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Nash Equilibrium as Conjectures in Public Belief*

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ABSTRACT. In Game Theory with its applications to Economics, the interpretation problem of a mixed strategy Nash equilibrium has been known to be important and a number of the interpretations has been proposed.

Aumann and Brandenburger [*Econometrica*, Vol.63(1995), No.5, 1161-1180] has succeeded in giving an epistemic interpretation of a mixed strategy Nash equilibrium as conjectures on the part of other players: When there is a common-prior, mutual knowledge of the payoff-functions and of rationality, and common-knowledge of the conjectures, imply that the conjectures form a mixed strategy Nash equilibrium. Where common-knowledge of something is the infinite recursion of mutual knowledge of it; that is, all players know it and they know that they know it and they know that they know that they know it and so on.

In the standard model of knowledge as like as the Aumann and Brandenburger model, the players are implicitly assumed to have *logically omniscient* ability; that is, they know every tautology and know all the implications of their knowledge. The assumptions about common-knowledge and about logically omniscient ability are evidently problematic in the sense that these are not realistic at all.

In this lecture presentation I propose a new model of *awareness* and *belief* with all players having no logically omniscient ability, where awareness and belief are weaker notions of knowledge. I say that an event is *public belief* if every player believes the event whenever it occurs. Rationality is the requirement that a mixed strategy of each player is optimal against a perturbation of his conjecture on the part of other players. I give the epistemic condition for a mixed strategy Nash equilibrium:

Theorem. *When there is a common-prior, public belief of payoff functions, of rationality and of a perturbation of conjectures imply that the conjectures induce a mixed strategy Nash equilibrium.*

I emphasize that I make no assumption about either common-belief or logically omniscient ability for players.

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1 INTRODUCTION

When a given game in strategic form is transformed into a decision problem, the uncertainty that a player faces in a game is the strategic choice of the other players' actions. Each player has therefore knowledge (or belief) of the other players' actions. In addition, each player is also uncertain about the knowledge of the other players' actions and must have knowledge of their knowledge, and so on. Hence, beginning with a game, the decision theoretical approach leads to the study of infinite recursion of knowledge (or belief) for the players; e.g., the study of common-knowledge (or common-belief) for the players. Once the transformation of a game into a decision problem has been completed, solution concepts may be explored from an epistemic point of view.

Since the pioneering contribution of Aumann(1976), game theorists and mathematical economists have investigated the concepts of common-knowledge (or common-belief) and the foundation of solution-concepts of games in different kinds of epistemic models. There are two important approaches among others: The first is the axiomatic approach; the syntactic models of knowledge and belief, and the second is the Bayesian approach of knowledge; the model of belief with probability 1. Bacharach(1985) and Samet(1990) adopted the first approach and extended the 'Agreeing to disagree' theorem of Aumann(1976). Aumann and Brandenburger(1995) adopted the second approach and succeeded in giving epistemic conditions for Nash equilibrium of a game as conjectures on the part of other players using mutual knowledge of players' rationality and common-knowledge of their conjectures.

In every approach, the players in model have been explicitly or implicitly required to be *logically omniscient*; that is, they can deduce all the logical implications of their knowledge (or belief) and they know (or believe) every tautology. However real people are not complete reasoners and the recent idea of 'bounded rationality' suggests dropping the problematic assumption. In regard to this Dekel, Lipman and Rustichini(1998) introduced a unawareness operator with axiom of plausibility and investigated the relation between the unawareness operator model and a possibility operator model.

The purpose of this lecture is to present a new model of awareness and belief in which the players are required neither to be men of complete perception nor to have the complete ability of logical reasoning.

I begin in Section 2 by reviewing the standard model of knowledge. Section 3 devotes to establish a model of awareness and belief without logical omniscience and to present the fundamental lemma. As consequence I extend the 'Agreeing to disagree' theorem of Aumann(1976) to the model of awareness and belief with a common-prior. In section 4 I give an epistemic condition for Nash equilibrium in a finite strategic game without common-belief assumption. In Appendix I give a proof of Fundamental Lemma.

2 STANDARD MODEL OF KNOWLEDGE

Let N be a set of finitely many *players* and i denote an player. A *state-space* is a finitely non-empty set, whose members are called *states*. An *event* is a subset of the state-space. If Ω is a state-space we denote by 2^Ω the field of all subsets of it. We say that an event F *occurs* at a state ω if ω belongs to F.

