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The construction of P-expansive maps of regular continua : A geometric approach

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1. Introduction and preliminaries.

In recent years, there has been a growing interest in the study of the dynamical behavior of continuous maps of a graph. Especially, one of the central questions in the theory of dynamical systems is how to recognize "chaos". The theme of this paper is how to describe visually the chaoticity of continuous maps of a graph. To do this, for each graph G and continuous map f of G, we shall construct a new subspace Z of the Euclidean 3-dimensional space and a continuous map g of Z which (G, f) is semi-conjugate to. And we shall use the notion of P-expansiveness in order to investigate how complicated the dynamical behavior of f is. The fractal and complicated structure of the new space Z implies the chaoticity of f.

In [2] and [1,Theorem 4.1] the following result has been shown. Let D be a dendrite, $f : D \longrightarrow D$ a continuous map and P a finite subset of D such that $f(P) \subset P$. Then there exist a dendrite E, a map $g : E \longrightarrow E$ and a semi-conjugacy $\pi : D \longrightarrow E$ (i.e., $\pi \circ f = g \circ \pi$) such that

(1) g is $\pi(P)$-expansive, and

(2) if $x, y, z \in P$ and $y \in [x, z]$ then $\pi(y) \in [\pi(x), \pi(z)]$.

If, in addition, the Markov graph of P has no basic intervals of order 0 and no loops of order 1, then $\pi|_P$ is one-to-one.

In this paper we expand the above result to a graph. Our main theorem is as follows :

THEOREM 3.4. *Let G be a graph, $f : G \longrightarrow G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Then there exist a regular continuum Z, a continuous map $g : Z \longrightarrow Z$ and a semi-conjugacy $\pi : G \longrightarrow Z$ such that*

(1) g is $\pi(P)$-expansive, and

(2) if $p, q \in P$ and Q is a subset of P with $A \cap Q \neq \emptyset$ for any arc A in G between p and q, then $A' \cap \pi(Q) \neq \emptyset$ for any arc A' in Z between $\pi(p)$ and $\pi(q)$.

In addition, f is point-wise P-expansive if and only if $\pi|_P$ is one-to-one.

We will show this by using a more geometrical method than that of Baldwin. Our interest is in what structure Z has. We can see visually that Z is a subset of a 3-dimensional space which has a fractal structure.

Let G be a graph, $f : G \longrightarrow G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Put $S(G, P) = P \bigcup \{C | C$ is a component of $G \setminus P\}$. Given $x \in G$, the *itinerary* of x with respect to P and f, written $I_{P,f}(x)$ (or just $I(x)$ if P and f are obvious from context), is defined to be the unique infinite sequence $(C_n)_{n \geq 0}$ from $S(G, P)$ given by the rule $f^n(x) \in C_n$ for all $n \geq 0$. If no two points of G have the same itinerary, then f will be called *P-expansive.* And f is *point-wise P-expansive* if for each $p, q \in P$, there exists some non-negative integer m such that $A \cap (P \setminus \{f^m(p), f^m(q)\}) \neq \emptyset$ for each arc A in G between $f^m(p)$ and $f^m(q)$.

Let K be a continuum and P a finite subset of K. Then we say that P *graph-separates* K if and only if there exists a finite set $S(K, P)$ of subsets of K such that

(1) the element of $S(K, P)$ partition K, i.e., every point of K is in exactly one member of $S(K, P)$,

(2) for each $p \in P$, $\{p\} \in S(K, P)$,

(3) for each $A \in S(K, P)$, the closure of A in K is arc-wise connected, and

(4) if $A, B \in S(K, P)$, then the closure of A and B either have empty intersection or intersect in only elements of P.

Note that we can also define P-expansive for a graph-separated continuum in a similar way.

2. Constructions of $X_{\rightharpoonup}$ and $X_{\rightharpoondown}$.

Let G be a graph, $f : G \longrightarrow G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. We will construct new spaces $X_{\rightharpoonup}$ and $X_{\rightharpoondown}$ from P and f.

