \times \mathbb{R})\) is locally Lipschitz continuous in \((t, y).\)

(A.2) For any \(t > 0\) fixed, \(f(t, y)\) is strictly increasing in \(y.\)

(A.3) For any \(t \in [0, \infty),\) there exists a continuous function \(\varphi(t)\) such that \(f(t, \varphi(t)) = 0.\)

**Example**

\[g(|x|) \Delta u = u |u|^{p-1} - f(|x|).\]

In this paper, we study (DP) in case \(N = 2.\)

Our aim is to prove the following

A) For \(\lambda_i > 0 \quad (i = 1, 2, \cdots, k) \quad \text{and} \quad \lambda^* \geq 0,\) there exists a unique standard and radial
viscosity solution of (DP).

B) If \( \lambda_i \geq 1 \) for all \( i = 1, 2, \cdots, k \), there exists a unique viscosity solution of (DP).

From A), every viscosity solution is radial and standard.

C) If \( 0 < \lambda_i < 1 \) for some \( i = 1, 2, \cdots, k \), there exist infinitely many viscosity solutions of (DP).

2. Structure of standard viscosity solutions.

Following Crandall and Huan [2], we call a viscosity solution \( u \) of (DP) a standard solution if \( u(x) = \varphi(a_i) \) (i.e., \( f(a_i, u(x)) = 0 \)) on \( |x| = a_i \) (\( i = 1, 2, \cdots, k \)).

In order to construct a standard viscosity solution we shall consider the following Dirichlet problems:

\[
\begin{aligned}
\text{(P}_0\text{)} & \quad \begin{cases}
-g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } B_{a_1}, \\
u(x) = b_1 & \text{on } |x| = a_1;
\end{cases} \\
\text{(P}_1\text{)} & \quad \begin{cases}
-g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } A(a_i, a_{i+1}), \\
u(x) = b_i & \text{on } |x| = a_i, \\
u(x) = b_{i+1} & \text{on } |x| = a_{i+1};
\end{cases} \\
\text{(P}_k\text{)} & \quad \begin{cases}
-g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } A(a_k, \infty), \\
u(x) = b_k & \text{on } |x| = a_k, \\
\lim_{|x| \to \infty} \frac{u(x)}{h(|x|)} = 1,
\end{cases}
\end{aligned}
\]

where \( A(a_i, a_{i+1}) = \{ x \in \mathbb{R}^N : a_i < |x| < a_{i+1} \}, \quad i = 1, 2, \cdots, k-1 \) and \( b_i = \varphi(a_i) \) (\( i = 1, 2, \cdots, k \)).

Let \( u_0 \in C(B_{a_1}) \cap C^2(B_{a_1}) \) (resp. \( u_i \in C(A(a_i, a_{i+1})) \cap C^2(A(a_i, a_{i+1})) \)) be a radial classical solution of (P_0) (resp. (P_i) (\( i = 1, 2, \cdots, k \))).

Put

\[
\tilde{u}(x) = \begin{cases}
u_0(x) & \text{for } x \in B_{a_1}, \\
u_1(x) & \text{for } x \in A(a_1, a_2), \\
\vdots & \\
u_k(x) & \text{for } x \in A(a_k, \infty)
\end{cases}
\]

It is easy to verify, by the definition of viscosity solutions, that \( \tilde{u} \) is a radial and standard
viscosity solution of (DP). An easy calculation shows that \( u(x) = y(t) \mid x \mid = t \) is a radial classical solution of (P\(_1\)) if and only if \( y(t) \) is a classical solution of the following boundary value problem (denoted by (BVP\(_1\))):

\[
g(t)\left( \frac{d^2y}{dt^2} + \frac{1}{t} \frac{dy}{dt} \right) = f(t, y) \quad \text{in} \quad a_i < t < a_{i+1}
\]

\[y(a_i) = b_i \quad \text{and} \quad y(a_{i+1}) = b_{i+1} \quad (i = 0, 1, \ldots, k),\]

where \( y(a_0) = b_0 \) and \( y(a_{k+1}) = b_{k+1} \) are replaced by \( \frac{dy}{dt}(0) = 0 \) and \( \lim_{t \to \infty} \frac{y(t)}{h(t)} = 1 \), respectively.

