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Note on Transitive Representations of Generalized Inverse $*$-Semigroups[1]

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Abstract

In [1], we obtained that an effective representation of a locally [generalized] inverse $*$-semigroup S is the sum of a uniquely determined family of transitive representations of S. In this paper, we will determine a transitive representation of a generalized inverse $*$-semigroup by using right ω-cosets. This is a generalization of Schein's result [5] for inverse semigroups.

1 Introduction

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies (i) $(x^*)^* = x$; (ii) $(xy)^* = y^*x^*$; (iii) $xx^*x = x$. Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. Denote the sets of idempotents and projections of S by $E(S)$ and $P(S)$, respectively.

Let S be a regular $*$-semigroup. If eSe is an inverse semigroup, for every $e \in E(S)$, S is called a *locally inverse $*$-semigroup.* If $E(S)$ is a normal band, that is, it satisfies the identity $xyzx = xzyx$, S is called a *generalized inverse $*$-semigroup.* A regular $*$-semigroup S is a generalized inverse $*$-semigroup if and only if it is a locally inverse $*$-semigroup and $E(S)$ forms a band.

Result 1.1 [3] *Let S be a regular $*$-semigroup. Define a relation $\leq$ on S by*

$$a \leq b \iff a = eb = bf \quad \text{for some} \quad e, f \in P(S).$$

Then $\leq$ is a partial order on S satisfying that $a \leq b$ implies $a^* \leq b^*$. If S is a generalized inverse $*$-semigroup, then $\leq$ is compatible.

For a subset A of a regular $*$-semigroup S, the set

$$A\omega = \{x \in S : \text{there exists } a \in A \text{ such that } a \leq x\}$$

is called the *closure* of A. The following statements are easily verified.

[1]This is the abstract and details will be published elsewhere.

$$(1)\ A \subseteq A\omega; \quad (2)\ A \subseteq B \Rightarrow A\omega \subseteq B\omega; \quad (3)\ (A\omega)\omega = A\omega.$$

We say that A is *closed* if $A\omega = A$.

Lemma 1.2 *If H is a regular $*$-subsemigroup of a generalized inverse $*$-semigroup S, then $H\omega$ is a closed generalized inverse $*$-subsemigroup of S.*

Let S be a regular $*$-semigroup and H a regular $*$-subsemigroup of S. If an element a in S satisfies $aa^* \in H$, then $(Ha)\omega$ is called a *right ω-coset* of H.

Lemma 1.3 *Let S be a generalized inverse $*$-semigroup, and let $(Ha)\omega$ and $(Hb)\omega$ be right ω-cosets of a regular $*$-subsemigroup H of S. Then*

$$(Ha)\omega \subseteq (Hb)\omega \iff a \in (Hb)\omega.$$

A non-empty set X with its reflexive and symmetric relation σ is called an *ι-set*, and denoted by $(X;\sigma)$. If σ is transitive, that is, it is an equivalence relation, then $(X;\sigma)$ is called a *transitive ι-set.*

Let $(X;\sigma)$ be an ι-set. A subset A of X is called an *ι-single subset* if, for any $x \in X$, there exists at most one element $y \in A$ such that $(x,y) \in \sigma$. If $(X;\sigma)$ is a transitive ι-set, A is an ι-single subset if and only if it satisfies that

$$(a,b) \in \sigma\ (a,b \in A) \implies a = b.$$

A mapping α in the symmetric inverse semigroup $\mathcal{I}_X$ is called a *partial one-to-one ι-mapping* of $(X;\sigma)$ if $d(\alpha)$ and $r(\alpha)$ are both ι-single subsets of $(X;\sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α, respectively. Denote the set of all partial one-to-one ι-mappings of $(X;\sigma)$ by $\mathcal{LI}_{(X;\sigma)}$. If σ is transitive, we denote it by $\mathcal{GI}_{(X;\sigma)}$ instead of $\mathcal{LI}_{(X;\sigma)}$. For any α, $\beta \in \mathcal{LI}_{(X;\sigma)}$, denote $\theta_{\alpha,\beta}$ by

$$\theta_{\alpha,\beta} = \{(a,b) \in r(\alpha) \times d(\beta) : (a,b) \in \sigma\} = (r(\alpha) \times d(\beta)) \cap \sigma.$$

Since a subset of an ι-single subset is also an ι-single subset, $\theta_{\alpha,\beta} \in \mathcal{LI}_{(X;\sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha,\ \beta \in \mathcal{LI}_{(X;\sigma)}\}$. Define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{LI}_{(X;\sigma)}$ as follows:

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $\mathcal{LI}_{(X;\sigma)}(\circ, *)$ by $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$ or simply by $\mathcal{LI}_{(X;\sigma)}$. In this paper, we use $\mathcal{LI}_{(X;\sigma)}$ rather than $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$.