An *information partition* (Π_i) is a class of mappings Π_i of Ω into 2^Ω in which $\{\Pi_i(\omega)|\omega \in \Omega\}$ makes a partition of Ω such that each image $\Pi_i(\omega)$ contains ω which is the set of states that i thinks are *possible* when ω occurs. The mapping Π_i is called the i's information partition and $\Pi_i(\omega)$ the *possibility set* of i at ω.

Given our interpretation of an information partition, an player i for whom $\Pi_i(\omega) \subseteq E$ knows, in the state ω, that some state in the event E has occurred. In this case we say that in the state ω the player i knows E. An i's *knowledge operator* is an operator K_i on 2^Ω such that K_iE is the set of states of Ω in which i knows that E has occurred; that is,

$$K_iE = \{\omega \in \Omega|\Pi_i(\omega) \subseteqq E\}.$$

I note that an i's knowledge operator satisfies the following axioms: For every E, F of 2^Ω,

N $\quad K_i\Omega = \Omega;$

K $\quad K_i(E \cap F) = K_iE \cap K_iF;$

T $\quad K_iF \subseteq F;$

4 $\quad K_iF \subseteq K_iK_iF;$

5 $\quad \Omega \setminus K_iF \subseteq K_i(\Omega \setminus K_iF).$

Definition. I call a pair $\langle \Omega, (K_i)\rangle$ the *standard model of knowledge* if K_i satisfies the five axioms **N**, **K**, **T**, **4** and **5**.

The information partition (Π_i) is then uniquely determined by

$$\Pi_i(\omega) = \bigcap_{\omega \in K_iE} E = \bigcap_{\omega \in T = K_iT} T .$$

The *common-knowledge* operator K_C is defined by

$$K_CX = \bigcap_{i_1, i_2, \dots, i_k \in N, k=1,2,\dots} K_{i_1}K_{i_2}\cdots K_{i_k}X.$$

We say that an event X is *common-knowledge at* ω if ω belongs to K_CX. That is, when ω occurs then for all k and for all players $i_1, i_2, \dots .i_k$ it is true that

i_1 knows that [i_2 knows that [$\dots i_{k-1}$ knows that [i_k knows X]] $\dots$].

This is an *iterated* notion of common-knowledge. We note that K_C satisfies the fixed point property:

$$K_CX \subseteq K_E(K_CX \cap X)$$

for every X of 2^{Ω}.

Let μ be a common-prior on Ω with $\mu(\omega) \gneqq 0$ for all $\omega \in \Omega$ and X an event. We denote by q_i the i's *posterior* of X at ω; that is,

$$q_i = \mu(X|\Pi_i(\omega)).$$

We set

$$[q] = \bigcap_{i \in N} \{\xi \in \Omega | \mu(X|\Pi_i(\xi)) = q_i\}.$$

We say that *all posteriors of the players are common-knowledge* at ω if ω belongs to $K_C([q])$, and we say that the *players cannot agree to disagree* if $q_i = q_j$ for all players i, j. Aumann(1976) showed the 'Agreeing to disagree' theorem:

Proposition 1. *In the standard model of knowledge with a common-prior, if all posteriors are common-knowledge at some state then all players cannot agree to disagree.*

Proof. See Proposition in Aumann (1976). □

Remark 1. Axiom **K** implies the *monotonicity* of player's knowledge:

M $\quad K_iE \subseteq K_iF \quad$ whenever $\quad E \subseteq F$.

Definition. I say that an player has *logically omniscient* ability if his knowledge operator satisfies Axioms **N** and **M**.

3 AWARENESS STRUCTURE

I present the notion of awareness structure that is a generalization of the standard model of knowledge. By a *state-space* I mean a non-empty (perhaps, infinite) set.

Definition. *A belief structure* is a pair $\langle \Omega, (B_i) \rangle$ in which Ω is a state-space and (B_i) is a class of i's *belief* operators on 2^{Ω}. The *mutual belief* operator is the operator B_E that assigns to each event F the intersection of B_iF for all i of N; that is,

$$B_EF = \bigcap_{i \in N} B_iF \ .$$

The *common-belief* operator B_C is defined in the following way(Lismont, 1993). We regard the class of all the operators on 2^W as a partially ordered set with the order $\sqsubset$ such that

$$B \sqsubset B' \quad \text{if and only if for every } F \text{ of } 2^W, \ BF \subseteqq B'F,$$

where B, B' are operators on 2^W . We define inductively the descending chain of operators, $\{B^m\}$, on non-negative integers m as follows:

$$\begin{aligned}
B^0F &:= B_EF, & \bar{B}^0F &:= B_E(B^0F \cap F);\\
B^1F &:= \bar{B}^0F \cap B^0F, & \bar{B}^1F &:= B_E(B^1F \cap F);\\
B^{m-1}F &:= \bar{B}^{m-2}F \cap B^{m-2}F, & \bar{B}^mF &:= B_E(B^{m-1}F \cap F);\\
B^mF &:= \bar{B}^{m-1}F \cap B^{m-1}F.
\end{aligned}$$

