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The surface diffusion flow equation does not preserve the convexity

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1 Introduction and main result

We consider the following geometric evolution law of the form

$$\begin{cases} V = -\Delta_{\Gamma(t)} H & \text{on } \Gamma(t),\, t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1}$$

Here t denotes the time variable and $\Gamma(t)$ denotes an unknown evolving hypersurface in $\mathbf{R}^n$ with $n \geq 3$; Γ_0 is a given initial hypersurface in $\mathbf{R}^n$. For each $t > 0$ and $x \in \Gamma(t)$, the quantities $V = V(t,x)$ and $H = H(t,x)$ denote the outward normal velocity and the outward mean curvature of $\Gamma(t)$ at x, respectively. The operator $\Delta_{\Gamma(t)}$ stands for the Laplace-Beltrami operator on $\Gamma(t)$. The law (1) is called the *surface diffusion flow equation.*

The solution $\Gamma(t)$ of (1) describes motion of interface in a binary alloy system. Equation (1) was first proposed by Mullins [15] to explain thermal grooving in material sciences. Also Davi and Gurtin [5] derived (1) from a view point of thermodynamics and continuum mechanics (see also Cahn and Taylor [1]). Recently, J. W. Cahn, C. M. Elliott and A. Novick-Cohen [2] linked (1) with the Cahn-Hilliard equation with a concentration dependent mobility via formal singular limit.

Parametrization of (1) tells us that (1) is a nonlinear fourth order parabolic equation. Generally speaking, behaviors of solutions of fourth order equations are less known than those of second order equations.

The purpose here is to study the qualitative behavior of the solution $\Gamma(t)$ of (1) in a short time.

Several mathematical and numerical studies for (1) show remarkable characteristic phenomena, for example, loss of embeddedness and loss of convexity. The loss of both embeddedness and convexity reflects the fact that a fourth order parabolic equation does not fulfill the maximum or comparison principle. In fact, this principle is satisfied in the second order model such as the mean curvature flow equation

$$\begin{cases} V = H & \text{on } \Gamma(t),\, t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{2}$$

The maximum principle prevents the developments of self-intersections (Grayson [11] for $n = 2$) and preserves convexity (Gage and Hamilton [8] for $n = 2$ and Huisken [12] for $n \geq 3$) of solutions of (2).

The loss of embeddedness for (1) was conjectured by Elliott and Garcke [6], numerically established by Escher, Mayer and Simonett [7], and proved by Giga and the author [9] for $n = 2$, and later by Mayer and Simonett [14] for $n \geq 2$.

On the other hand, J. Escher proposed a conjecture in the conference "Nonlinear Evolution Equation" held in the end of June of 1997 in Oberwolfach that the convexity of the solution $\Gamma(t)$ of (1) is not necessarily preserved. This phenomenon was also suggested by numerical studies by B. D. Coleman, R. S. Falk and M. Moakher [3, 4].

Our problem here is to prove the loss of convexity for (1) from a theoretical point of view. The rigorous proofs for the loss of convexity were obtained by Giga and the author [10] for closed curves and by the author [13] for compact hypersurfaces. Our main result is the following.

Theorem 1 . ([10] *for* $n = 2$, [13] *for* $n \geq 3$). *There is a strictly convex closed compact initial hypersurface* Γ_0 *such that the smooth solution* $\Gamma(t)$ *of (1) starting from* Γ_0 *loses its convexity during a time interval* (t_0, t_1) *with* $t_0 > 0$ *determined by* Γ_0.

In the following we only treat the case that $\Gamma(t)$ are compact hypersurfaces. Here we summarize the strategy to prove Theorem 1. Intuitively we can imagine that the solution hypersurface of (1) starting from an initial hypersurface with sufficiently weak convexity easily creats a loss of convexity. From this observation, in the first step, we introduce a deformation depending on a small parameter $\varepsilon > 0$ for strictly convex hypersurfaces Γ_0. Let us denote by Γ_0^ε the deformed hypersurface. This deformation is constructed to weaken the convexity of the original surface Γ_0 such as one of the principal curvatures of Γ_0^ε has the order $0(-\varepsilon)$ locally. Then the smooth solution $\Gamma^\varepsilon(t)$ of (1) starting from Γ_0^ε exists for $t \in [0, T^\varepsilon]$ for some $T^\varepsilon > 0$. But we should be afraid that T^ε may shrink to 0 as $\varepsilon \to 0$. In the second step we present a fact that there is a time $T > 0$ such that $T^\varepsilon \geq T$ for any sufficiently small ε. This means that $\Gamma^\varepsilon(t)$ exists uniformly in ε. Finally, using the results of the previous two steps, we prove that if $\varepsilon > 0$ is sufficiently small, then one of the principal curvatures of $\Gamma^\varepsilon(t)$ becomes positive after a finite time, which means that $\Gamma^\varepsilon(t)$ loses its convexity.

Sections 2 and 3 are devoted to state local existence results for (1) which are constructed in a differente framework from the result in [7]. Our result clarifies how the existence time of local solutions depends on initial data. This enables us to establish the uniform local existence result in ε. In Section 4 we state the precise description of the proof of Theorem 1.

