The construction of $P$-expansive maps of regular continua: A geometric approach (Unsolved Problems and its Progress in General・Geometric Topology)

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The construction of $P$-expansive maps of regular continua: A geometric approach

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1. Introduction and preliminaries.

In recent years, there has been a growing interest in the study of the dynamical behavior of continuous maps of a graph. Especially, one of the central questions in the theory of dynamical systems is how to recognize "chaos". The theme of this paper is how to describe visually the chaoticity of continuous maps of a graph. To do this, for each graph $G$ and continuous map $f$ of $G$, we shall construct a new subspace $Z$ of the Euclidean 3-dimensional space and a continuous map $g$ of $Z$ which $(G, f)$ is semi-conjugate to. And we shall use the notion of $P$-expansiveness in order to investigate how complicated the dynamical behavior of $f$ is. The fractal and complicated structure of the new space $Z$ implies the chaoticity of $f$.

In [2] and [1, Theorem 4.1] the following result has been shown. Let $D$ be a dendrite, $f : D \to D$ a continuous map and $P$ a finite subset of $D$ such that $f(P) \subset P$. Then there exist a dendrite $E$, a map $g : E \to E$ and a semi-conjugacy $\pi : D \to E$ (i.e., $\pi \circ f = g \circ \pi$) such that

(1) $g$ is $\pi(P)$-expansive, and
(2) if $x, y, z \in P$ and $y \in [x, z]$ then $\pi(y) \in [\pi(x), \pi(z)]$.

If, in addition, the Markov graph of $P$ has no basic intervals of order 0 and no loops of order 1, then $\pi|_{P}$ is one-to-one.

In this paper we expand the above result to a graph. Our main theorem is as follows:

**Theorem 3.4.** Let $G$ be a graph, $f : G \to G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Then there exist a regular continuum $Z$, a continuous map $g : Z \to Z$ and a semi-conjugacy $\pi : G \to Z$ such that

(1) $g$ is $\pi(P)$-expansive, and
(2) if $p, q \in P$ and $Q$ is a subset of $P$ with $A \cap Q \neq \emptyset$ for any arc $A$ in $G$ between $p$ and $q$, then $A' \cap \pi(Q) \neq \emptyset$ for any arc $A'$ in $Z$ between $\pi(p)$ and $\pi(q)$.

In addition, $f$ is point-wise $P$-expansive if and only if $\pi|_{P}$ is one-to-one.
We will show this by using a more geometrical method than that of Baldwin. Our interest is in what structure $Z$ has. We can see visually that $Z$ is a subset of a 3-dimensional space which has a fractal structure.

Let $G$ be a graph, $f : G \rightarrow G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Put $S(G, P) = P \cup \{C|C$ is a component of $G \setminus P\}$. Given $x \in G$, the itinerary of $x$ with respect to $P$ and $f$, written $I_{P,f}(x)$ (or just $I(x)$ if $P$ and $f$ are obvious from context), is defined to be the unique infinite sequence $(C_{n})_{n \geq 0}$ from $S(G, P)$ given by the rule $f^{n}(x) \in C_{n}$ for all $n \geq 0$. If no two points of $G$ have the same itinerary, then $f$ will be called $P$-expansive. And $f$ is point-wise $P$-expansive if for each $p, q \in P$, there exists some non-negative integer $m$ such that $A \cap (P \setminus \{f^{m}(p), f^{m}(q)\}) \neq \emptyset$ for each arc $A$ in $G$ between $f^{m}(p)$ and $f^{m}(q)$.

Let $K$ be a continuum and $P$ a finite subset of $K$. Then we say that $P$ graph-separates $K$ if and only if there exists a finite set $S(K, P)$ of subsets of $K$ such that

1. the element of $S(K, P)$ partition $K$, i.e., every point of $K$ is in exactly one member of $S(K, P)$,
2. for each $p \in P$, $\{p\} \in S(K, P)$,
3. for each $A \in S(K, P)$, the closure of $A$ in $K$ is arc-wise connected, and
4. if $A, B \in S(K, P)$, then the closure of $A$ and $B$ either have empty intersection or intersect in only elements of $P$.

Note that we can also define $P$-expansive for a graph-separated continuum in a similar way.

2. Constructions of $X_{\rightarrow}$ and $X_{\leftarrow}$

Let $G$ be a graph, $f : G \rightarrow G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. We will construct new spaces $X_{\rightarrow}$ and $X_{\leftarrow}$ from $P$ and $f$.

