Note on Transitive Representations of Generalized Inverse Semigroups (Languages, Algebra and Computer Systems)

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Note on Transitive Representations of Generalized Inverse \( \ast \)-Semigroups

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Abstract

In [1], we obtained that an effective representation of a locally [generalized] inverse \( \ast \)-semigroup \( S \) is the sum of a uniquely determined family of transitive representations of \( S \). In this paper, we will determine a transitive representation of a generalized inverse \( \ast \)-semigroup by using right \( \omega \)-cosets. This is a generalization of Schein’s result [5] for inverse semigroups.

1 Introduction

A semigroup \( S \) with a unary operation \( \ast : S \rightarrow S \) is called a regular \( \ast \)-semigroup if it satisfies (i) \((x^*)^* = x\); (ii) \((xy)^* = y^*x^*\); (iii) \(xx^*x = x\). Let \( S \) be a regular \( \ast \)-semigroup. An idempotent \( e \) in \( S \) is called a projection if \( e^* = e \). Denote the sets of idempotents and projections of \( S \) by \( E(S) \) and \( P(S) \), respectively.

Let \( S \) be a regular \( \ast \)-semigroup. If \( eSe \) is an inverse semigroup, for every \( e \in E(S) \), \( S \) is called a locally inverse \( \ast \)-semigroup. If \( E(S) \) is a normal band, that is, it satisfies the identity \( xyzz = zzyx \), \( S \) is called a generalized inverse \( \ast \)-semigroup. A regular \( \ast \)-semigroup \( S \) is a generalized inverse \( \ast \)-semigroup if and only if it is a locally inverse \( \ast \)-semigroup and \( E(S) \) forms a band.

Result 1.1 [3] Let \( S \) be a regular \( \ast \)-semigroup. Define a relation \( \leq \) on \( S \) by

\[
a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).
\]

Then \( \leq \) is a partial order on \( S \) satisfying that \( a \leq b \) implies \( a^* \leq b^* \). If \( S \) is a generalized inverse \( \ast \)-semigroup, then \( \leq \) is compatible.

For a subset \( A \) of a regular \( \ast \)-semigroup \( S \), the set

\[
A\omega = \{ x \in S : \text{there exists } a \in A \text{ such that } a \leq x \}
\]

is called the closure of \( A \). The following statements are easily verified.

\footnote{This is the abstract and details will be published elsewhere.}
(1) \( A \subseteq A\omega; \) \( A \subseteq B \Rightarrow A\omega \subseteq B\omega; \) \( (A\omega)\omega = A\omega. \)

We say that \( A \) is closed if \( A\omega = A. \)

Lemma 1.2 If \( H \) is a regular \(*\)-subsemigroup of a generalized inverse \(*\)-semigroup \( S, \) then \( H\omega \) is a closed generalized inverse \(*\)-subsemigroup of \( S. \)

Let \( S \) be a regular \(*\)-semigroup and \( H \) a regular \(*\)-subsemigroup of \( S. \) If an element \( a \) in \( S \) satisfies \( aa* \in H, \) then \( (Ha)\omega \) is called a right \( \omega \)-coset of \( H. \)

Lemma 1.3 Let \( S \) be a generalized inverse \(*\)-semigroup, and let \( (Ha)\omega \) and \( (Hb)\omega \) be right \( \omega \)-cosets of a regular \(*\)-subsemigroup \( H \) of \( S. \) Then

\[
(Ha)\omega \subseteq (Hb)\omega \iff a \in (Hb)\omega.
\]

A non-empty set \( X \) with its reflexive and symmetric relation \( \sigma \) is called an \( \iota \)-set, and denoted by \( (X; \sigma). \) If \( \sigma \) is transitive, that is, it is an equivalence relation, then \( (X; \sigma) \) is called a transitive \( \iota \)-set.

Let \( (X; \sigma) \) be an \( \iota \)-set. A subset \( A \) of \( X \) is called an \( \iota \)-single subset if, for any \( x \in X, \) there exists at most one element \( y \in A \) such that \( (x, y) \in \sigma. \) If \( (X; \sigma) \) is a transitive \( \iota \)-set, \( A \) is an \( \iota \)-single subset if and only if it satisfies that

\[
(a, b) \in \sigma \, (a, b \in A) \implies a = b.
\]

A mapping \( \alpha \) in the symmetric inverse semigroup \( \mathcal{I}_X \) is called a partial one-to-one \( \iota \)-mapping of \( (X; \sigma) \) if \( d(\alpha) \) and \( r(\alpha) \) are both \( \iota \)-single subsets of \( (X; \sigma), \) where \( d(\alpha) \) and \( r(\alpha) \) are the domain and the range of \( \alpha, \) respectively. Denote the set of all partial one-to-one \( \iota \)-mappings of \( (X; \sigma) \) by \( \mathcal{L}\mathcal{I}_{(X;\sigma)}. \) If \( \sigma \) is transitive, we denote it by \( \mathcal{G}\mathcal{I}_{(X;\sigma)} \) instead of \( \mathcal{L}\mathcal{I}_{(X;\sigma)}. \) For any \( \alpha, \beta \in \mathcal{L}\mathcal{I}_{(X;\sigma)}, \) denote \( \theta_{\alpha,\beta} \) by

\[
\theta_{\alpha,\beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\} = r(\alpha) \times d(\beta) \cap \sigma.
\]