2 Local solutions for motion by surface diffusion near a convex hypersurface

Following [7], we introduce a parametrization for (1). Let Σ be a smooth compact convex closed embedded oriented hypersurface in $\mathbf{R}^n$ and let $\{U_\beta, \psi_\beta\}_{\beta=1}^{\bar\beta}$ be an atlas on Σ. For $s \in U_\beta \subset \Sigma$, $\psi_\beta(s) = (u_\beta^1, \cdots, u_\beta^{n-1}) \in U'_\beta := \psi_\beta(U_\beta) \subset \mathbf{R}^{n-1}$ is called the local coordinate of s. Let z be the induced metric on Σ from the Euclidean metric in $\mathbf{R}^n$ and let h be the second fundamental quantity of Σ. The sign convention adopted here is that

$$h[\xi,\xi] \leq 0 \quad \text{for any tangent vector field } \xi \text{ on } \Sigma.$$

In local coordinates they are written as

$$z = \sum_{i,j=1}^{n-1} z_{\beta,ij} du_\beta^i \otimes du_\beta^j, \quad h = \sum_{i,j=1}^{n-1} h_{\beta,ij} du_\beta^i \otimes du_\beta^j$$

at $s \in U_\beta$. Hereafter, if any confusion may not be caused, then using Einstein's convention and omitting the index β we often simply write them as

$$z = z_{ij} du^i \otimes du^j, \quad h = h_{ij} du^i \otimes du^j.$$

Throughout this paper we regard Σ as a Riemannian manifold with the metric z. We call Σ the reference hypersurface.

Let $\rho : [0,T) \times \Sigma \to \mathbf{R}_+ := \{r \in \mathbf{R}; r \geq 0\}$ be a smooth scalar field and we assume that $\Gamma(t)$, $t \in [0,T)$, is written in terms of $\rho(t,s)$ with $\rho(0,s) = \rho_0(s)$ as

$$\Gamma(t) = \{s + \rho(t,s)\nu(s) \in \mathbf{R}^n; s \in \Sigma\},$$

where $\nu(s)$ is the outward normal of Σ at $s \in \Sigma$. We define two geometric quantities of Σ:

$$\begin{aligned} w(r) &= w_{ij}(r) du^i \otimes du^j := (z_{ij} - 2h_{ij}r + z^{kl}h_{ki}h_{lj}r^2) du^i \otimes du^j \quad \text{for } r \geq 0, \\ \sigma(\rho, d\rho) &:= w(\rho) + d\rho \otimes d\rho, \end{aligned}$$

where $(z^{ij}) = (z_{ij})^{-1}$ and $d\rho = \frac{\partial \rho}{\partial u^i} du^i$.

Lemma 2 . *Let Σ be a smooth compact convex closed embedded oriented hypersurface in $\mathbf{R}^n$. Then,*
(i) $w(r)$ is a metric on Σ for $r \geq 0$,
(ii) $\sigma(f, df)$ is a metric on Σ for nonnegative (at least) C^1-scalar field $f = f(s)$ on Σ.

By Lemma 2 we can consider the geometric operators acting on scalar fields Ψ on Σ such as $\mathrm{grad}_{w(f)}$, $\mathrm{Hess}_{w(f)}$, $\Delta_{w(f)}$, and $\Delta_{\sigma(f,df)}$ with a nonnegative scalar field f on Σ. They are given in local coordinates by

$$\begin{aligned} \mathrm{grad}_{w(f)}\Psi &= w^{ij}(f) \frac{\partial \Psi}{\partial u^j} \frac{\partial}{\partial u^i}, \\ \mathrm{Hess}_{w(f)}\Psi &= \left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - W_{ij}^k(f, df) \frac{\partial \Psi}{\partial u^k}\right) du^i \otimes du^j, \\ \Delta_{w(f)}\Psi &= w^{ij}(f)\left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - W_{ij}^k(f, df) \frac{\partial \Psi}{\partial u^k}\right), \\ \Delta_{\sigma(f,df)}\Psi &= \sigma^{ij}(f, df)\left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - \gamma_{ij}^k(f, df, \nabla df) \frac{\partial \Psi}{\partial u^k}\right), \end{aligned}$$

where $(w^{ij}(f)) := (w_{ij}(f))^{-1}$, $(\sigma^{ij}(f,df)) := (\sigma_{ij}(f,df))^{-1}$; $W^k_{ij}(f,df)$ and $\gamma^k_{ij}(f,df,\nabla df)$ with $i,j,k = 1,2,\cdots,n-1$ are the Christoffel symbols of $w(f)$ and $\sigma(f,df)$, respectively. We also define

$$L = L(\rho,d\rho) := (1 + w(\rho)[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho])^{1/2}, \tag{3}$$
$$w'(r) = w'_{ij}(r)du^i \otimes du^j := (-2h_{ij} + 2z^{kl}h_{ki}h_{lj}r)du^i \otimes du^j,$$
$$w'(r)w^*(r) := w'_{ij}(r)w^{ij}(r) \tag{4}$$

for $r \geq 0$.