Result 1.4 [4] *For an ι-set $(X;\sigma)$, we have the following:*

(1) *The $*$-groupoid $\mathcal{LI}_{(X;\sigma)}$, defined above, is a locally inverse $*$-semigroup. Moreover, any locally inverse $*$-semigroup can be embedded (up to $*$-isomorphism) in $\mathcal{LI}_{(X;\sigma)}$ on some ι-set $(X;\sigma)$.*

(2) $E(\mathcal{LI}_{(X;\sigma)}) = \mathcal{M}$ *and* $P(\mathcal{LI}_{(X;\sigma)}) = \{1_A : A$ *is an ι-single subset of* $(X;\sigma)\}$.

(3) *If $(X;\sigma)$ is a transitive ι-set, then $\mathcal{LI}_{(X;\sigma)}$ is a generalized inverse $*$-semigroup. Moreover, any generalized inverse $*$-semigroup can be embedded (up to $*$-isomorphism) in $\mathcal{GI}_{(X;\sigma)}$ on some transitive ι-set $(X;\sigma)$.*

(4) *If σ is the identity relation on X, then $\mathcal{LI}_{(X;\sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X.*

We call $\mathcal{LI}_{(X;\sigma)}$ [$\mathcal{GI}_{(X;\sigma)}$] the *$\iota$-symmetric locally [generalized] inverse $*$-semigroup* on the ι-set [the transitive ι-set] $(X;\sigma)$ with the *structure sandwich set $\mathcal{M}$*.

Result 1.5 [1] *Let H be a locally [generalized] inverse $*$-subsemigroup of $\mathcal{LI}_{(X;\sigma)}$ [$\mathcal{GI}_{(X;\sigma)}$] on a [transitive] ι-set $(X;\sigma)$, and define a relation τ_H on X by*

$$(x,y) \in \tau_H \iff \text{there exists } \alpha \in H \text{ such that } x \in d(\alpha) \text{ and } x\alpha = y.$$

Then τ_H is a symmetric and transitive relation on X.

The subset $\{x \in X : (x,x) \in \tau_H\} = d(\tau_H)$, say, of X is called the *domain* of τ_H. If $d(\tau_H) = X$, that is, τ_H is an equivalence relation on X, then H is said to be *effective.* If τ_H is the universal relation on X, then H is said to be *transitive.*

A representation $\phi : S \to \mathcal{LI}_{(X;\sigma)}$ of a locally inverse $*$-semigroup S is called a *effective* [*transitive*] representation if $S\phi$ is an effective [transitive] locally inverse $*$-subsemigroup of $\mathcal{LI}_{(X;\sigma)}$. Similarly, the effectivity and the transitivity for a representation $\phi : S \to \mathcal{GI}_{(X;\sigma)}$ of a generalized inverse $*$-semigroup S are defined.

Result 1.6 [1] *An effective representation of a locally [generalized] inverse $*$-semigroup S is the sum of a uniquely determined family of transitive representations of S.*

The purpose of this paper is to characterize a transitive representation of a generalized inverse $*$-semigroup. The notation and the terminology are those of [1] and [2], unless otherwise stated.

2 Transitive representations

Let S be a generalized inverse $*$-semigroup, and let $(X;\sigma)$ be a transitive ι-set and $\psi : S \to \mathcal{GI}_{(X;\sigma)}$ $(s \mapsto \psi^s)$ a transitive representation of S. Fix an element z in X and set

$$H = \{s \in S : z\psi^s = z\}.$$

Lemma 2.1 *The set H, defined above, is a closed generalized inverse $*$-subsemigroup of S.*

Define a relation δ on S by

$$\delta = \{(a,b) \in S \times S : z\psi^a = z\psi^b\}.$$

We also assume that $(a,b) \in \delta$ if $z \notin d(\psi^a) \cup d(\psi^b)$.