First we want to define an equivalence relation $\sim_1$ on P. Let $p, q \in P$. If for any non-negative integer i, there exists an arc A_i in G between $f^i(p)$ and $f^i(q)$ such that $A_i \cap P = \{f^i(p), f^i(q)\}$, then we put $p \sim_1' q$, where A_i may now consist of a single point. Now, if for $p, q \in P$, there exist some points $p_1, p_2, \ldots, p_k$ of P such that $p \sim_1' p_1 \sim_1' p_2 \sim_1' \cdots \sim_1' p_k \sim_1' q$, then we set $p \sim_1 q$. This relation $\sim_1$ is an equivalence relation on P. Let $[p]_1$ be the equivalence class of p, $P_1 = \{[p]_1 | p \in P\}$ and $G_1 = G / \sim_1$ the space obtained from G by identifying each equivalence class of P. Then we define a continuous map $f_1 : G_1 \longrightarrow G_1$ such that $f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_1([p]_1) = [f(p)]_1$ for $[p]_1 \in P_1$. Similarly, if for any $p, q \in P_1$ and non-negative integer i, there exists an arc A_i in G_1 between $f_1^i(p)$ and $f_1^i(q)$ such that $A_i \cap P_1 = \{f_1^i(p), f_1^i(q)\}$, then we put $p \sim_2' q$. And if there exist some points $p_1, p_2, \cdots, p_k$

of P_1 such that $p \sim_2' p_1 \sim_2' p_2 \sim_2' \cdots \sim_2' p_k \sim_2' q$, then we set $p \sim_2 q$. This relation $\sim_2$ is also an equivalence relation on P_1. Let $[p]_2 = \{q | p \sim_2 q \text{ and } p, q \in P_1\}$, $P_2 = \{[p]_2 | p \in P_1\}$ and $G_2 = G_1 / \sim_2$ the space obtained from G_1 by identifying each equivalence class of P_1. Then we define a continuous map $f_2 : G_2 \longrightarrow G_2$ such that $f_2|_{G_2 \setminus P_2} = f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_2([p]_2) = [f_1(p)]_2$ for $[p]_2 \in P_2$. In the same way, we can obtain the space G_ℓ and a continuous map $f_\ell : G_\ell \longrightarrow G_\ell$ for $\ell \geq 1$. Since P is finite, there is some natural number m such that $f_m : G_m \longrightarrow G_m$ is point-wise P-expansive. There exists a semi-conjugacy π_i between (G_{i-1}, f_{i-1}) and (G_i, f_i) for $i = 1, 2, \ldots, m$, where $(G_0, f_0) = (G, f)$. We will construct Z and π' in Theorem 3.4 by the use of the point-wise P_m-expansiveness of f_m.

$$
\begin{array}{ccc}
G & \xrightarrow{f} & G \\
\pi_1 \downarrow & & \downarrow \pi_1 \\
G_1 & \xrightarrow[f_1]{} & G_1 \\
\pi_2 \downarrow & & \downarrow \pi_2 \\
G_2 & \xrightarrow[f_2]{} & G_2 \\
\pi_3 \downarrow & & \downarrow \pi_3 \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\pi_m \downarrow & & \downarrow \pi_m \\
G_m & \xrightarrow[f_m]{} & G_m \\
\pi' \downarrow & & \downarrow \pi' \\
Z & \xrightarrow[g]{} & Z
\end{array}
$$

By the argument above, we may proceed with our construction, under the assumption that f *is point-wise P-expansive*, in the rest part of this section.

Let $S(G,P) \setminus P = \{C_1, C_2, \ldots, C_n\}$ and $P = \{p_1, p_2, \ldots, p_k\}$. We will express the relation of elements of $S(G,P)$ as follows : If $p, q \in P$ and $f(p) = q$, then $p \longrightarrow q$. This arrow $\longrightarrow$ defines the Markov graph $P_\rightarrow$ on P (See section 4). If $C_i, C_j \in S(G,P) \setminus P$ and $C_j \subset f(C_i)$, then $C_i \longrightarrow C_j$. If $f(C_i) \cap C_j \neq \emptyset$, then $C_i \rightharpoonup C_j$. These arrows $\longrightarrow$ and $\rightharpoonup$ define the Markov graphs $M_\rightarrow$ and $M_\rightharpoonup$ of elements of $S(G,P) \setminus P$ respectively. Note that $\longrightarrow$ implies $\rightharpoonup$.