Then, as computed in [7] we can have the partial differential equation of $\rho(t,s)$ described by local coordinates which parametrizes (1). But this equation can be rewritten in more intrinsic way by making further computations. We present here the equation of $\rho(t,s)$ in the intrinsic manner in the following:

$$\begin{cases} \rho_t &= -L\Delta_{\sigma(\rho,d\rho)}\Big[L^{-3}\{L^2\Delta_{w(\rho)}\rho - \mathrm{Hess}_{w(\rho)}\rho[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] \\ & \quad -\frac{1}{2}w'(\rho)[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] - \frac{1}{2}L^2 w'(\rho)w^*(\rho)\}\Big], \quad t > 0, \ s \in \Sigma, \\ \rho(0,s) &= \rho_0(s), \quad s \in \Sigma. \end{cases} \tag{5}$$

Remark 3 . *The equation (5) is still valid on any smooth compact closed oriented hypersurface Σ (not necessarily convex) as long as $w(\rho)$ and $\sigma(\rho,d\rho)$ are metrics on Σ.*

Next we state the local existence theorem of (5) for positive initial data. The differences between the local existence theorem in [7] and ours is that our theorem uses the convex reference hypersurface Σ, which enable us to treat large magnitude of positive initial data $\rho_0(s)$. This property is needed in the proof of our main theorem (Theorem 11). Another difference is that our local existence theorem is established in a different category from that of [7], that is, the local existence theorem in [7] is established in the framework of t-continuous and s-little Hölder continuous regularity whereas our theorem is in the framework of the type $C^{[m/4]+\alpha,m+4\alpha}(\Sigma)$ for an integer $m \geq 4$ and $0 < \alpha < 1/4$, where the symbol $[q]$ denotes the largest integer less than or equal to q. Our theorem also explicitly gives the information how the existence time depends on initial data, which is also useful in the proof of Theorem 11.

Theorem 4 . *Let $m \geq 4$ be an integer and $0 < \alpha < 1/4$. Let $\rho_0 \in C^{m+4\alpha}(\Sigma)$ with $\rho_0(s) > 0$ for $s \in \Sigma$. Set*

$$m_0 = \min_{s\in\Sigma} \rho_0(s) > 0. \tag{6}$$

Then there are positive constants $T(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}, m_0)$ and $G(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)})$ such that (5) has a unique solution $\rho(t,s)$ satisfying

$$\rho \in C^{[m/4]+\alpha,m+4\alpha}([0,T_0]\times\Sigma),$$
$$\|\rho\|_{C^{[m/4]+\alpha,m+4\alpha}([0,T_0]\times\Sigma)} \leq G(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}),$$
$$\min_{(t,s)\in[0,T_0]\times\Sigma} \rho(t,s) \geq m_0/2 > 0,$$

where $T_0 := T(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}, m_0)$.

Remark 5 . *Here $T(M_0, m_0)$ is nonincreasing in M_0 and nondecreasing in m_0; $G(M_0)$ is nondecreasing in M_0.*

The essential task of the proof of Theorem 4 is to show that the linearized differential operator at initial data ρ_0 is sectorial in $C(\Sigma)$. Once this is verified, we can use the iteration method in the linearized equation to obtain the desired unique local solution of (5). The details are omitted.

3 Local existence of rotationally axisymmetric hypersurfaces for motion by surface diffusion

In this section we consider solutions of (1) which are rotationally axisymmetric hypersurfaces. First of all we specify the reference hypersurface Σ embedded in $\mathbf{R}^n$ with the orthogonal cordinate $(x^1, x^2, \cdots, x^n)$ as a surface of rotation. For this purpose we prepare a fixed smooth convex closed curve M embedded in $\mathbf{R}^2_{x^1,x^2} := \{(x^1, x^2, \cdots, x^n) \in \mathbf{R}^n; x^3 = \cdots = x^n = 0\}$. We assume that M is symmetric with respect to both the x^1 and x^2-axes and also assume that M contains two straight line segments which are parallel to the x^1-axis and intersect the x^2-axis. To be precise we put

$$M = \{\mu = (p(u^1), q(u^1)) \in \mathbf{R}^2_{x^1,x^2}; u^1 \in \mathbf{T}\},$$

where $\mathbf{T} := \mathbf{R}/2l\mathbf{Z}$ and $2l$ is the total length of M; u^1 denotes the arclength parameter of M and is assumed to move clock-wise. We assume that functions p and q are smooth and satisfy

$$p(u^1) = -p(-u^1), \quad q(u^1) = q(-u^1) \quad \text{for } u^1 \in \mathbf{T},$$

$$(p(u^1), q(u^1)) = \begin{cases} (u^1, b) & \text{for } |u^1| \leq a, \\ (l - u^1, -b) & \text{for } |u^1 - l| \leq a, \end{cases}$$

$$\kappa(u^1) < 0 \quad \text{for } a < u^1 < l - a \text{ and } l + a < u^1 < 2l - a$$

for some $a, b \in (0, l/2)$, where $\kappa(u^1)$ is the outward curvature of M.

We define a reference hypersurface Σ as the surface of rotation of M about the x^1-axis. This is given by

$$\Sigma = \{s = (p(u^1), q(u^1)\omega) \in \mathbf{R}^n; u^1 \in [-l/2, l/2], \omega \in S^{n-2}\},$$

where S^{n-2} is the unit sphere in $\mathbf{R}^{n-1}_{x^2,\cdots,x^{n-1}}$ centered at $(x^2, \cdots, x^{n-1}) = (0, \cdots, 0)$. We introduce a specified atlas of Σ as follows. Let $\{V_\gamma, \theta_\gamma\}_{\gamma=1}^{\bar{\gamma}}$ be an atlas of S^{n-2}. Here we assume that the coordinate functions θ_γ

$$\begin{array}{rccl} \theta_\gamma : & V_\gamma & \to & V'_\gamma \subset \mathbf{R}^{n-1} \\ & \omega & \mapsto & (u^2_\gamma, \cdots, u^{n-1}_\gamma) \end{array}$$

gives polar coordinates for V_γ for $\gamma = 1, \cdots, \bar{\gamma}$. Set $U'_\gamma := (-l/2, l/2) \times V'_\gamma$ for $\gamma = 1, \cdots, \bar{\gamma}$. Define the mappings φ_γ $(\gamma = 1, \cdots, \bar{\gamma})$ by

$$\begin{array}{rccl} \varphi_\gamma : & U'_\gamma & \to & \Sigma \\ & (u^1, u^2_\gamma, \cdots, u^{n-1}_\gamma) & \mapsto & (p(u^1), q(u^1)\theta^{-1}_\gamma(u^2_\gamma, \cdots, u^{n-1}_\gamma)). \end{array}$$