First we want to define an equivalence relation $\sim_{1}$ on $P$. Let $p, q \in P$. If for any non-negative integer $i$, there exists an arc $A_{i}$ in $G$ between $f^{i}(p)$ and $f^{i}(q)$ such that $A_{i} \cap P = \{f^{i}(p), f^{i}(q)\}$, then we put $p \sim_{1}^{i} q$, where $A_{i}$ may now consist of a single point. Now, if for $p, q \in P$, there exist some points $p_{1}, p_{2}, \ldots, p_{k}$ of $P$ such that $p \sim_{1}^{1} p_{1} \sim_{1}^{1} p_{2} \sim_{1}^{1} \cdots \sim_{1}^{1} p_{k} \sim_{1}^{1} q$, then we set $p \sim_{1} q$. This relation $\sim_{1}$ is an equivalence relation on $P$. Let $[p]_{1}$ be the equivalence class of $p$, $P_{1} = \{[p]_{1}|p \in P\}$ and $G_{1} = G/\sim_{1}$ the space obtained from $G$ by identifying each equivalence class of $P$. Then we define a continuous map $f_{1} : G_{1} \rightarrow G_{1}$ such that $f_{1}|_{G_{1}\setminus P_{1}} = f|_{G\setminus P}$ and $f_{1}([p]_{1}) = [f(p)]_{1}$ for $[p]_{1} \in P_{1}$. Similarly, if for any $p, q \in P_{1}$ and non-negative integer $i$, there exists an arc $A_{i}$ in $G_{1}$ between $f_{1}^{i}(p)$ and $f_{1}^{i}(q)$ such that $A_{i} \cap P_{1} = \{f_{1}^{i}(p), f_{1}^{i}(q)\}$, then we put $p \sim_{2}^{i} q$. And if there exist some points $p_{1}, p_{2}, \ldots, p_{k}$
of $P_1$ such that $p \sim_2 p_1 \sim_2 p_2 \sim_2 \cdots \sim_2 p_k \sim_2 q$, then we set $p \sim_2 q$. This relation $\sim_2$ is also an equivalence relation on $P_1$. Let $[p]_2 = \{q|p \sim_2 q \text{ and } p, q \in P_1\}$ and $G_2 = G_1/ \sim_2$ the space obtained from $G_1$ by identifying each equivalence class of $P_1$. Then we define a continuous map $f_2 : G_2 \rightarrow G_2$ such that $f_2|_{G_2 \backslash P_2} = f_1|_{C_1 \backslash P_1} = f|_{G \backslash P}$ and $f_2([p]_2) = [f_1(p)]_2$ for $[p]_2 \in P_2$. In the same way, we can obtain the space $G_\ell$ and a continuous map $f_\ell : G_\ell \rightarrow G_\ell$ for $\ell \geq 1$. Since $P$ is finite, there is some natural number $m$ such that $f_m : G_m \rightarrow G_m$ is point-wise $P$-expansive. There exists a semi-conjugacy $\pi_i$ between $(G_{i-1}, f_{i-1})$ and $(G_i, f_i)$ for $i = 1, 2, \ldots, m$, where $(G_0, f_0) = (G, f)$. We will construct $Z$ and $\pi'$ in Theorem 3.4 by the use of the point-wise $P_m$-expansiveness of $f_m$.

$$
\begin{array}{cccc}
G & \xrightarrow{f} & G \\
\pi_1 \downarrow & & \downarrow \pi_1 \\
G_1 & \xrightarrow{f_1} & G_1 \\
\pi_2 \downarrow & & \downarrow \pi_2 \\
G_2 & \xrightarrow{f_2} & G_2 \\
\pi_3 \downarrow & & \downarrow \pi_3 \\
\vdots & & \vdots \\
\pi_m \downarrow & & \downarrow \pi_m \\
G_m & \xrightarrow{f_m} & G_m \\
\pi' \downarrow & & \downarrow \pi' \\
Z & \xrightarrow{g} & Z 
\end{array}
$$

By the argument above, we may proceed with our construction, under the assumption that $f$ is point-wise $P$-expansive, in the rest part of this section.

Let $S(G, P) \backslash P = \{C_1, C_2, \ldots, C_n\}$ and $P = \{p_1, p_2, \ldots, p_k\}$. We will express the relation of elements of $S(G, P)$ as follows: If $p, q \in P$ and $f(p) = q$, then $p \rightarrow q$. This arrow $\rightarrow$ defines the Markov graph $P_\rightarrow$ on $P$ (See section 4). If $C_i, C_j \in S(G, P) \backslash P$ and $C_j \subset f(C_i)$, then $C_i \rightarrow C_j$. If $f(C_i) \cap C_j \neq \emptyset$, then $C_i \rightarrow C_j$. These arrows $\rightarrow$ and $\leftarrow$ define the Markov graphs $M_\rightarrow$ and $M_\leftarrow$ of elements of $S(G, P) \backslash P$ respectively. Note that $\rightarrow$ implies $\leftarrow$.