Now we will construct a new space $X_\rightharpoonup$ by using the Markov graphs $M_\rightharpoonup$ and $P_\rightarrow$. First we will construct a subspace X which is the union of 3-dimensional balls $B_1, B_2, \ldots, B_n$ in the Euclidean 3-dimensional space $\mathbf{E}^3$ by regarding elements $C_1, C_2, \ldots, C_n$ of $S(G,P) \setminus P$ as 3-dimensional balls $B_1, B_2, \ldots, B_n$ of $\mathbf{E}^3$. That is to say, $X = \bigcup_{i=1}^n B_i$, where the relationship of B_i and B_j is decided as follows : If $cl(C_i) \cap cl(C_j) = \emptyset$ for $C_i, C_j \in S(G,P) \setminus P$, then

$B_i \cap B_j = \emptyset$. And if $cl(C_i) \cap cl(C_j) = \{q_1, q_2, \ldots, q_\ell\} \subset P$, then $B_i \cap B_j = Bd(B_i) \cap Bd(B_j) = \{q'_1, q'_2, \ldots, q'_\ell\}$, where $Bd(B)$ is the boundary of B. Without confusion, we can express elements of $cl(C_i) \cap cl(C_j)$ and $B_i \cap B_j$ in a similar way. And for each $p \in (P \cap cl(C_i)) \setminus \bigcup\{cl(C_j) \cap cl(C_{j'}) | j \neq j' \text{ and } 1 \leq j, j' \leq n\}$, we take a corresponding point $p' \in Bd(B_i) \setminus \bigcup\{B_j \cap B_{j'} | j \neq j' \text{ and } 1 \leq j, j' \leq n\}$. For simplicity, we set $p' = p \in P$ (see Figure 1).

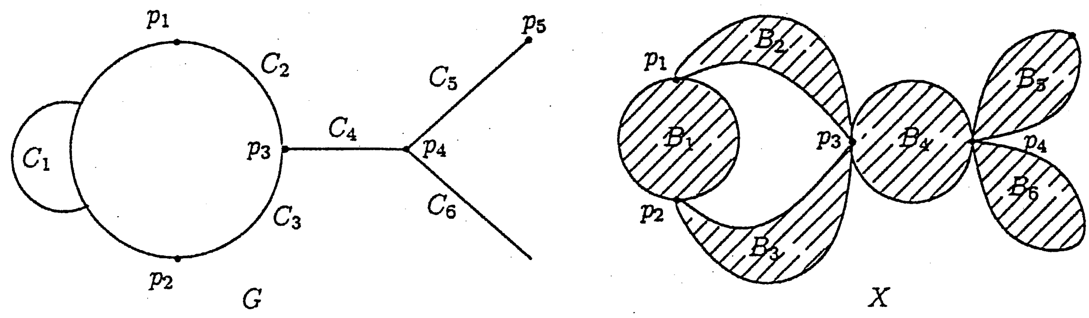

Figure 1:

Put $X_0 = X$. We will construct a subspace X_1 contained in X_0 by using the Markov graph $M_\rightharpoonup$ and $P_\rightarrow$. For each $i = 1, 2, \ldots, n$, we have an embedding $h_i : X \hookrightarrow B_i$ such that

(1) $h_i(X) \cap Bd(B_i) \subset P$, and

(2) for each $p, q \in P$ with $p \in Bd(B_i)$ and $p \longrightarrow q$, $h_i(q) = p \in Bd(B_i)$.

If $C_i \rightharpoonup C_j$ $(C_i, C_j \in S(G, P) \setminus P)$ in the Markov graph $M_\rightharpoonup$, then let $B_{i,j} = h_i(B_j)$ which is a copy of B_j. If $C_i \not\rightharpoonup C_j$, then $B_{i,j} = \emptyset$. Let $Y_i = \bigcup_{j=1}^{n} B_{i,j}$, $\mathrm{B}_i = \{B_j | C_i \rightharpoonup C_j\}$ and $(\bigcup \mathrm{B}_i) \cap P = \{p_{t(i:1)}, p_{t(i:2)}, \ldots, p_{t(i:k(i))}\}$, where $t(i : \ell)$ and $k(i)$ are natural numbers with $1 \leq t(i : \ell), k(i) \leq k$ $(1 \leq \ell \leq k(i))$. And put $h_i(p_{t(i:\ell)}) = p_{i,t(i:\ell)}$. Then we obtain a connected subset $X_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ (see Figure 2).