Let $U'_\gamma := \varphi_\gamma(U'_\gamma)$ and define the mappings $\psi_\gamma : U_\gamma \to U'_\gamma$ by $\psi_\gamma := \varphi_\gamma^{-1}$ for $\gamma = 1, \cdots, \bar{\gamma}$. Moreover set

$$\begin{aligned} U_+ &= \{s = (p(u^1), q(u^1)\omega) \in \Sigma; a < u^1 \le l/2, \omega \in S^{n-2}\}, \\ U_- &= \{s = (p(u^1), q(u^1)\omega) \in \Sigma; -l/2 \le u^1 < -a, \omega \in S^{n-2}\}. \end{aligned}$$

Define the mappings $\psi_\pm$ by

$$\begin{array}{rccl} \psi_\pm : & U_\pm & \to & \mathbf{R}^{n-1}_{x^2,\cdots,x^n} \\ & (x^1, x^2, \cdots, x^n) & \mapsto & (u^1_\pm, \cdots, u^{n-1}_\pm) := (x^2, \cdots, x^n). \end{array}$$

Then $\{U_\beta, \psi_\beta\}_{\beta=\pm,1,\cdots,\bar{\gamma}}$ gives an atlas of Σ.

Next we show the explicit way to give the initial hypersurface of rotation Γ_0. For an integer $m \ge 4$ and $0 < \alpha < 1/4$, let $E^{m+4\alpha}(M)$ be

$$E^{m+4\alpha}(M) = \{\lambda_0 \in C^{m+4\alpha}(M); \lambda_0(-u^1) = \lambda_0(u^1) = \lambda_0(l - u^1) > 0 \quad \text{for } u^1 \in \mathbf{T}\}.$$

Here we regard $\lambda_0(u^1)$ as $\lambda_0(p(u^1), q(u^1))$. Hereafter, if any confusion may not be caused, we will always identify a scalar field f on a manifold $\mathcal{M}$ as a function of the (local) coordinates of $\mathcal{M}$. For $\lambda_0 \in E^{m+4\alpha}(M)$ we associate a curve

$$\gamma[\lambda_0] := \{\mu + \lambda_0(\mu)N(\mu) \in \mathbf{R}^2; \mu \in M\},$$

where $N(\mu)$ is the outward normal of M at $\mu \in M$. By symmetry of M and $\lambda_0 \in E^{m+4\alpha}(M)$, $\gamma[\lambda_0]$ is simple and closed. Let $\mathcal{C}^{m+4\alpha}(M)$ be the set of all $\gamma[\lambda_0]$ with $\lambda_0 \in E^{m+4\alpha}(M)$ such that the outward curvature κ_0 of $\gamma[\lambda_0]$ is negative everywhere. Of course, for each $\gamma_0 \in \mathcal{C}^{m+4\alpha}(M)$ there is a unique $\lambda_0 \in E^{m+4\alpha}(M)$ such that $\gamma_0 = \gamma[\lambda_0]$. For $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ let $\Gamma[\lambda_0]$ be the hypersurface of rotation of $\gamma_0 \cap \{(x^1, x^2); x^2 \ge 0\}$ which can be written as

$$\Gamma[\lambda_0] = \{s + \rho_0(s)\nu(s) \in \mathbf{R}^n; s \in \Sigma\}.$$

Here $\rho_0(s) := \lambda_0(\zeta(s))$ and $\zeta(s)$ is a mapping from Σ to M defined by

$$\begin{array}{rccl} \zeta : & \Sigma & \to & M \\ & (p(u^1), q(u^1)\omega) & \mapsto & (p(u^1), q(u^1)) \end{array} \tag{7}$$

and $\nu(s)$ is the outward normal of $\Gamma[\lambda_0]$ given by

$$\nu(s) = (-q'(u^1), p'(u^1)\omega).$$

We call γ_0 the generator of $\Gamma[\lambda_0]$. Note that the followings hold

$$\min_{s\in\Sigma} \rho_0(s) = \min_{\mu\in M} \lambda_0(\mu) > 0, \tag{8}$$

$$\|\rho_0\|_{C^{m+4\alpha}(\Sigma)} \le K_0 \|\lambda_0\|_{C^{m+4\alpha}(M)} \tag{9}$$

with some constant $K_0 > 0$. Let $\mathcal{S}^{m+4\alpha}(\Sigma)$ be the set of all $\Gamma[\lambda_0]$ with $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$. Then every element belonging to $\mathcal{S}^{m+4\alpha}(\Sigma)$ is strictly convex closed compact hypersurface of rotation about the x^1-axis with a generator symmetric with respect to the x^2-axis. It is clear that for each $\Gamma_0 \in \mathcal{S}^{m+4\alpha}(\Sigma)$ there is a unique generator $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ such that $\Gamma_0 = \Gamma[\lambda_0]$. Then for $\Gamma_0 \in \mathcal{S}^{m+4\alpha}(\Sigma)$ as the initial hypersurface we have the following local existence result for (1), which assures that the solution is also rotationally axisymmetric about the x^1-axis with a generator symmetric with respect to the x^2-axis.