Now we will construct a new space $X_\rightarrow$ by using the Markov graphs $M_\leftarrow$ and $P_\rightarrow$. First we will construct a subspace $X$ which is the union of 3-dimensional balls $B_1, B_2, \ldots, B_n$ in the Euclidean 3-dimensional space $\mathbb{E}^3$ by regarding elements $C_1, C_2, \ldots, C_n$ of $S(G, P) \backslash P$ as 3-dimensional balls $B_1, B_2, \ldots, B_n$ of $\mathbb{E}^3$. That is to say, $X = \bigcup_{i=1}^n B_i$, where the relationship of $B_i$ and $B_j$ is decided as follows: If $cl(C_i) \cap cl(C_j) = \emptyset$ for $C_i, C_j \in S(G, P) \backslash P$, then
$B_i \cap B_j = \emptyset$. And if $cl(C_i) \cap cl(C_j) = \{q_1, q_2, \ldots, q_t\} \subset P$, then $B_i \cap B_j = Bd(B_i) \cap Bd(B_j) = \{q'_1, q'_2, \ldots, q'_t\}$, where $Bd(B)$ is the boundary of $B$. Without confusion, we can express elements of $cl(C_i) \cap cl(C_j)$ and $B_i \cap B_j$ in a similar way. And for each $p \in (P \cap cl(C_i)) \setminus \bigcup \{cl(C_j) \cap cl(C_j')|j \neq j' \text{ and } 1 \leq j, j' \leq n\}$, we take a corresponding point $p' \in Bd(B_i) \setminus \bigcup \{B_j \cap B_j'|j \neq j' \text{ and } 1 \leq j, j' \leq n\}$. For simplicity, we set $p' = p \in P$ (see Figure 1).

Put $X_0 = X$. We will construct a subspace $X_1$ contained in $X_0$ by using the Markov graph $M_-$ and $P_-$. For each $i = 1, 2, \ldots, n$, we have an embedding $h_i : X \hookrightarrow B_i$ such that
1. $h_i(X) \cap Bd(B_i) \subset P$, and
2. for each $p, q \in P$ with $p \in Bd(B_i)$ and $p \rightarrow q$, $h_i(q) = p \in Bd(B_i)$.
If $C_i \rightarrow C_j$ ($C_i, C_j \in S(G, P) \setminus P$) in the Markov graph $M_-$, then let $B_{i,j} = h_i(B_j)$ which is a copy of $B_j$. If $C_i \not\rightarrow C_j$, then $B_{i,j} = \emptyset$. Let $Y_i = \bigcup_{i=1}^{n} B_{i,j}$, $B_i = \{B_j|C_i \rightarrow C_j\}$ and $(\bigcup B_i) \cap P = \{p_{t(i;\ell)}, p_{t(i;\ell+1)}, \ldots, p_{t(i;k(i))}\}$, where $t(i; \ell)$ and $k(i)$ are natural numbers with $1 \leq t(i; \ell), k(i) \leq k(1 \leq p \leq k(i))$ and put $h_i(p_{t(i;\ell)}) = p_{t(i;\ell)}$. Then we obtain a connected subset $X_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ (see Figure 2).
Similarly, we will construct a subspace $X_2$ in $X_1$. Let $h_{i_0,i_1} : X \hookrightarrow B_{i_0,i_1}$ be an embedding such that

1. $h_{i_0,i_1}(X) \cap Bd(B_{i_0,i_1}) \subset h_{i_0}(P)$, and
2. for each $p_{i_0,j} \in Bd(B_{i_0,i_1}) \cap h_{i_0}(P)$ and $q \in P$ with $p_j \rightarrow q$, $h_{i_0,i_1}(q) = p_{i_0,j} \in Bd(B_{i_0,i_1})$.

If $C_{i_1} \rightarrow C_j$ in the Markov graph $M_{\rightarrow}$, then let $B_{i_0,i_1,j} = h_{i_0,i_1}(B_j)$. And if $C_{i_1} \not\rightarrow C_j$, then $B_{i_0,i_1,j} = \emptyset$. Let $Y_{i_0,i_1} = \bigcup_{j=1}^{n} B_{i_0,i_1,j}$, $B_{i_1} = \{B_j | C_{i_1} \rightarrow C_j\}$ and $(\bigcup B_{i_1}) \cap P = \{p_{t(i_0,i_1:1)}, p_{t(i_0,i_1:2)}, \ldots, p_{t(i_0,i_1:k(i_0,i_1))}\}$. Put $h_{i_0,i_1}(p_{t(i_0,i_1:j)}) = p_{t(i_0,i_1:i_1)}, 1 \leq j \leq t(i_0,i_1 : k(i_0,i_1))$. Then we obtain $X_2 = \bigcup \{Y_{i_0,i_1}|1 \leq i_0, i_1 \leq n\}$ (see Figure 3).