Lemma 2.2 *The relation δ, defined above, is a right congruence on S satisfying the following conditions:*
(1) $\delta \cap (H \times H) = H \times H$,
(2) *For $a \in S$ and $h \in H$, $(a,h) \in \delta$ implies $a \in H$.*

Let $\mathcal{X}$ be the set of all right ω-cosets of H. Define a relation $\sim$ on $\mathcal{X}$ by

$$(Ha)\omega \sim (Hb)\omega \iff (a,b) \in \delta.$$

Lemma 2.3 *The relation $\sim$, defined above, is an equivalence relation on $\mathcal{X}$.*

Let $\mathcal{X}/\sim\ = \mathcal{Y}$, say, and denote the $\sim$-class containing $(Ha)\omega$ by $(Ha)\tilde{\omega}$. For any $a \in S$, define a partial mapping ϕ_H^a on $\mathcal{Y}$ by

$$d(\phi_H^a) = \{(Hxaa^*)\tilde{\omega} : xaa^*x^* \in H\} \ \ and \ \ \phi_H^a : (Hxaa^*)\tilde{\omega} \mapsto (Hxa)\tilde{\omega},$$

Lemma 2.4 *For any $a \in S$ and $(Ha)\tilde{\omega} \in \mathcal{Y}$, we have*

$$(Hx)\tilde{\omega} \in d(\phi_H^a) \iff (x, xaa^*) \in \delta$$

Lemma 2.5 *For any $a \in S$, $\phi_H^a \in \mathcal{I}_\mathcal{Y}$ and $(\phi_H^a)^{-1} = \phi_H^{a^*}$.*

Define a relation Ω on $\mathcal{Y}$ by

$$\Omega = \{((Hx)\tilde{\omega}, (Hy)\tilde{\omega}) : (Hx)\tilde{\omega}\phi_H^e = (Hy)\tilde{\omega} \text{ for some } e \in E(S)\}.$$

Lemma 2.6 *The relation Ω, defined above, is an equivalence relation on $\mathcal{Y}$, that is, $(\mathcal{Y};\Omega)$ is a transitive ι-set.*

Now we can consider the ι-symmetric generalized inverse $*$-semigroup $\mathcal{GI}_{(\mathcal{Y};\Omega)}$ on the transitive ι-set $(\mathcal{Y};\Omega)$.

Lemma 2.7 *For any $a \in S$, $d(\phi_H^a)$ and $r(\phi_H^a)$ are ι-single subsets of $(\mathcal{Y};\Omega)$.*

Lemma 2.8 *For any $a, b \in S$, $\theta_{\phi_H^a,\phi_H^b} = \phi_H^{a^*abb^*}$.*

Lemma 2.9 *The mapping ϕ_H: $S \to \mathcal{GI}_{(\mathcal{Y};\Omega)}$ $(a \mapsto \phi_H^a)$ is a transitive representation of S.*

Let $\varphi : S \to \mathcal{GI}_{(X;\sigma)}$ and $\xi : S \to \mathcal{GI}_{(Y;\tau)}$ be two representations of a generalized inverse $*$-semigroup S. Then φ and ξ are *equivalent* if there exists a bijection $\theta : X \to Y$ such that, for $s \in S$ and $x \in X$,

$$x \in d(\varphi^s) \iff x\theta \in d(\xi^s) \text{ and } (x\varphi^s)\theta = (x\theta)\xi^s.$$

Lemma 2.10 *The transitive representation $\psi : S \to \mathcal{GI}_{(X;\sigma)}$ is equivalent to ϕ_H, defined above.*

From result 1.5, lemma 3.1 and 3.2, we obtain a following theorem.

Theorem 2.11 *Every effective representation of a generalized inverse $*$-semigroup S is uniquely a sum of transitive representations ψ_α, each of which is equivalent to ϕ_{H_α} for some closed generalized inverse $*$-subsemigroup H_α of S.*

References

[1] T. E. Hall and T. Imaoka, *Representations and amalgamation of generalized inverse $*$-semigroups*, Semigroup Forum **58**(1999), 126 – 141.

[2] J. M. Howie, *Introduction to semigroup theory*, Academic Press, London, 1976.

[3] T. Imaoka, *Prehomomorphisms on regular $*$-semigroups*, Mem. Fac. Sci. Shimane Univ. **15**(1981), 23 – 27.

[4] T. Imaoka and M. Katsura, *Representations of locally inverse $*$-semigroups* II, Semigroup Forum **55**(1997), 247 – 255.

[5] B. M. Schein, *Representations of generalized groups*, Izv. Vysš. Učebn. Zaved Matematica No. 3 (**28**) (1962), 164 – 176.

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