On noting that the operators on 2^Ω are at most finite, there is a sufficiently large number M such that for all $k \geq M$, $B^k = B^M$. We denote $B_C := B^M$, and say that an event E is *common-belief in* ω if ω belongs to $B_C E$.

Worthy noticing is that B_C satisfies the fixed point property:

FP $\quad B_C F \subseteq B_E(B_C F \cap F) \quad$ for every F of 2^Ω.

Definition. An *awareness structure* is a triple $\langle \Omega, (A_i), (B_i) \rangle$ in which $\langle \Omega, (B_i) \rangle$ is a belief structure and (A_i) is a class of i's *awareness* operators on 2^Ω such that Axiom **PL** (axiom of plausibility) is valid:

PL $\quad B_i F \cup B_i(\Omega \setminus B_i F) \subseteq A_i F \quad$ for every F of 2^Ω.

The awareness structure is called *finite* if the state-space is a finite set.

The axiom **PL** due to Dekel, Lipman and Rustichini (1998) says that i is aware of F if he believes it or if he believes that he dose not believe it.

The *mutual awareness* operator is the operator A_E on 2^Ω that assigns to each event F the intersection of $A_i F$ for all i of N; that is,

$$A_E F = \bigcap_{i \in N} A_i F .$$

The interpretation of $A_i F$ is the event that 'i is aware of F,'where as $A_E F$ is interpreted as the event 'everybody is aware of F.'

Definition. Let $\langle \Omega, (A_i), (B_i) \rangle$ be an awareness structure. I say that an event F is *self-aware* of i if $F \subseteq A_i F$ and it is said to be *publicly aware* if $F \subseteq A_E F$. An event T is said to be *i's evident belief* if $T \subseteq B_i T$, and it is said to be *public belief* at state ω if $\omega \in T \subseteq B_E T$.

An event is public belief (or respectively, it is publicly aware) if whenever it occurs all players believe it (or they are all aware of it.) We can think of public belief as embodying the essence of what is involved in all players making their direct observations.

Definition. The *associated* information structure (P_i) is a class of the mappings P_i of Ω into 2^Ω in which P_i assigns to each ω the intersection of all the i's evident beliefs T to which ω belongs; that is,

$$P_i(\omega) = \bigcap_{T \in 2^\Omega} \{T \mid \omega \in T \subseteq B_i T\}.$$

(If there is no event T for which $\omega \in T \subseteq B_i T$ then we take $P_i(\omega)$ to be non-defined.) We call $P_i(\omega)$ the i's *evidence set* at ω.

An evidence set is interpreted as the basis for all i's evident beliefs since each i's evident belief T is decomposed into a union of all evidence sets contained in T.

Definition. A non-empty event H is said to be *P_i-invariant* if for every ξ of H, $P_i(\xi)$ is defined and is contained in H.

Remark 2. The standard model of knowledge can be interpreted as an awareness structure $\langle\Omega, (A_i), (B_i)\rangle$ such that Ω is finite, B_i satisfies **N**, **K**, **T**, **4** and **5**, and A_i is the *trivial* awareness operator; i.e. $A_i(E) = \Omega$ for every $E \in 2^\Omega$. In fact, the associated information structure (P_i) with the standard model of knowledge coincides with the information partition (Π_i) of the model. This says that an awareness structure is an extension of the standard model of knowledge. In this regard we note that every event is publicly aware in the standard model of knowledge.

Example 1. Consider the following situation. Player 1 believes the theory that "the earth is not flat and it moves around the sun," while player 2 believes the theory that "the earth is neither flat nor it moves around the sun"; the former theory is an 1's evident belief and the latter is an 2's evident belief. Furthermore it is public belief that "the earth is not flat."