Since a subset of an \( \iota \)-single subset is also an \( \iota \)-single subset, \( \theta_{\alpha,\beta} \in \mathcal{L}\mathcal{I}_{(X;\sigma)}. \) Let \( \mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{L}\mathcal{I}_{(X;\sigma)}\}. \) Define a multiplication \( \circ \) and a unary operation \( \ast \) on \( \mathcal{L}\mathcal{I}_{(X;\sigma)} \) as follows:

\[
\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},
\]

where the multiplication of the right side of the first equality is that of \( \mathcal{I}_X. \) Denote \( \mathcal{L}\mathcal{I}_{(X;\sigma)}(\circ, \ast) \) by \( \mathcal{L}\mathcal{I}_{(X;\sigma)}(\mathcal{M}) \) or simply by \( \mathcal{L}\mathcal{I}_{(X;\sigma)}. \) In this paper, we use \( \mathcal{L}\mathcal{I}_{(X;\sigma)}(\mathcal{M}) \) rather than \( \mathcal{L}\mathcal{I}_{(X;\sigma)}. \)
Result 1.4 [4] For an \( \iota \)-set \( (X; \sigma) \), we have the following:

1. The \(*\)-groupoid \( \mathcal{G}\mathcal{I}_{(X;\sigma)} \), defined above, is a locally inverse \(*\)-semigroup. Moreover, any locally inverse \(*\)-semigroup can be embedded (up to \(*\)-isomorphism) in \( \mathcal{L}\mathcal{I}_{(X;\sigma)} \) on some \( \iota \)-set \( (X; \sigma) \).

2. \( E(\mathcal{L}\mathcal{I}_{(X;\sigma)}) = \mathcal{M} \) and \( P(\mathcal{L}\mathcal{I}_{(X;\sigma)}) = \{1_A : A \text{ is an } \iota \text{-single subset of } (X; \sigma)\} \).

3. If \( (X; \sigma) \) is a transitive \( \iota \)-set, then \( \mathcal{L}\mathcal{I}_{(X;\sigma)} \) is a generalized inverse \(*\)-semigroup. Moreover, any generalized inverse \(*\)-semigroup can be embedded (up to \(*\)-isomorphism) in \( \mathcal{G}\mathcal{I}_{(X;\sigma)} \) on some transitive \( \iota \)-set \( (X; \sigma) \).

4. If \( \sigma \) is the identity relation on \( X \), then \( \mathcal{L}\mathcal{I}_{(X;\sigma)} \) is the symmetric inverse semigroup \( \mathcal{I}_X \) on \( X \).

We call \( \mathcal{L}\mathcal{I}_{(X;\sigma)} [\mathcal{G}\mathcal{I}_{(X;\sigma)}] \) the \( \iota \)-symmetric locally [generalized] inverse \(*\)-semigroup on the \( \iota \)-set [the transitive \( \iota \)-set] \( (X; \sigma) \) with the structure sandwich set \( \mathcal{M} \).

Result 1.5 [1] Let \( H \) be a locally [generalized] inverse \(*\)-subsemigroup of \( \mathcal{L}\mathcal{I}_{(X;\sigma)} [\mathcal{G}\mathcal{I}_{(X;\sigma)}] \) on a [transitive] \( \iota \)-set \( (X; \sigma) \), and define a relation \( \tau_H \) on \( X \) by

\[
(x, y) \in \tau_H \iff \text{there exists } \alpha \in H \text{ such that } x \in d(\alpha) \text{ and } x\alpha = y.
\]

Then \( \tau_H \) is a symmetric and transitive relation on \( X \).

The subset \( \{x \in X : (x, x) \in \tau_H\} = d(\tau_H) \), say, of \( X \) is called the domain of \( \tau_H \). If \( d(\tau_H) = X \), that is, \( \tau_H \) is an equivalence relation on \( X \), then \( H \) is said to be effective. If \( \tau_H \) is the universal relation on \( X \), then \( H \) is said to be transitive.

A representation \( \phi : S \to \mathcal{L}\mathcal{I}_{(X;\sigma)} \) of a locally inverse \(*\)-semigroup \( S \) is called an effective [transitive] representation if \( S\phi \) is an effective [transitive] locally inverse \(*\)-subsemigroup of \( \mathcal{L}\mathcal{I}_{(X;\sigma)} \). Similarly, the effectivity and the transitivity for a representation \( \phi : S \to \mathcal{G}\mathcal{I}_{(X;\sigma)} \) of a generalized inverse \(*\)-semigroup \( S \) are defined.

