Note on Transitive Representations of Generalized Inverse \(*\)-Semigroups\(^1\)

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Abstract

In [1], we obtained that an effective representation of a locally [generalized] inverse \(*\)-semigroup \(S\) is the sum of a uniquely determined family of transitive representations of \(S\). In this paper, we will determine a transitive representation of a generalized inverse \(*\)-semigroup by using right \(\omega\)-cosets. This is a generalization of Schein’s result [5] for inverse semigroups.

1 Introduction

A semigroup \(S\) with a unary operation \(\ast : S \rightarrow S\) is called a regular \(*\)-semigroup if it satisfies (i) \((x^\ast)^\ast = x\); (ii) \((xy)^\ast = y^\ast x^\ast\); (iii) \(xx^\ast x = x\). Let \(S\) be a regular \(*\)-semigroup. An idempotent \(e\) in \(S\) is called a projection if \(e^\ast = e\). Denote the sets of idempotents and projections of \(S\) by \(E(S)\) and \(P(S)\), respectively.

Let \(S\) be a regular \(*\)-semigroup. If \(eSe\) is an inverse semigroup, for every \(e \in E(S)\), \(S\) is called a locally inverse \(*\)-semigroup. If \(E(S)\) is a normal band, that is, it satisfies the identity \(xyzx = xzyx\), \(S\) is called a generalized inverse \(*\)-semigroup. A regular \(*\)-semigroup \(S\) is a generalized inverse \(*\)-semigroup if and only if it is a locally inverse \(*\)-semigroup and \(E(S)\) forms a band.

Result 1.1 [3] Let \(S\) be a regular \(*\)-semigroup. Define a relation \(\leq\) on \(S\) by

\[ a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S). \]

Then \(\leq\) is a partial order on \(S\) satisfying that \(a \leq b\) implies \(a^\ast \leq b^\ast\). If \(S\) is a generalized inverse \(*\)-semigroup, then \(\leq\) is compatible.

For a subset \(A\) of a regular \(*\)-semigroup \(S\), the set

\[ A\omega = \{ x \in S : \text{there exists } a \in A \text{ such that } a \leq x \} \]

is called the closure of \(A\). The following statements are easily verified.

\(^1\)This is the abstract and details will be published elsewhere.
(1) $A \subseteq A\omega$;  (2) $A \subseteq B \Rightarrow A\omega \subseteq B\omega$;  (3) $(A\omega)\omega = A\omega$.

We say that $A$ is closed if $A\omega = A$.

**Lemma 1.2** If $H$ is a regular $\ast$-subsemigroup of a generalized inverse $\ast$-semigroup $S$, then $H\omega$ is a closed generalized inverse $\ast$-subsemigroup of $S$.

Let $S$ be a regular $\ast$-semigroup and $H$ a regular $\ast$-subsemigroup of $S$. If an element $a$ in $S$ satisfies $aa^\ast \in H$, then $(Ha)\omega$ is called a right $\omega$-coset of $H$.

**Lemma 1.3** Let $S$ be a generalized inverse $\ast$-semigroup, and let $(Ha)\omega$ and $(Hb)\omega$ be right $\omega$-cosets of a regular $\ast$-subsemigroup $H$ of $S$. Then

$$(Ha)\omega \subseteq (Hb)\omega \iff a \in (Hb)\omega.$$ 

A non-empty set $X$ with its reflexive and symmetric relation $\sigma$ is called an $\iota$-set, and denoted by $(X; \sigma)$. If $\sigma$ is transitive, that is, it is an equivalence relation, then $(X; \sigma)$ is called a transitive $\iota$-set.

Let $(X; \sigma)$ be an $\iota$-set. A subset $A$ of $X$ is called an $\iota$-single subset if, for any $x \in X$, there exists at most one element $y \in A$ such that $(x, y) \in \sigma$. If $(X; \sigma)$ is a transitive $\iota$-set, $A$ is an $\iota$-single subset if and only if it satisfies that

$$(a, b) \in \sigma (a, b \in A) \implies a = b.$$ 

A mapping $\alpha$ in the symmetric inverse semigroup $\mathcal{I}_X$ is called a partial one-to-one $\iota$-mapping of $(X; \sigma)$ if $d(\alpha)$ and $r(\alpha)$ are both $\iota$-single subsets of $(X; \sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of $\alpha$, respectively. Denote the set of all partial one-to-one $\iota$-mappings of $(X; \sigma)$ by $\mathcal{LI}(X; \sigma)$. If $\sigma$ is transitive, we denote it by $\mathcal{GI}(X; \sigma)$ instead of $\mathcal{LI}(X; \sigma)$. For any $\alpha, \beta \in \mathcal{LI}(X; \sigma)$, denote $\theta_{\alpha, \beta}$ by

$$\theta_{\alpha, \beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\} = (r(\alpha) \times d(\beta)) \cap \sigma.$$ 