This can be represented as follows: The state-space Ω consists of four states $\alpha, \beta, \gamma, \delta$, where state α represents the proposition "the earth is not flat but it moves around the sun," state β "the earth is neither flat nor it moves around the sun," state γ "not α," and state δ represents "not β." The belief operators are given by:

$$B_1(\{\alpha\}) = \Omega, B_1(\{\alpha, \beta\}) = \{\alpha, \beta\}, B_1(\Omega) = \{\alpha\} \text{ and } B_1(E) = \emptyset \text{ otherwise;}$$
$$B_2(\{\beta\}) = \Omega, B_2(\{\alpha, \beta\}) = \{\alpha, \beta\}, B_2(\Omega) = \{\beta\} \text{ and } B_2(E) = \emptyset \text{ otherwise.}$$

The associated information structure is given by:

$$P_1(\alpha) = \{\alpha\}, P_1(\beta) = \{\alpha, \beta\} \text{ and } P_1(\omega) \text{ is not defined otherwise;}$$
$$P_2(\alpha) = \{\alpha, \beta\}, P_2(\beta) = \{\beta\} \text{ and } P_2(\omega) \text{ is not defined otherwise.}$$

Let μ be the equal probability measure on Ω: $\mu(\omega) = 1/4$. Now if we denote by $q_i(X, \omega)$ the posterior of X in ω defined by $\mu(X|P_i(\omega))$ then we obtain that $q_2(\{\alpha\}, \alpha) = 1/2$; that is, in the true state α player 2's posterior of the event $\{\alpha\}$ (the earth is not flat and it moves around the sun) is 1/2 when player 2 believes that β is true and never believes that α is so ($B_2(\alpha) = \emptyset$), contrary to the spirit of the example. $\square$

I improve on the definition of posterior as follows:

Definition. Let $\langle\Omega, (A_i), (B_i), \mu\rangle$ be an awareness structure with a common-prior μ. I define the mapping $\mathbf{q}_i$ of $2^\Omega \times \Omega$ into $[0, 1]$ that assigns to each (X, ω) the conditional probability $\mu(X \cap A_i(X)\ |P_i(\omega))$. For every real number q_i, I denote

$$[q_i] = \{\omega \in \Omega | \mathbf{q}_i(X, \omega) = q_i\}.$$

An interpretation of $\mathbf{q}_i(X; \omega)$ is the conditional probability of the i's *awareness section* of X under his evidence set at ω.

I say q_i to be the i's *posterior* of X at ω if ω belongs to $[q_i]$. I denote by q the profile $(q_i)_{i\in N}$. An event $[q]$ is the intersection of the sets $[q_i]$ for all i of N; that is,

$$[q] = \bigcap_{i\in N} [q_i].$$

For Example 1, letting $A_i(E) = B_i(E) \cup B_i(-B_i(E))$ I obtain that $A_2(\{\alpha\}) = \{\beta\}$. Therefore it follows from the new definition that player 2's posterior of $\{\alpha\}$ at state α is $\mathbf{q}_2(\{\alpha\}, \alpha) = 0$, as desired.

The following lemma is the key to proving Theorems 1 and 2.

Fundamental Lemma. *Let (P_i) be the associated information structure with a finite awareness structure and μ a common-prior. Let q_i be an i's posterior of an event X at a state ω. If there is an event H such that the following two properties* (a), (b) *are true. Then we obtain that*

$$\mu(X \cap A_i(X)\,|H) = q_i :$$

(a) *H is non-empty and it is P_i-invariant,*
(b) *H is contained in $[q_i]$.*

Proof. See Appendix.

I say that the players *commonly believe their posteriors q_i* of X at ω if $[q]$ is common-belief at ω; that is, $\omega \in B_C([q])$. I can prove the generalized version of Aumann's theorem:

Theorem 1. *In a finite awareness structure with a common-prior, if all players commonly believe their posteriors q_i of a publicly aware event X at a state ω then they cannot agree to disagree; that is, $q_i = q_j$ for every i, j, even when they are not logically omniscient.*

Proof. I set $[q] \cap B_C([q])$ by H. I note that H is P_i-invariant for every i. It follows that H satisfies the conditions (a) and (b) in Fundamental Lemma. Therefore $\mu(X|H) = \mu(X \cap A_i(X)\,|H) = q_i$ for every i. □

Remark 3. In Matsuhisa(1998) the logic of awareness and belief is introduced and it is shown that the logic have the finite model property. From this syntactical point of view it suffices to explore the class of all finite awareness structures.