Result 1.6 [1] An effective representation of a locally [generalized] inverse \(*\)-semigroup \( S \) is the sum of a uniquely determined family of transitive representations of \( S \).

The purpose of this paper is to characterize a transitive representation of a generalized inverse \(*\)-semigroup. The notation and the terminology are those of [1] and [2], unless otherwise stated.
2 Transitive representations

Let $S$ be a generalized inverse $\ast$-semigroup, and let $(X;\sigma)$ be a transitive $\iota$-set and $\psi : S \rightarrow G\mathcal{I}(X;\sigma) (s \mapsto \psi^s)$ a transitive representation of $S$. Fix an element $z$ in $X$ and set

$$H = \{ s \in S : z\psi^s = z \}.$$

**Lemma 2.1** The set $H$, defined above, is a closed generalized inverse $\ast$-subsemigroup of $S$.

Define a relation $\delta$ on $S$ by

$$\delta = \{ (a, b) \in S \times S : z\psi^a = z\psi^b \}.$$

We also assume that $(a, b) \in \delta$ if $z \not\in d(\psi^a) \cup d(\psi^b)$.

**Lemma 2.2** The relation $\delta$, defined above, is a right congruence on $S$ satisfying the following conditions:

1. $\delta \cap (H \times H) = H \times H$,
2. For $a \in S$ and $h \in H$, $(a, h) \in \delta$ implies $a \in H$.

Let $\mathcal{X}$ be the set of all right $\omega$-cosets of $H$. Define a relation $\sim$ on $\mathcal{X}$ by

$$(Ha)\omega \sim (Hb)\omega \iff (a, b) \in \delta.$$

**Lemma 2.3** The relation $\sim$, defined above, is an equivalence relation on $\mathcal{X}$.

Let $\mathcal{X}/\sim = \mathcal{Y}$, say, and denote the $\sim$-class containing $(Ha)\omega$ by $(Ha)\tilde{\omega}$. For any $a \in S$, define a partial mapping $\phi^a_H$ on $\mathcal{Y}$ by

$$d(\phi^a_H) = \{ (Hxaa^*)\tilde{\omega} : xaa^*x^* \in H \} \quad \text{and} \quad \phi^a_H : (Hxaa^*)\tilde{\omega} \mapsto (Hxa)\tilde{\omega},$$

**Lemma 2.4** For any $a \in S$ and $(Ha)\tilde{\omega} \in \mathcal{Y}$, we have

$$(Hx)\tilde{\omega} \in d(\phi^a_H) \iff (x, xaa^*) \in \delta.$$

**Lemma 2.5** For any $a \in S$, $\phi^a_H \in \mathcal{I}_y$ and $(\phi^a_H)^{-1} = \phi^{a^*}_H$.

Define a relation $\Omega$ on $\mathcal{Y}$ by

$$\Omega = \{ ((Hx)\tilde{\omega}, (Hy)\tilde{\omega}) : (Hx)\tilde{\omega}\phi^e_H = (Hy)\tilde{\omega} \text{ for some } e \in E(S) \}.$$
Lemma 2.6 The relation $\Omega$, defined above, is an equivalence relation on $\mathcal{Y}$, that is, $(\mathcal{Y}; \Omega)$ is a transitive $\iota$-set.

Now we can consider the $\iota$-symmetric generalized inverse $*$-semigroup $\mathcal{GI}(\mathcal{Y}; \Omega)$ on the transitive $\iota$-set $(\mathcal{Y}; \Omega)$.

Lemma 2.7 For any $a \in S$, $d(\phi^a_H)$ and $r(\phi^a_H)$ are $\iota$-single subsets of $(\mathcal{Y}; \Omega)$.

Lemma 2.8 For any $a, b \in S$, $\theta_{\phi^a_H, b}$ is $\phi^{ab*}_H$.

Lemma 2.9 The mapping $\phi_H : S \rightarrow \mathcal{GI}(\mathcal{Y}; \Omega)$ is a transitive representation of $S$.

Let $\varphi : S \rightarrow \mathcal{GI}(X, \sigma)$ and $\xi : S \rightarrow \mathcal{GI}(Y, \tau)$ be two representations of a generalized inverse $*$-semigroup $S$. Then $\varphi$ and $\xi$ are equivalent if there exists a bijection $\theta : X \rightarrow Y$ such that, for $s \in S$ and $x \in X$,

$$x \in d(\varphi^s) \iff x\theta \in d(\xi^s) \text{ and } (x\varphi^s)\theta = (x\theta)\xi^s.$$ 

Lemma 2.10 The transitive representation $\psi : S \rightarrow \mathcal{GI}(X, \sigma)$ is equivalent to $\phi_H$, defined above.

From result 1.5, lemma 3.1 and 3.2, we obtain a following theorem.

Theorem 2.11 Every effective representation of a generalized inverse $*$-semigroup $S$ is uniquely a sum of transitive representations $\psi_\alpha$, each of which is equivalent to $\phi_{H_\alpha}$ for some closed generalized inverse $*$-subsemigroup $H_\alpha$ of $S$.

References