![Figure 3](image)

When this operation is repeated inductively, we obtain $X_0 \supset X_1 \supset X_2 \supset \cdots$ and a subspace $X_- = \bigcap_{i=0}^{\infty} X_i$ of $\mathbb{E}^3$. Note that $X_-$ is connected.

Next let $X'_1, X'_2, \ldots$ be subspaces constructed in a similar way on basis of the Markov graph $M_{\rightarrow}$. Then we obtain a subspace $X_- = \bigcap_{i=1}^{\infty} X'_i$ of $\mathbb{E}^3$. Note that $X_-$ is not always connected.

3. Construction of $Z$.

Let $G$ be a graph, $f : G \rightarrow G$ a continuous map, $P = \{p_1, p_2, \ldots, p_k\}$ a finite subset of $G$ such that $f(P) \subset P$ and $S(G, P) \setminus P = \{C_1, C_2, \ldots, C_n\}$. We may also assume that $f$ is point-wise $P$-expansive in this section from the argument in section 2. And let $X_-, X_{\rightarrow}$ be the above spaces constructed by the Markov graphs $(M_{\rightarrow}, P_{\rightarrow})$, $(M_{\rightarrow}, P_{\rightarrow})$ on $S(G, P)$ respectively.

**Theorem 3.1.** The subspace $X_-$ of $\mathbb{E}^3$ is a regular continuum.
Since $f$ is point-wise $P$-expansive, $\lim_{m \to \infty} \text{diam}(B_{i_0, i_1, \ldots, i_m}) = 0$. Thus we can define a map $\pi : G \to X_\infty$ as follows: Given $x \in G$, if $f^\ell(x) \in cl(C_i)$ for any $\ell = 0, 1, 2, \ldots$, then $\pi(x) = \bigcap_{\ell=0}^\infty B_{i_0, i_1, \ldots, i_\ell}$.

**Lemma 3.2.** $\pi : G \to X_\infty$ is continuous.

Now we will put $Z = \pi(G)$. Then $X_\infty \subset Z \subset X_\infty$. In general it is difficult to recognize the precise structure of $Z$, but by the above relation $X_\infty \subset Z \subset X_\infty$, we can realize the approximate structure of $Z$. Since $X_\infty$ is regular, $Z$ is also regular.

Note that by the construction, if for any element $C \in S(G, P) \setminus P$, there exist finitely many elements $C_1, C_2, \ldots, C_m$ of $S(G, P)$ such that $f(C) = \bigcup_{i=1}^m C_i$, then $X_\infty = Z = X_\infty$.

Define a map $g : X_\infty \to X_\infty$ as follows: If $\{x\} = \bigcap_{\ell=0}^\infty B_{i_0, i_1, \ldots, i_\ell}$, then $\{g(x)\} = \pi\{x\}$. We can investigate the uniqueness of $g$ as we did that of $\pi$. Note that $g(Z) \subset Z$.

**Lemma 3.3.** $g : X_\infty \to X_\infty$ is continuous.

**Theorem 3.4.** Let $G$ be a graph, $f : G \to G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Then there exist a regular continuum $Z$, a continuous map $g : Z \to Z$ and a semi-conjugacy $\pi : G \to Z$ such that

1. $g$ is $\pi(P)$-expansive, and
2. if $p, q \in P$ and $Q$ is a subset of $P$ with $A \cap Q \neq \emptyset$ for any arc $A$ in $G$ between $p$ and $q$, then $A' \cap \pi(Q) \neq \emptyset$ for any arc $A'$ in $Z$ between $\pi(p)$ and $\pi(q)$.

In addition, $f$ is point-wise $P$-expansive if and only if $\pi|_P$ is one-to-one.

**Proposition 3.5.** Let $G$ be a graph, $f : G \to G$ a continuous map and $P$ the set of vertices of $G$ with $f(P) \subset P$. If $f$ is point-wise $P$-expansive and $f|_{[p, q]}$ is one-to-one for each edge $[p, q]$ between $p$ and $q$, then $Z$ is homeomorphic to $G$.

**Remark.** In Theorem 3.4, we can obtain the same result by using a graph-separated continuum instead of a graph.

References