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Two properties added to $M_3$-spaces

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1. Introduction.

In this article, we talk about what we recently obtained on the outstanding open problem whether $M_3$-spaces are $M_1$. Assume here that all spaces are regular $T_2$.

As well-known, $M_i$-spaces ($i=1, 2, 3$) are defined by Ceder in terms of special bases as follows:

Definition 1.1. (1) A space $X$ is $M_1$ if and only if there exists a $\sigma$-CP (=closure-preserving) base for $X$.

(2) A space $X$ is $M_2$ if and only if there exists a $\sigma$-CP quasi-base for $X$.

(3) A space $X$ is $M_3$ if and only if there exists a $\sigma$-cushioned pair-base.

The equality $M_2 = M_3$ is obtained independently by Gruenhage and Junnila, and $M_3$-spaces are studied by Borges under the name of stratifiable spaces. But the implication $M_3 \Rightarrow M_1$ is not answered yet.

The partial answers to this problem have been obtained. The first is due to Slaughter that every Lasnev space is $M_1$. The sequence of positive answers which insists that every $M_3$-space with some special property is $M_1$ is shown by the diagram below. Among them, we note that the results od Ito and Tamano are applicable to our cases here.

Ito showed that every Nagata space, more widely every $M_3$-space whose every point has a CP open neighborhood base is $M_1$. Choosing the essential part from his discussion, Tamano showd that every Baire, Frechet $M_3$-space is $M_1$. The difference between them is one little thing, but acute. Tamano proposed ther the following question: Is every Frechet $M_3$-space an $M_1$-space? We clear it by adding some property to $M_3$-spaces.

2. Two properties.

To solve his question positively, we introduce the following property (*), which is weaker than that of Frechet spaces:

Definition 2.1([1, Definition 2.3]). A space $X$ has property (*) if for any open subset $O$ of $X$ and any point of $B(O)$, the boundary of $O$ in $X$, there exists a CP closed network $B$ at $p$ in $X$ such that for each $B \in B$,
$B \subset \overline{O}, B \cap B(O) = \{p\}$.

With the aid of property ($\ast$), we can show the following:

Theorem 2.2([1, Theorem 3.1]). Let $X$ be an M$_3$-space with property ($\ast$). Then every closed subset of $X$ has a CP open neighborhood base in $X$, and hence $X$ is an M$_1$-space.

Corollary 2.3. Every Frechet M$_3$-space is M$_1$.

Does every M$_3$-space have property ($\ast$)? But unfortunately this is not the case, because there exists an M$_1$-space which does not satisfy property ($\ast$) ([1, Example 1]). This example leads us to define another property added to M$_3$-spaces that is weaker than the former one.

Definition 2.4([2, Definition 8]). A space $X$ satisfies property (P) if for each open subset $O$ of $X$ and each point $p$ of $B(O)$, there exists a CP closed local network $B$ at $p$ in $X$ such that for each $B \in B$, $\overline{B \cap O} = B$.

In this case again, we can show the following:

Theorem 2.5([2, Theorem 15]). If $X$ is an M$_3$-space with property (P), then every closed subset of $X$ has a CP open neighborhood base in $X$.

What kind of spaces except for Frechet M$_3$-spaces have this property (P)? As for this question we can show the following:

Theorem 2.6([2, Theorem 16]). Every sequential M$_3$-space have property (P), and hence is an M$_1$-space.

But we do not know whether every M$_3$-space have property (P). Maybe this is a question equivalent to the well-known M$_3$.M$_1$problem.

References

\[ \sigma \text{-discrete } \mathcal{M}_3\text{-space} \]
\[ \downarrow \]
\[ F_\sigma \text{-metrizable } \mathcal{M}_3\text{-space} \]
\[ \downarrow \]
\[ \mathcal{L}\text{-space} \]
\[ \downarrow \]
\[ \text{free } \mathcal{L}\text{-space} \]
\[ \downarrow \]
\[ \mathcal{M}_3\mu \text{-space} \]
\[ \downarrow \]
\[ \text{perfect image of an } \mathcal{M}_0\text{-space} \]
\[ \downarrow \]
\[ \mathcal{M}_3\text{-space with property(}P\text{)} \]
\[ \downarrow \]
\[ \mathcal{M}_1\text{-space} \]

\[ \text{Nagata space} \]

\[ \text{Baire } \mathcal{M}_3\text{-space} \]
\[ \downarrow \]
\[ \text{Fréchet } \mathcal{M}_3\text{-space} \]
\[ \downarrow \]
\[ \text{sequential } \mathcal{M}_3\text{-space} \]