Theorem 6 . *Let $m \geq 4$ be an integer and $0 < \alpha < 1/4$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$ with $\rho_0(s) = \lambda_0(\zeta(s))$ and $\gamma_0 = \gamma[\lambda_0] \in C^{m+4\alpha}(M)$. Then:*
(i) There is a positive constant $T_1 := T(\|\lambda_0\|_{C^{m+4\alpha}(M)}, \min_{\mu\in M}\lambda_0(\mu))$ such that (1) has a unique solution

$$\Gamma(t) = \{s + \rho(t,s)\nu(s); s \in \Sigma\} \quad \textit{for } t \in [0, T_1]$$

with $\rho(t,s)$ established in Theorem 4.
(ii) There is a unique $\lambda \in C^{[m/4]+\alpha, m+4\alpha}([0,T_1] \times M)$ such that $\rho(t,s) = \lambda(t, \zeta(s))$ with $\lambda(0,\zeta(s)) = \lambda_0(\zeta(s))$.
(iii) $\rho(t,s)$ is even in the u^1-coordinate.

Remark 7 . *Let $\rho(t,s)$ be the unique solution of (5) obtained in Theorem 6 with the initial data $\rho_0(s) = \lambda_0(\zeta(s))$. Since $\rho(t,s)$ depends only on t and the u^1-coordinate in $\Sigma\backslash\{s_\pm\}$, we simply write it as $\rho(t,u^1)$ instead of $\rho(t, \psi_\gamma^{-1}(u^1, u_\gamma^2, \cdots, u_\gamma^{n-1}))$ for $\gamma \in \{1, \cdots, \bar{\gamma}\}$. Set $\Sigma_c = \{(u^1, b\omega) \in \Sigma; |u^1| < a, \omega \in S^{n-1}\}$. Then, when s is in Σ_c, (5) is simply written as*

$$\begin{cases} \rho_t = -L^{-1}\Big[\dfrac{\partial^2 H}{\partial u^1 \partial u^1} + \Big(\dfrac{n-2}{b+\rho} - L^{-2}\dfrac{\partial^2 \rho}{\partial u^1\partial u^1}\Big)\dfrac{\partial \rho}{\partial u^1}\dfrac{\partial H}{\partial u^1}\Big], & t \in [0,T_1],\ |u^1| < a, \\ \rho(0,u^1) = \lambda_0(u^1), \quad |u^1| < a, \end{cases} \tag{10}$$

where $L = (1 + (\frac{\partial \rho}{\partial u^1})^2)^{1/2}$ and H is the mean curvature which is given by

$$H = (1 + (\frac{\partial \rho}{\partial u^1})^2)^{-3/2}\Big\{\frac{\partial^2 \rho}{\partial u^1 \partial u^1} - \frac{n-2}{b+\rho}(1 + (\frac{\partial \rho}{\partial u^1})^2)\Big\}.$$

This fact will be used in the proof of Theorem 1.

4 Proof of Theorem 1

4.1 Deformation (step 1)

For a given strictly convex compact smooth rotationally axisymmetric hypersurface, using the results of [10], we shall give an explicit way of its deformation which weaken the local convexity of the original surface. First we summarize the results [10] for a deformation operator for concave functions and for strictly convex closed curve. Next we give a deformation of strictly convex compact hypersurface of rotation by deforming its generator.

Deformation of concave functions. Let f be smooth concave functions defined on the interval $(-1,1)$. For parameter $\delta \in (0, 1/8)$, let χ_δ be a smooth cut-off function such as

$$\chi_\delta(y) = \begin{cases} 1 & \text{for } |y| < \delta, \\ 0 & \text{for } 2\delta < |y| < 1 \end{cases}$$

and $0 \leq \chi_\delta(y) \leq 1$ for $y \in (-1,1)$. Then, for parameters $\varepsilon \in (0,1]$ and $\delta \in (0,1/8)$ we define a deformation of f by

$$M^{\varepsilon,\delta}(y) = f(\frac{1}{2}) + \int_0^y v^{\varepsilon,\delta}(\xi)d\xi \quad \text{for } y \in (-1,1),$$

where

$$v^{\varepsilon,\delta}(y) = \int_0^y w^{\varepsilon,\delta}(\xi)d\xi\chi_{1/4}(y) + f'(y)(1-\chi_{1/4}(y)),$$

$$w^{\varepsilon,\delta}(y) = (-\varepsilon - \frac{y^4}{4!})\chi_\delta(y) + f''(y)(1-\chi_\delta(y))$$

for $y \in (-1,1)$.