From now on we briefly explain that the existence and uniqueness of classical solutions of (BVP\(_1\)) \((i = 1, 2, \ldots, k)\) play an essential role to prove our assertion stated in Introduction. Assume \( \lambda_i \geq 1 \) for all \( i = 1, 2, \ldots, k \). Let \( u(x) \) be an arbitrary viscosity solution of (DP). Define

\[
\overline{U}(x) = \sup_{|y| = |x|} u(y) \quad \text{and} \quad \underline{U}(x) = \inf_{|y| = |x|} u(y).
\]

We observe that \( \overline{U}(x) \) (resp. \( \underline{U}(x) \)) is continuous and radial viscosity subsolution (resp. supersolution) and \( \overline{U}(x) = \underline{U}(x) = b_i \) on \( |x| = a_i \) (by \( \lambda_i \geq 1 \)). By the well-known comparison theorem, we have

\[y_i(|x|) \leq \underline{U}(x) \leq \overline{U}(x) \leq y_i(|x|)\]

for \( a_i \leq |x| \leq a_{i+1} \) \((i = 0, 1, 2, \ldots, k)\), where \( y_i \) is the unique solution of (BVP\(_1\)).

3. Existence and uniqueness for (BVP\(_1\)).

In order to study (BVP\(_1\)), we introduce the following integral equations:

\[
y(t) = \alpha + \int_{0}^{t} (\log s) g(s)^{-1} f(s, y(s)) ds,
\]

\[
y(t) = \alpha + t_0 \beta \log(t/t_0) + \int_{t_0}^{t} \log(s) g(s)^{-1} f(s, y(s)) ds,
\]

where \( 0 < t_0 \notin \{a_1, a_2, \ldots, a_k\} \), \( \alpha \) and \( \beta \) are real parameters. Applying a fixed point theorem, we can prove the local existence of solutions of (5) and (6).
First, to solve (BVP$_0$), we define

\[ S_0^+ = \{ \alpha \in \mathbb{R}; \lim_{t \to T_\alpha} y_\alpha(t) = +\infty \} \]
\[ S_0 = \{ \alpha \in \mathbb{R}; \lim_{t \to \alpha_1} y_\alpha(t) = \text{exists} \} \]
\[ S_0^- = \{ \alpha \in \mathbb{R}; \lim_{t \to T_\alpha} y_\alpha(t) = -\infty \}, \]

where $y_\alpha$ is a classical solution of (5) obtained by prolonging local solutions of (5) and (6).

We see that (i) in case $0 < \lambda_1 < 2$,

\[ S_0^+ = [\overline{\alpha}, \infty), \quad S_0 = (\underline{\alpha}, \overline{\alpha}), \quad S_0^- = (-\infty, \underline{\alpha}] \]

and \( \{y_\alpha(a_1) = \lim_{t \to a_1} y_\alpha(t); \alpha \in S_0\} = \mathbb{R} \);

and (ii) in case $\lambda_1 \geq 2$,

\[ S_0^+ = (\alpha_0, \infty), \quad S_0 = \{\alpha_0\}, \quad S_0^- = (-\infty, \alpha_0) \quad \text{and} \quad y_{\alpha_0}(a_1) = b_1. \]

Consequently we have

**PROPOSITION 1.** There exists a unique classical solution $y_0$ of (BVP$_0$).

Next, to solve (BVP$_i$) ($i = 1, 2, \ldots, k-1$), we fix $t_0 \in (a_i, a_{i+1})$ and define for each $\alpha \in \mathbb{R}$

\[ B_i^+ = \{ \beta \in \mathbb{R}; \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = +\infty \} \]
\[ B_i = \{ \beta \in \mathbb{R}; \lim_{t \downarrow \alpha_i} y_{\alpha\beta}(t) = \text{exists} \} \]
\[ B_i^- = \{ \beta \in \mathbb{R}; \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = -\infty \}, \]

where $y_{\alpha\beta}(t)$ is a solution of (6) on $(T_{\alpha\beta}, t_0]$ ($a_i \leq T_{\alpha\beta} < t_0$).

