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Almost $n$-dimensional spaces

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We consider only separable metric spaces. A space $X$ is said to be almost $n$-dimensional if it has a basis $\{U_i\}$ such that if $\text{cl}U_i \cap \text{cl}U_j = \emptyset$ then $X = G \cup H$ where $G$ and $H$ are closed sets, $U_i \subset G \setminus H$, $U_j \subset H \setminus G$ and $\dim G \cap H \leq n - 1$ and $n$ is the smallest natural number such that such a basis exists for $n$. It is clear that $n$-dimensional spaces are at most almost $n$-dimensional.

Oversteegen and Tymchatyn [9] proved that almost 0-dimensional spaces are at most 1-dimensional. The Erdős space of irrational sequences in Hilbert space is known to be a universal almost 0-dimensional space [5]. Erdős space is 1-dimensional. Homeomorphism groups of positive dimensional Menger compacta are almost 0-dimensional [9] and at least 1-dimensional by classical results of Brechner [2] and Bestvina [1].

Almost 0-dimensional spaces are at most 1-dimensional and the 1-dimensionality cannot be improved. Our first result shows that this interesting behaviour does not occur in higher dimensions and the following one points out an interesting property of almost 0-dimensional spaces.

**Theorem 1** (Levin-Tymchatyn [7]) If $X$ is almost $n$-dimensional, $n \geq 1$ then $X$ is $n$-dimensional.

**Theorem 2** (Levin-Tymchatyn [7]) Let $X = X_1 \cup X_2$ where $X_1$ is almost 0-dimensional and $X_2$ is 0-dimensional. Then $\dim X \leq 1$.

The proof of these theorems employs so-called $L$-embeddings. A subset $X$ of a compactum $K$ is $L$-embedded in $K$ if for every open cover $\mathcal{U}$ of $K$ there is a neighbourhood $U$ of $X$ in $K$ such that the continua in $U$ refine $\mathcal{U}$. An almost 0-dimensional space is $L$-embeddable in a compactum [6] and

**Theorem 3** (Levin-Pol [6]) If a space $X$ is $L$-embeddable in a compactum $K$ then $\dim X \leq 1$.

As an application of almost 1-dimensional spaces we will consider an old question of R. Duda about the dimension of a hereditarily locally connected,
non-degenerate space \( X \). Nishiura and Tymchatyn [8] showed that each pair of disjoint, closed, connected subsets of \( X \) can be separated by a closed countable subset of \( X \). Hence each basis for \( X \) of open connected sets witnesses the almost 1-dimensionality of \( X \). Then Theorem 1 implies:

**Theorem 4** (Levin-Tymchatyn [7]) *If \( X \) is a hereditarily locally connected, non-degenerate space then \( \dim X = 1 \).*

A partial solution to the question of R. Duda was given in [9] where it was proved that hereditarily locally connected spaces are at most 2-dimensional. Finally let us note that Theorem 2 does not hold if \( X_2 \) is almost 0-dimensional. Indeed, let \( Y \) be 1-dimensional and almost 0-dimensional, let \( M \) be a 1-dimensional compactum and let \( M = M_1 \cup M_2 \), \( \dim M_1 = \dim M_2 = 0 \). Then \( X_1 = Y \times M_1 \) and \( X_2 = Y \times M_2 \) are almost 0-dimensional, and by a theorem of Hurewicz [4] (see also [3], p. 78, 1.9.E(b)) \( X = X_1 \cup X_2 = Y \times M \) is 2-dimensional.

**References**