Next we deform smooth concave functions f defined on the interval $(-\alpha_0, \alpha_0)$ with $\alpha_0 > 0$. For $c > 0$ set

$$(L_c f)(y) = f(cy) \quad \text{for } |y| < \alpha_0/c.$$

This L_c gives the dilation operator for f and $L_c f$ becomes smooth concave functions from $(-\alpha_0/c, \alpha_0/c)$ to $\mathbf{R}$. Then we define the deformation of f by

$$(M^{\varepsilon,\delta}_{\alpha_0} f)(y) = ((L_{1/\alpha_0} M^{\varepsilon,\delta} L_{\alpha_0})f)(y) \quad \text{for } y \in (-\alpha_0, \alpha_0).$$

Deformation of strictly convex closed curve. We shall deform $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ with an integer $m \geq 4$ and $\alpha \in (0, 1/4)$. Let $r_a\lambda_0$ be the restriction of λ_0 on the interval $(-a, a)$. Set

$$\xi^{\varepsilon,\delta}(u^1) = \begin{cases} ((M^{\varepsilon,\delta}_a r_a)\lambda_0)(u^1) & \text{for } |u^1| < a, \\ \lambda_0(u^1) & \text{for } a \leq |u^1| \leq l/2, \end{cases}$$

and

$$\lambda_0^{\varepsilon,\delta}(u^1) = \begin{cases} \xi^{\varepsilon,\delta}(u^1), & \text{for } |u^1| \leq l/2, \\ \xi^{\varepsilon,\delta}(l - u^1), & \text{for } l/2 \leq u^1 \leq 3l/2. \end{cases} \tag{11}$$

We define a deformation of γ_0 by $\gamma_0^{\varepsilon,\delta} := \gamma[\lambda_0^{\varepsilon,\delta}]$.

We state several properties for the deformed curve $\gamma_0^{\varepsilon,\delta}$.

Lemma 8 . *Let $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ with an integer $m \geq 4$ and $\alpha \in (0,1/4)$. let $\varepsilon \in (0,1]$ and $\delta \in (0, 1/8)$ and let $\lambda_0^{\varepsilon,\delta}$ be given in (11).*
(i) $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}]$ becomes a closed curve and is symmetric with respect to the both x^1 and x^2-axes.
(ii) $\lambda_0^{\varepsilon,\delta} \in E^{m+4\alpha}(M)$ and there is a constant $\Lambda^\delta > 0$ (which is also depends on λ_0, m, α and a but independent of ε) such that

$$\|\lambda_0^{\varepsilon,\delta}\|_{\mathcal{C}^{m+4\alpha}(M)} \leq \Lambda^\delta \quad \textit{for all } \varepsilon \in (0,1] \tag{12}$$

and Λ^δ is unbounded when $\delta \to 0$.
(iii) Outside the set $\{(x^1, x^2); |x^1| < \delta a, x^2 > 0\}$, the set $\gamma[\lambda_0^{\varepsilon,\delta}]$ agrees with $\gamma[\lambda_0]$.
(iv) $((\frac{d}{du^1})^2\lambda_0^{\varepsilon,\delta})(0) = -\varepsilon/a^2$, $((\frac{d}{du^1})^4\lambda_0^{\varepsilon,\delta})(0) = 0$, $((\frac{d}{du^1})^6\lambda_0^{\varepsilon,\delta})(0) = -1/a^6$.
(v) There exists a $\delta_0 > 0$ depending on λ_0 such that

$$\sup_{|u^1| \leq a} ((\frac{d}{du^1})^2 \lambda_0^{\varepsilon,\delta})(u^1) < 0,$$

$$\inf_{|u^1| \leq a} \lambda_0^{\varepsilon,\delta}(u^1) \geq \lambda_0(\pm a) > 0$$

for all $\varepsilon \in (0,1]$, $\delta \in (0, \delta_0)$. In particular, $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}] \in \mathcal{C}^{m+4\alpha}(M)$ with

$$\inf_{\mu \in M} \lambda_0^{\varepsilon,\delta}(\mu) \geq \inf_{\mu \in M} \lambda_0(\mu) > 0 \quad \textit{for all } \varepsilon \in (0,1],\ \delta \in (0, \delta_0). \tag{13}$$

This lemma is almost directly follows from Theorem 3 in [10].

Deformation of strictly convex compact closed hypersurface of rotation. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$. For $\delta \in (0,\delta_0)$ with δ_0 established in Lemma 8 (v), we define a deformation of Γ_0 by $\Gamma_0^{\varepsilon,\delta} := \Gamma[\lambda_0^{\varepsilon,\delta}]$. By virtue of Lemma 8, $\Gamma_0^{\varepsilon,\delta}$ is a strictly convex closed compact hypersurface of rotation about the x^1-axis with the generator $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}]$. We summarize the properties of $\Gamma_0^{\varepsilon,\delta}$ in the next lemma which follows from Lemma 8, (8), and (9).

Lemma 9 . *If $\delta \in (0,\delta_0)$, then $\Gamma_0^{\varepsilon,\delta}$ belongs to $\mathcal{S}^{m+4\alpha}(\Sigma)$. Moreover, the following uniform estimates hold:*

$$\begin{aligned} &\|\rho_0^{\varepsilon,\delta}\|_{C^{m+4\alpha}(\Sigma)} \le K_0\Lambda^\delta, \\ &\min_{s\in\Sigma} \rho_0^{\varepsilon,\delta}(s) \ge \min_{\mu\in M} \lambda_0(\mu) > 0 \end{aligned} \tag{14}$$

for all $\varepsilon \in (0,1]$ and $\delta \in (0,\delta_0)$, where $\rho_0^{\varepsilon,\delta}(s) := \lambda_0^{\varepsilon,\delta}(\zeta(s))$ and $\zeta(s)$ is in (7), and K_0 is in (9).

4.2 Uniform local existence in ε (step 2)

Combining Theorems 4 and 6 and Lemma 9, we have the following uniform local existence result in $\varepsilon \in (0,1]$, which assures that the existence time of the solution does not shrink to 0 and the magnitude of the solution remains bounded when $\varepsilon \to 0$.