We can prove that (i) in case $0 < \lambda_i < 2$,

\[ B_i^- = [\overline{\beta}, \infty), \quad B_i = (\underline{\beta}, \overline{\beta}), \quad B_i^+ = (-\infty, \underline{\beta}] \]

and \( \{y_{\alpha\beta}(a_i) = \lim_{t \downarrow \alpha_i} y_{\alpha\beta}(t); \beta \in B_i\} = \mathbb{R} \);
and (ii) in case \( \lambda_i \geq 2 \),

\[
B_i^- = (\beta_i, \infty), \quad B_i = \{\beta_i\}, \quad B_i^0 = (-\infty, \beta_i) \text{ for some } \beta_i = \beta(\alpha) \quad \text{and} \quad y_{\alpha \beta(\alpha)}(a_i) = b_i.
\]

And then we solve

\[
\begin{cases}
g(t)\left(\frac{d^2y}{dt^2} + \frac{1}{t} \frac{dy}{dt}\right) = f(t, y) \quad \text{in} \quad [t_0, a_{i+1}) \\
y(t_0) = \alpha, \quad \frac{dy}{dt}(t_0) = \beta(\alpha).
\end{cases}
\]

Define

\[
A_i^+ = \{\alpha \in \mathbb{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = +\infty\} \\
A_i = \{\alpha \in \mathbb{R}; \lim_{t \in [a_{i+1}]} y_\alpha(t) \text{ exists}\} \\
A_i^- = \{\alpha \in \mathbb{R}; \lim_{t \downarrow T_\alpha} y_\alpha(t) = -\infty\},
\]

where \( y_\alpha(t) := y_{\alpha \beta(\alpha)}(t) \) is a solution of (7) on \([t_0, T_\alpha) \quad (t_0 < T_\alpha \leq a_{i+1}).\)

We observe that (i) in case \( 0 < \lambda_{i+1} < 2 \),

\[
A_i^+ = [\overline{\alpha}, \infty), \quad A_i = (\alpha, \overline{\alpha}), \quad A_i^- = (-\infty, \alpha] \\
\text{and} \quad \{y_\alpha(a_{i+1}) = \lim_{t \uparrow a_{i+1}} y_\alpha(t); \alpha \in A_i\} = \mathbb{R};
\]

and (ii) in case \( \lambda_{i+1} \geq 2 \),

\[
A_i^+ = (\alpha_i, \infty), \quad A_i = \{\alpha_i\}, \quad A_i^- = (-\infty, \alpha_i) \text{ for some } \alpha_i, \quad \text{and} \quad y_{\alpha_i}(a_{i+1}) = b_{i+1}.
\]

Therefore we have

**Proposition 2.** There exists a unique classical solution \( y_i(t) \) of \((\text{BVP}_i) \quad (i = 1, 2, \ldots, k-1)\).

(Nota the uniqueness in Propositions 1 and 2 follows immediately from the maximum principle.)

Finally we shall prove the existence and uniqueness of solutions of \((\text{BVP}_k)\). It should be noted that we have to introduce several boundary conditions at \( \infty \) corresponding to the structure of (1). To state our result, we introduce some notation:

\[
\ell := \lambda_1 + \lambda_2 + \cdots + \lambda_k - \lambda^* \quad \text{and} \quad \gamma := (\ell - 2)/(p - 1),
\]
where $p > 1$ is assumed. For $(\text{BVP}_k)$, we make the following assumptions:

(A.4) \[ \lim_{|x| \to \infty} \varphi(|x|) = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{t^p(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = 0. \]

(A.5) There exist positive constants $k_0$ and $K_0$ such that

\[ k_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1}) \leq f(t, y_1) - f(t, y_2) \leq K_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1}) \]

for every $y_1 > y_2$ and $t \gg 1$.

(A.6) \[ f(|x|, y) \]

has the following form:

\[ f(|x|, y) = y|y|^{p-1} - \varphi(|x|)\varphi(|x|)^{p-1}. \]

**Remark**

(i) It is easy to verify that (A.6) $\Rightarrow$ (A.5) $\Rightarrow$ \{(A.1), (A.2)\}.

(ii) If \[ \lim_{t \to \infty} \frac{t^p(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = \delta > 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\varphi(t)}{t^{\gamma}} = \infty, \]

then $\varphi(t)$ blows up in a finite interval.