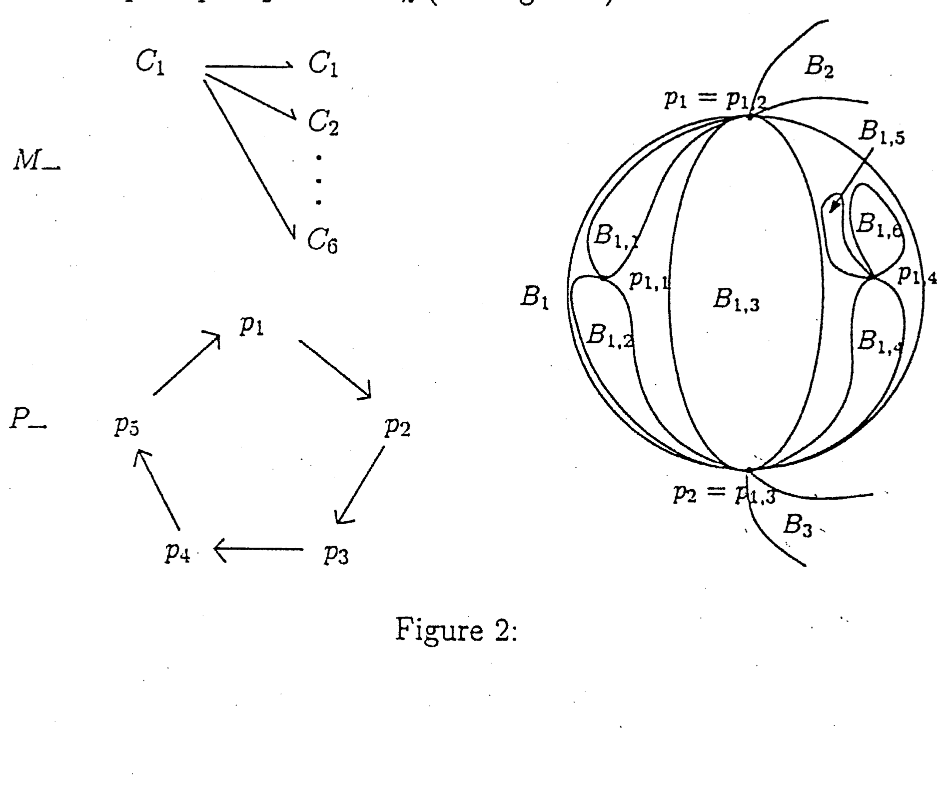

Figure 2:

Similarly, we will construct a subspace X_2 in X_1. Let $h_{i_0,i_1} : X \hookrightarrow B_{i_0,i_1}$ be an embedding such that

(1) $h_{i_0,i_1}(X) \cap Bd(B_{i_0,i_1}) \subset h_{i_0}(P)$, and

(2) for each $p_{i_0,j} \in Bd(B_{i_0,i_1}) \cap h_{i_0}(P)$ and $q \in P$ with $p_j \longrightarrow q$, $h_{i_0,i_1}(q) = p_{i_0,j} \in Bd(B_{i_0,i_1})$.

If $C_{i_1} \rightharpoonup C_j$ in the Markov graph $M_{\rightharpoondown}$, then let $B_{i_0,i_1,j} = h_{i_0,i_1}(B_j)$. And if $C_{i_1} \not\rightharpoonup C_j$, then $B_{i_0,i_1,j} = \emptyset$. Let $Y_{i_0,i_1} = \bigcup_{j=1}^{n} B_{i_0,i_1,j}$, $\mathbf{B}_{i_1} = \{B_j | C_{i_1} \rightharpoonup C_j\}$ and $(\bigcup \mathbf{B}_{i_1}) \cap P = \{p_{t(i_0,i_1:1)}, p_{t(i_0,i_1:2)}, \dots, p_{t(i_0,i_1:k(i_0,i_1))}\}$. Put $h_{i_0,i_1}(p_{t(i_0,i_1:j)}) = p_{i_0,i_1,t(i_0,i_1:j)}$ $(1 \le j \le t(i_0, i_1 : k(i_0, i_1))$. Then we obtain $X_2 = \bigcup\{Y_{i_0,i_1} | 1 \le i_0, i_1 \le n\}$ (see Figure 3).

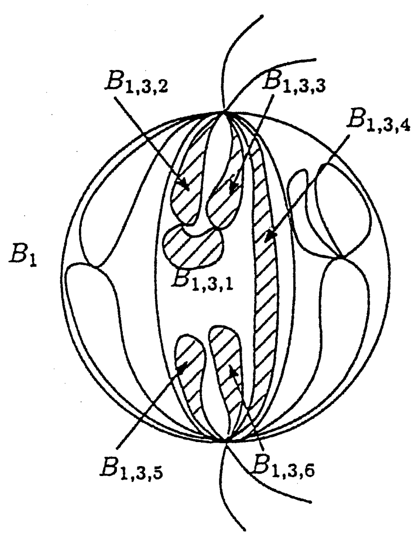

Figure 3:

When this operation is repeated inductively, we obtain $X_0 \supset X_1 \supset X_2 \supset \cdots$ and a subspace $X_{\rightharpoondown} = \bigcap_{i=0}^{\infty} X_i$ of $\mathbf{E}^3$. Note that $X_{\rightharpoondown}$ is connected.