Proposition 10 . *Let $m \ge 4$ be an integer and $\alpha \in (0,1/4)$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$ with $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$. Let δ_0 be positive constant determined by λ_0 in Lemma 8 (v) and let $\delta \in (0,\delta_0)$. Set $T^\delta := T(K_0\Lambda^\delta, \min_{\mu\in M}\lambda_0(\mu)) > 0$, where $K_0\Lambda^\delta$ is in (14). Then the followings hold.*
(i) For any $\varepsilon \in (0,1]$ there exists a unique solution $\rho^{\varepsilon,\delta}(t,s)$ of (5) with initial data $\rho_0^{\varepsilon,\delta}(s)$ (defined in Lemma 9) which satisfies

$$\rho^{\varepsilon,\delta} \in C^{[m/4]+\alpha,m+4\alpha}([0,T^\delta]\times\Sigma),$$

$$\|\rho^{\varepsilon,\delta}\|_{C^{[m/4]+\alpha,m+4\alpha}([0,T^\delta]\times\Sigma)} \le G(K_0\Lambda^\delta),$$

$$\min_{(t,s)\in[0,T^\delta]\times\Sigma} \rho^{\varepsilon,\delta}(t,s) \ge \frac{1}{2}\min_{\mu\in M}\lambda_0(\mu) > 0$$

for all $\varepsilon \in (0,1]$ and $\delta \in (0,\delta_0)$.
(ii) There is a unique $\lambda^{\varepsilon,\delta} \in C^{[m/4]+\alpha,m+4\alpha}([0,T^\delta]\times M)$ such that $\rho^{\varepsilon,\delta}(t,s) = \lambda^{\varepsilon,\delta}(t,\zeta(s))$ with $\lambda^{\varepsilon,\delta}(0,\mu) = \lambda_0^{\varepsilon,\delta}(\mu)$ for $\mu \in M \cap \{x^2 \ge 0\}$.
(iii) $\rho^{\varepsilon,\delta}(t,s)$ is even in the u^1-coordinate.

4.3 Loss of convexity (step 3)

In this step we prove:

Theorem 11 . *(Loss of convexity). Let $m \ge 10$ be an integer and let $\alpha \in (0,1/4)$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$. Let δ_0 be a positive constant established in Lemma 8 and let δ be in $(0,\delta_0)$. Let $\lambda_0^{\varepsilon,\delta}$ be the deformed function of λ_0 (defined in (11)) for $\varepsilon \in (0,1]$. Let T^δ be the time (established in Proposition 10) such that there is a unique solution $\rho^{\varepsilon,\delta}(t,s) =$*

$\lambda^{\varepsilon,\delta}(t,\zeta(s))$ of (5) in $[0,T^{\delta}]\times\Sigma$ with initial data $\rho_0^{\varepsilon,\delta}(s)=\lambda_0^{\varepsilon,\delta}(\zeta(s))$, where $\zeta(s)$ is in (7). Then there is an $\varepsilon_0^{\delta}>0$ such that for any $\varepsilon\in(0,\varepsilon_0^{\delta})$ there are positive times $t_0^{\varepsilon,\delta}$ and $t_1^{\varepsilon,\delta}$ with $t_0^{\varepsilon,\delta}<\min(T^{\delta},t_1^{\varepsilon,\delta})$ having the property that the solution $\Gamma^{\varepsilon,\delta}(t):=\Gamma[\lambda^{\varepsilon,\delta}(t,\cdot)]$ of (1) starting from $\Gamma[\lambda_0^{\varepsilon,\delta}]\in\mathcal{S}^{m+4\alpha}(\Sigma)$ loses its convexity for at least $t\in(t_0^{\varepsilon,\delta},\min(T^{\delta},t_1^{\varepsilon,\delta}))$. Moreover, $t_0^{\varepsilon,\delta}\to0$ as $\varepsilon\to0$ for any $\delta\in(0,\delta_0)$.

Proof. Recall that by Proposition 10 $\rho^{\varepsilon,\delta}(t,s)$ does not depend on the $(u_\gamma^2,\cdots,u_\gamma^{n-1})$-coordinate for $s\in U_\gamma$ with $\gamma\in\{1,\cdots,\bar\gamma\}$. So we simply write $\rho^{\varepsilon,\delta}(t,u^1)$ instead of $\rho^{\varepsilon,\delta}(t,\psi_\gamma^{-1}(u^1,u_\gamma^2,\cdots,u_\gamma^{n-1}))$. Our purpose is to show that $((\partial/\partial u^1)^2\rho^{\varepsilon,\delta})(t,0)$ becomes positive after a finite time in its smooth evolution by (5). We devide the proof into two steps.