**Proposition 3.** Let $\ell \leq 2$. Assume (A.4) and (A.5). Then there exists a unique solution of $(\text{BVP}_k)$ with $h(t) \approx \varphi(t)$. Moreover, if $h(t) \not\approx \varphi(t)$ then $(\text{BVP}_k)$ does not possess any solution, where $h(t) \approx \varphi(t)$ means that \[ \lim_{t \to \infty} \frac{h(t)}{\varphi(t)} = 1. \]

**Sketch of proof of Proposition 3.** Let $y_\alpha(t)$ be a classical solution of (4) in $[a_k, T_\alpha)$ satisfying $y_\alpha(a_k) = b_k$. Then, it is important to note that \[ \lim_{t \to T_\alpha} y_\alpha(t) = +\infty \]

or \[ \lim_{t \to T_\alpha} y_\alpha(t) = -\infty. \] (In other words, equation (4) does not possess any bounded solution.) Therefore, as before, we define

\[ A^+ = \{\alpha \in \mathbb{R}; \lim_{t \to T_\alpha} y_\alpha(t) = +\infty\} \]

\[ A^- = \{\alpha \in \mathbb{R}; \lim_{t \to T_\alpha} y_\alpha(t) = -\infty\}, \]

where $T_\alpha \leq \infty$. It is shown that (1)$A^+ \neq \emptyset$, (2)$A^- \neq \emptyset$, (3)$A^+ \cup A^- = \mathbb{R}$, and (4)$\alpha_1 < \alpha_2$ if $\alpha_1 \in A^-$ and $\alpha_2 \in A^+$. Hence, the cut $\overline{\alpha} = (A^-, A^+)$ is determined. Using (A.4), we have $A^- = (-\infty, \overline{\alpha})$, $A^+ = [\overline{\alpha}, \infty)$ and $T_{\overline{\alpha}} = \infty$. We can show that \[ \lim_{t \to \infty} \frac{y_{\overline{\alpha}}(t)}{\varphi(t)} = 1 \]

and the uniqueness of solutions of $(\text{BVP}_k)$ with $h(t) \approx \varphi(t)$ holds.

In a similar spirit, we have
PROPOSITION 4. Let $\ell > 2$. Assume (A.4), (A.5) and $\lim_{t \to \infty} \frac{\varphi(t)}{t^\gamma} = \infty$. Then the assertions as in Proposition 3 are valid.

Now, it remains to consider the case $\lim_{t \to \infty} \frac{\varphi(t)}{t^\gamma} = \kappa$ ($0 < \kappa < \infty$) under the assumptions (A.4) and (A.6). In this case we may assume

$$g(t)^{-1} = t^{-\ell} + g_1(t)t^{-\ell}, \quad |g_1(t)| \leq K_1/t$$

$$\varphi(t)^p = \kappa^p t^{\gamma p} + \varphi_1(t)t^{\gamma p}, \quad |\varphi_1(t)| \leq K_1/t$$

for every $t >> 1$. Putting $y(t) = t^\gamma w(t)$, we get a new ODE for $w(t)$:

$$(8) \quad \frac{d^2 w}{dt^2}(t) + \frac{2\gamma + 1}{t} \frac{dw}{dt}(t) = \frac{1}{t^2} \{w|w|^{p-1} - \kappa^p - \gamma^2 w\} + \text{(lower term)},$$

where

$$\text{lower term} = \frac{1}{t^2} \{g_1(t)(w|w|^{p-1} - \kappa^p) - (1 + g_1(t))\varphi_1(t)\}.$$ 

Then we have 3 types such that

We have to introduce various boundary functions $h(|x|)$ corresponding to Type (1) - Type (3). In what follows, we will focus on Type (3), because Type (3) is the most interesting case. In this case, we first note that every solution $w(t)$ of (8) with infinite life span converges to the one of $\{w_{-1}, w_0, w_1\}$. From this it follows that if $y(t)$ is a
solution of (4) in $(a_k, \infty)$ with infinite life span, then $y(t)/t^\gamma$ converges to the one of \{w_{-1}, w_0, w_1\} as $t \to \infty$. Define

$$A^+ = \{\alpha \in \mathbb{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = +\infty \text{ and } T_\alpha < \infty\}$$

$$A_1 = \{\alpha \in \mathbb{R}; \lim_{t \to \infty} y_\alpha(t)/t^\gamma = w_1\}$$

$$A_0 = \{\alpha \in \mathbb{R}; \lim_{t \to \infty} y_\alpha(t)/t^\gamma = w_0\}$$

$$A_{-1} = \{\alpha \in \mathbb{R}; \lim_{t \uparrow \infty} y_\alpha(t)/t^\gamma = w_{-1}\}$$

$$A^- = \{\alpha \in \mathbb{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = -\infty \text{ and } T_\alpha < \infty\}.$$

**Lemma 5.** $A^- = (-\infty, \alpha_*), A_{-1} = \{\alpha_*\}, A_0 = (\alpha_*, \alpha^*)$, $A_1 = \{\alpha^*\}$ and $A^+ = (\alpha^*, \infty)$.