Since a subset of an $\iota$-single subset is also an $\iota$-single subset, $\theta_{\alpha, \beta} \in \mathcal{LI}(X; \sigma)$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{LI}(X; \sigma)\}$. Define a multiplication $\circ$ and a unary operation $\ast$ on $\mathcal{LI}(X; \sigma)$ as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta \quad \text{and} \quad \alpha^\ast = \alpha^{-1},$$ 

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $\mathcal{LI}(X; \sigma)(\circ, \ast)$ by $\mathcal{LI}(X; \sigma)(\mathcal{M})$ or simply by $\mathcal{LI}(X; \sigma)$. In this paper, we use $\mathcal{LI}(X; \sigma)$ rather than $\mathcal{LI}(X; \sigma)(\mathcal{M})$. 

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**Result 1.4** [4] For an $\iota$-set $(X; \sigma)$, we have the following:

1. The $*$-groupoid $\mathcal{L}(X; \sigma)$, defined above, is a locally inverse $*$-semigroup. Moreover, any locally inverse $*$-semigroup can be embedded (up to $*$-isomorphism) in $\mathcal{L}(X; \sigma)$ on some $\iota$-set $(X; \sigma)$.

2. $E(\mathcal{L}(X; \sigma)) = \mathcal{M}$ and $P(\mathcal{L}(X; \sigma)) = \{1_A : A \text{ is an } \iota\text{-single subset of } (X; \sigma)\}$.

3. If $(X; \sigma)$ is a transitive $\iota$-set, then $\mathcal{L}(X; \sigma)$ is a generalized inverse $*$-semigroup. Moreover, any generalized inverse $*$-semigroup can be embedded (up to $*$-isomorphism) in $\mathcal{G}(X; \sigma)$ on some transitive $\iota$-set $(X; \sigma)$.

4. If $\sigma$ is the identity relation on $X$, then $\mathcal{L}(X; \sigma)$ is the symmetric inverse semigroup $\mathcal{I}_X$ on $X$.

We call $\mathcal{L}(X; \sigma)$ [$\mathcal{G}(X; \sigma)$] the $\iota$-symmetric locally [generalized] inverse $*$-semigroup on the $\iota$-set [the transitive $\iota$-set] $(X; \sigma)$ with the structure sandwich set $\mathcal{M}$.

**Result 1.5** [1] Let $H$ be a locally [generalized] inverse $*$-subsemigroup of $\mathcal{L}(X; \sigma)$ [$\mathcal{G}(X; \sigma)$] on a [transitive] $\iota$-set $(X; \sigma)$, and define a relation $\tau_H$ on $X$ by

$$(x, y) \in \tau_H \iff \text{there exists } \alpha \in H \text{ such that } x \in d(\alpha) \text{ and } x\alpha = y.$$  

Then $\tau_H$ is a symmetric and transitive relation on $X$.

The subset $\{x \in X : (x, x) \in \tau_H\} = d(\tau_H)$, say, of $X$ is called the domain of $\tau_H$. If $d(\tau_H) = X$, that is, $\tau_H$ is an equivalence relation on $X$, then $H$ is said to be effective. If $\tau_H$ is the universal relation on $X$, then $H$ is said to be transitive.

A representation $\phi : S \to \mathcal{L}(X; \sigma)$ of a locally inverse $*$-semigroup $S$ is called a effective [transitive] representation if $S\phi$ is an effective [transitive] locally inverse $*$-subsemigroup of $\mathcal{L}(X; \sigma)$. Similarly, the effectivity and the transitivity for a representation $\phi : S \to \mathcal{G}(X; \sigma)$ of a generalized inverse $*$-semigroup $S$ are defined.

**Result 1.6** [1] An effective representation of a locally [generalized] inverse $*$-semigroup $S$ is the sum of a uniquely determined family of transitive representations of $S$.