Next let $X_1', X_2', \dots$ be subspaces constructed in a similar way on basis of the Markov graph $M_{\rightarrow}$. Then we obtain a subspace $X_{\rightarrow} = \bigcap_{i=1}^{\infty} X_i'$ of $\mathbf{E}^3$. Note that $X_{\rightarrow}$ is not always connected.

3. Construction of Z.

Let G be a graph, $f : G \longrightarrow G$ a continuous map, $P = \{p_1, p_2, \dots, p_k\}$ a finite subset of G such that $f(P) \subset P$ and $S(G, P) \setminus P = \{C_1, C_2, \dots, C_n\}$. We may also assume that f is point-wise P-expansive in this section from the argument in section 2. And let $X_{\rightarrow}$, $X_{\rightharpoondown}$ be the above spaces constructed by the Markov graphs $(M_{\rightarrow}, P_{\rightarrow})$, $(M_{\rightharpoondown}, P_{\rightharpoondown})$ on $S(G, P)$ respectively.

THEOREM 3.1. *The subspace $X_{\rightharpoondown}$ of $\mathbf{E}^3$ is a regular continuum.*

Since f is point-wise P-expansive, $lim_{m\to\infty}\ diam(B_{i_0,i_1,\ldots,i_m}) = 0$. Thus we can define a map $\pi : G \longrightarrow X_{\rightharpoondown}$ as follows : Given $x \in G$, if $f^\ell(x) \in cl(C_{i_\ell})$ for any $\ell = 0, 1, 2, \ldots$, then $\pi(x) = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,i_2,\ldots,i_\ell}$.

LEMMA 3.2. *$\pi : G \longrightarrow X_{\rightharpoondown}$ is continuous.*

Now we will put $Z = \pi(G)$. Then $X_{\rightarrow} \subset Z \subset X_{\rightharpoondown}$. In general it is difficult to recognize the precise structure of Z, but by the above relation $X_{\rightarrow} \subset Z \subset X_{\rightharpoondown}$, we can realize the approximate structure of Z. Since $X_{\rightharpoondown}$ is regular, Z is also regular.

Note that by the construction, if for any element $C \in S(G, P) \setminus P$, there exist finitely many elements $C_1, C_2, \ldots, C_m$ of $S(G, P)$ such that $f(C) = \bigcup_{i=1}^{m} C_i$, then $X_{\rightarrow} = Z = X_{\rightharpoondown}$.

Define a map $g : X_{\rightharpoondown} \longrightarrow X_{\rightharpoondown}$ as follows : If $\{x\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}$, then $\{g(x)\} = g(\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}) = \bigcap_{\ell=1}^{\infty} B_{i_1,i_2,\ldots,i_\ell}$. We can investigate the uniqueness of g as we did that of π. Note that $g(Z) \subset Z$.

LEMMA 3.3. *$g : X_{\rightharpoondown} \longrightarrow X_{\rightharpoondown}$ is continuous.*

THEOREM 3.4. *Let G be a graph, $f : G \longrightarrow G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Then there exist a regular continuum Z, a continuous map $g : Z \longrightarrow Z$ and a semi-conjugacy $\pi : G \longrightarrow Z$ such that*

(1) g is $\pi(P)$-expansive, and

(2) if $p, q \in P$ and Q is a subset of P with $A \cap Q \neq \emptyset$ for any arc A in G between p and q, then $A' \cap \pi(Q) \neq \emptyset$ for any arc A' in Z between $\pi(p)$ and $\pi(q)$.

In addition, f is point-wise P-expansive if and only if $\pi|_P$ is one-to-one.

PROPOSITION 3.5. *Let G be a graph, $f : G \longrightarrow G$ a continuous map and P the set of vertices of G with $f(P) \subset P$. If f is point-wise P-expansive and $f|_{[p,q]}$ is one-to-one for each edge $[p, q]$ between p and q, then Z is homeomorphic to G.*

REMARK. In Theorem 3.4, we can obtain the same result by using a graph-separated continuum instead of a graph.

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