Using this lemma, we have

**Proposition 6.** Type (1) $\implies \exists$ a unique solution of (BVP)$_k$ with $h(t) \approx w_1 t^\gamma$.

Type (2) $\implies \exists$ a unique solution of (BVP)$_k$ with $h(t) \approx w_1 t^\gamma$ and $\exists$ infinitely many solutions of (BVP)$_k$ with $h(t) \approx w_0 t^\gamma$.

Type (3) $\implies \exists$ a unique solution of (BVP)$_k$ with $h(t) \approx w_1 t^\gamma$, $\exists$ infinitely many solutions of (BVP)$_k$ with $h(t) \approx w_0 t^\gamma$, and $\exists$ a unique solution of (BVP)$_k$ with $h(t) \approx w_{-1} t^\gamma$.

**Remark** In the case where $\lim_{t \to \infty} y(t)/t^\gamma = w_0$, if the above boundary condition is replaced by stronger another condition, then we can prove the uniqueness of solutions of (BVP)$_k$. In fact, let $\alpha \in A_0 = (\alpha_*, \alpha^*)$ and $y_\alpha$ be a solution of (BVP)$_k$. Then, for every $\sigma \in \mathbb{R}$, there exists a unique $q_{\sigma}(t) = O(t^{\gamma-1})$ (as $t \to \infty$) such that

$$\lim_{t \to \infty} \frac{y_\alpha(t) - \{w_0 t^\gamma + q_{\sigma}(t)\}}{t^{\delta_1}} = \sigma,$$

where $\delta_1 = \sqrt{p|w_0|^{p-1}}, \delta_1 \neq 0$.

Let $\delta_1 = 0$. For for every $\sigma \in \mathbb{R}$, there exists a unique $q_{\sigma}(t) = O(t^{\gamma-1})$ (as $t \to \infty$) such that

$$\lim_{t \to \infty} \frac{y_\alpha(t) - q_{\sigma}(t)}{\log t} = \sigma.$$

**Main result**

(I) Let $l \leq 2$ and $\lambda_i > 0 \quad (i = 1, 2, \cdots, k)$. Assume (A.4) and (A.5). Then there exists a unique standard radial viscosity solution $u$ of (DP) with boundary function
$h(|x|) \approx \varphi(|x|)$. Moreover, if $h(|x|) \not\approx \varphi(|x|)$ then (DP) does not possess any standard radial viscosity solution of (DP).

(II) Let $l > 2$ and $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$. Assume (A.4). Assume (A.5) and \[ \lim_{t \to \infty} \varphi(t)/t^\gamma = +\infty. \] Then the assertions of (I) are also valid.

(III) Let $l > 2$ and $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$. Assume (A.4). Assume (A.6) and \[ \lim_{t \to \infty} \varphi(t)/t^\gamma = \kappa(> 0). \] Then the same results for standard radial viscosity solutions of (DP) as those in Proposition 6 hold. Of course, boundary functions are replaced by

$$
\begin{align*}
& h(|x|) \approx w_1 |x|^\gamma \quad \text{in case Type(1)}; \\
& h(|x|) \approx w_i |x|^\gamma \quad (i \in \{0, 1\}) \quad \text{in case Type(2)}; \\
& h(|x|) \approx w_i |x|^\gamma \quad (i \in \{-1, 0, 1\}) \quad \text{in case Type(3)}.
\end{align*}
$$

In particular, in the case where $h(|x|) \approx w_0 |x|^\gamma$, the boundary condition at $\infty$ is replaced by

$$
\lim_{|x| \to \infty} \frac{u(x) - \{w_0 |x|^\gamma + q\sigma(|x|)\}}{|x|^{\delta_1}} = \sigma,
$$

where $\delta_1 = \sqrt{p|w_0|p^{-1}}, \delta_1 \neq 0$.

If $\delta_1 = 0$, then the boundary condition at $\infty$ is represented with

$$
\lim_{|x| \to \infty} \frac{u(x) - q\sigma(|x|)}{\log |x|} = \sigma.
$$

(IV) If $\lambda_i \geq 1$ for all $i \in \{1, 2, \cdots, k\}$, then the uniqueness of viscosity solutions of (DP) holds. Hence, every viscosity solution of (DP) is radial.

References