First step. We shall prove that for any $\delta\in(0,\delta_0)$ there is a positive constant $C_{\varepsilon,\delta}$ such that

$$(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)=a^{-6}+C_{\varepsilon,\delta},\quad C_{\varepsilon,\delta}=O(\varepsilon^2)\quad\text{as }\varepsilon\to0 \tag{15}$$

for each $\delta\in(0,\delta_0)$. To show (15) we recall that $\rho^{\varepsilon,\delta}(t,u^1)$ satisfies (10) for $t\in[0,T^{\delta}]$ and $|u^1|<a$ with $\rho^{\varepsilon,\delta}(0,u^1)=\lambda_0^{\varepsilon,\delta}(u^1)$ for $|u^1|<a$. Differentiating (10) in u^1 twice and using the property $\rho^{\varepsilon,\delta}(t,u^1)=\rho^{\varepsilon,\delta}(t,-u^1)$ for $|u^1|<a$ of Proposition 10 (iii), we get

$$\begin{aligned}
&(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)\\
&=\Big[-D_1^6\rho^{\varepsilon,\delta}-\{45(D_1^2\rho^{\varepsilon,\delta})^4(-\frac{n-2}{b+\rho^{\varepsilon,\delta}}+D_1^2\rho^{\varepsilon,\delta})\\
&-18(D_1^2\rho^{\varepsilon,\delta})^2(\frac{(n-2)D_1^2\rho^{\varepsilon,\delta}}{(b+\rho^{\varepsilon,\delta})^2}-\frac{2(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{b+\rho^{\varepsilon,\delta}})\\
&-\frac{6(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{(b+\rho^{\varepsilon,\delta})^3}+\frac{12(n-2)(D_1^2\rho^{\varepsilon,\delta})^3}{(b+\rho^{\varepsilon,\delta})^2}\}\\
&+\{(D_1^2\rho^{\varepsilon,\delta})^2-2D_1^2\rho^{\varepsilon,\delta}(\frac{n-2}{b+\rho^{\varepsilon,\delta}}-D_1^2\rho^{\varepsilon,\delta})\}\cdot\\
&\cdot\{3(D_1^2\rho^{\varepsilon,\delta})^2(\frac{n-2}{b+\rho^{\varepsilon,\delta}}-D_1^2\rho^{\varepsilon,\delta})+\frac{(n-2)D_1^2\rho^{\varepsilon,\delta}}{(b+\rho^{\varepsilon,\delta})^2}-\frac{2(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{b+\rho^{\varepsilon,\delta}}\}\Big](0,0),
\end{aligned}$$

where $D_1:=\partial/\partial u^1$. We use values of derivatives of $\lambda_0^{\varepsilon,\delta}$ in Lemma 8 (iv) and use the inequality in Lemma 8 (v) to get (15).

(15) guarantees that for any $\delta\in(0,\delta_0)$ there is an $\varepsilon_1^{\delta}>0$ such that

$$(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)\geq1/2a^6\quad\text{for }\varepsilon\in(0,\varepsilon_1^{\delta}]. \tag{16}$$

We take $\varepsilon\in(0,\varepsilon_1^{\delta}]$ in the following.

Second step. We shall complete the proof of Theorem 11. We note that it follows from Proposition 10 that there is a constant $B^{\delta}>0$ such that

$$\sup_{t\in[0,T^{\delta}]}|((\frac{\partial}{\partial t})^2(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0)|\leq B^{\delta}. \tag{17}$$

By Taylor's expansion, Lemma 8, (16), and (17), we obtain

$$((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0)\;=\;((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)+(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)t$$

$$
\begin{aligned}
& + \int_0^t \int_0^\sigma ((\frac{\partial}{\partial t})^2 (\frac{\partial}{\partial u^1})^2 \rho^{\varepsilon,\delta})(\tau, 0) d\tau d\sigma \\
\geq\ & ((\frac{d}{du^1})^2 \rho_0^{\varepsilon,\delta})(0) + (\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2 \rho^{\varepsilon,\delta})(0,0)t - B^\delta t^2 \\
\geq\ & -a^{-2}\varepsilon + \frac{1}{2} a^{-6} t - B^\delta t^2 \qquad (18)
\end{aligned}
$$

for $0 \leq t \leq T^\delta$. We can take ε small so that the quadratic polynomial $B^\delta t^2 - \frac{1}{2}a^{-6}t + a^{-2}\varepsilon$ has the smallest positive zero less than T^δ. In fact, we can take $\varepsilon_2^\delta > 0$ small so that

$$
0 < \frac{\frac{1}{2}a^{-6} - (\frac{1}{4}a^{-12} - 4B^\delta a^{-2}\varepsilon_2^\delta)^{1/2}}{2B^\delta} < T^\delta.
$$

Set $\varepsilon_0^\delta = \min(\varepsilon_1^\delta, \varepsilon_2^\delta)$. Then for each $\varepsilon \in (0, \varepsilon_0^\delta)$ the polynomial $B^\delta t^2 - \frac{1}{2}a^{-6}t + a^{-2}\varepsilon$ has two positive zeros $t_0^{\varepsilon,\delta} < t_1^{\varepsilon,\delta}$ with $t_0^{\varepsilon,\delta} < T^\delta$. By (18),

$$
((\frac{\partial}{\partial u^1})^2 \rho^{\varepsilon,\delta})(t, 0) \geq -(t - t_0^{\varepsilon,\delta})(t - t_1^{\varepsilon,\delta}) \quad \text{for } t \in [0, T^\delta].
$$

This implies

$$
((\frac{\partial}{\partial u^1})^2 \rho^{\varepsilon,\delta})(t, 0) > 0 \quad \text{for } t \in (t_0^{\varepsilon,\delta}, \min(T^\delta, t_1^{\varepsilon,\delta})).
$$

This shows that $\Gamma^{\varepsilon,\delta}(t) = \Gamma[\lambda^{\varepsilon,\delta}(t, \cdot)]$ loses its convexity for at least $t \in (t_0^{\varepsilon,\delta}, \min(T^\delta, t_1^{\varepsilon,\delta}))$. The assertion $t_0^{\varepsilon,\delta} \to 0$ as $\varepsilon \to 0$ follows from its definition. $\square$

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