The purpose of this paper is to characterize a transitive representation of a generalized inverse $*$-semigroup. The notation and the terminology are those of [1] and [2], unless otherwise stated.
2 Transitive representations

Let $S$ be a generalized inverse $*$-semigroup, and let $(X; \sigma)$ be a transitive $\iota$-set and $\psi: S \to \mathcal{GI}(X; \sigma) (s \mapsto \psi^s)$ a transitive representation of $S$. Fix an element $z$ in $X$ and set

$$H = \{ s \in S : z\psi^s = z \}.$$

**Lemma 2.1** The set $H$, defined above, is a closed generalized inverse $*$-subsemigroup of $S$.

Define a relation $\delta$ on $S$ by

$$\delta = \{ (a, b) \in S \times S : z\psi^a = z\psi^b \}.$$

We also assume that $(a, b) \in \delta$ if $z \not\in d(\psi^a) \cup d(\psi^b)$.

**Lemma 2.2** The relation $\delta$, defined above, is a right congruence on $S$ satisfying the following conditions:

1. $\delta \cap (H \times H) = H \times H$,
2. For $a \in S$ and $h \in H$, $(a, h) \in \delta$ implies $a \in H$.

Let $\mathcal{X}$ be the set of all right $\omega$-cosets of $H$. Define a relation $\sim$ on $\mathcal{X}$ by

$$(Ha)\omega \sim (Hb)\omega \iff (a, b) \in \delta.$$

**Lemma 2.3** The relation $\sim$, defined above, is an equivalence relation on $\mathcal{X}$.

Let $\mathcal{X}/\sim = \mathcal{Y}$, say, and denote the $\sim$-class containing $(Ha)\omega$ by $(Ha)\tilde{\omega}$. For any $a \in S$, define a partial mapping $\phi^a_H$ on $\mathcal{Y}$ by

$$d(\phi^a_H) = \{ (Hxaa^*)\tilde{\omega} : xaa^*x^* \in H \} \quad \text{and} \quad \phi^a_H : (Hxaa^*)\tilde{\omega} \mapsto (Hxa)\tilde{\omega},$$

**Lemma 2.4** For any $a \in S$ and $(Ha)\tilde{\omega} \in \mathcal{Y}$, we have

$$(Hx)\tilde{\omega} \in d(\phi^a_H) \iff (x, xaa^*) \in \delta$$

**Lemma 2.5** For any $a \in S$, $\phi^a_H \in \mathcal{T}_Y$ and $(\phi^a_H)^{-1} = \phi^{a^{*}}_H$.

Define a relation $\Omega$ on $\mathcal{Y}$ by

$$\Omega = \{ ((Hx)\tilde{\omega}, (Hy)\tilde{\omega}) : (Hx)\tilde{\omega} \phi^e_H = (Hy)\tilde{\omega} \text{ for some } e \in E(S) \}. $$
Lemma 2.6 The relation $\Omega$, defined above, is an equivalence relation on $\mathcal{Y}$, that is, $(\mathcal{Y};\Omega)$ is a transitive $\iota$-set.

Now we can consider the $\iota$-symmetric generalized inverse *-semigroup $\mathcal{GI}_{(\mathcal{Y};\Omega)}$ on the transitive $\iota$-set $(\mathcal{Y};\Omega)$.

Lemma 2.7 For any $a \in S$, $d(\phi_H^a)$ and $r(\phi_H^a)$ are $\iota$-single subsets of $(\mathcal{Y};\Omega)$.

Lemma 2.8 For any $a, b \in S$, $\theta_{\phi_H^a}, b = \phi_H^{a^*ab}$.

Lemma 2.9 The mapping $\phi_H: S \rightarrow \mathcal{GI}_{(\mathcal{Y};\Omega)}$ is a transitive representation of $S$.

Let $\varphi: S \rightarrow \mathcal{GI}_{(X;\sigma)}$ and $\xi: S \rightarrow \mathcal{GI}_{(Y;\tau)}$ be two representations of a generalized inverse *-semigroup $S$. Then $\varphi$ and $\xi$ are equivalent if there exists a bijection $\theta: X \rightarrow Y$ such that, for $s \in S$ and $x \in X$,

$$x \in d(\varphi^s) \iff x\theta \in d(\xi^s) \text{ and } (x\varphi^s)\theta = (x\theta)\xi^s.$$ 

Lemma 2.10 The transitive representation $\psi: S \rightarrow \mathcal{GI}_{(X;\sigma)}$ is equivalent to $\phi_H$, defined above.

From result 1.5, lemma 3.1 and 3.2, we obtain a following theorem.

Theorem 2.11 Every effective representation of a generalized inverse *-semigroup $S$ is uniquely a sum of transitive representations $\psi_\alpha$, each of which is equivalent to $\phi_{H_\alpha}$ for some closed generalized inverse *-subsemigroup $H_\alpha$ of $S$.

References


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