4 PUBLIC BELIEF AND NASH EQUILIBRIUM

By a *game G* I mean a *finite* game in strategic form $< N, (S_i), (g_i) >$ in which N is a finite set of players $\{1, 2, \ldots, n\}$ and for every player i, S_i is a finite set of i's actions and g_i is an i's payoff-function of S into $\boldsymbol{R}$, where S denotes the product $S_1 \times S_2 \times \cdots \times S_n$, S_{-i} the product $S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ and g denote the n-tuple $(g_1, g_2, \ldots g_n)$. For every s of S denote $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

A probability distribution ϕ_i on S_{-i} is said to be an i's *overall conjecture* (or simply i's *conjecture*). For each player j other than i,this induces the marginal on j's actions; we call it i's *individual conjecture* about j (or simply i's conjecture about j.) Functions on Ω are viewed like random variables in a probability space (Ω, μ). If $\mathbf{x}$ is a such function and x is a value of it, I denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega |\, \mathbf{x}(\omega) = x\}$.

An awareness structure with a common-prior μ yields the two overall conjectures as follows.

Definition. The i's *normal conjecture* ϕ_i is given by $\mu([s_{-i}] \cap A_i([s_{-i}])|P_i(\omega))$; and the i's *ε-perturbed conjecture* ϕ_i^ε is given by $(1-\varepsilon)\mu([s_{-i}] \cap A_i([s_{-i}])|P_i(\omega)) + \varepsilon p_i([s_{-i}] \cap A_i([s_{-i}]))$, where $\varepsilon \in (0,1)$ and p_i is a state-independent probability measure on Ω.

The probability distribution $p_i([s_{-i}] \cap A_i([s_{-i}]))$ on S_{-i} is a perturbation unable to be controlled by i when the other players $-i$ play actions s_{-i} . I denote $\phi := (\phi_1, \phi_2, \dots, \phi_n)$ and $\phi^\varepsilon := (\phi_1^\varepsilon, \phi_2^\varepsilon, \dots, \phi_n^\varepsilon)$.

Definition. Let $\mathcal{A} = <\Omega, (A_i), (B_i)>$ be an awareness structure. I say that G is an *$\mathcal{A}$-game with ε-perturbed conjectures* if there is a common-prior μ on Ω and if for every player i there are two random variables $\mathbf{g}_i$ of Ω into the class of real valued functions $\{g_i\}_{i\in N}$ on S and $\mathbf{s}_i$ of Ω into S_i such that the four conditions are valid:

(i) $[g] = \cap_{i\in N}[\mathbf{g}_i = g_i]$ is i's evident belief;
(ii) $[s_i] = [\mathbf{s}_i = s_i]$ is i's evident belief for every s_i of S_i;
(iii) $[s_{-i}] = [\mathbf{s}_{-i} = s_{-i}]$ is self-aware of i for every s_{-i} of S_{-i}; and
(iv) $[\phi] = \cap_{i\in N}[\phi_i^\varepsilon = \phi_i]$ is i's evident belief for every n-tuple of conjectures $\phi = (\phi_i)_{i\in N}$,

where $[\mathbf{s}_{-i} = s_{-i}] := \cap_{j\neq i}[\mathbf{s}_j = s_j]$ and $[\phi_i^\varepsilon = \phi_i] := \cap_{s_{-i}\in S_{-i}}[\phi_i^\varepsilon(s_{-i}) = \phi_i(s_{-i})]$.

In an $\mathcal{A}$-game G the pay-off functions $g = (g_1, g_2, \dots, g_n)$ is said to be *actually played* at a state ω if ω belongs to $[\mathbf{g} = g]$. An i's action s_i is said to be *actual* at a state ω if ω belongs to the set $[\mathbf{s}_i = s_i]$.

Definition. An player i is said to be *ε-rational* at ω if each i's actual action s_i maximizes the expectation of his actually played pay-off function g_i at ω when the other players actions are distributed according to his ε-perturbed conjecture $\phi_i^\varepsilon(\omega)$: Formally, letting $g_i = \mathbf{g}_i(\omega)$ and $s_i = \mathbf{s}_i(\omega)$,

$$\mathrm{Exp}^\varepsilon(g_i(s_i, \mathbf{s}_{-i}); \omega) \geqq \mathrm{Exp}^\varepsilon(g_i(t_i, \mathbf{s}_{-i}); \omega)$$

for all t_i in S_i.[2] An player i is said to be *rational* at ω if he is 0-rational at ω.

Let R_j^ε denote the set of all the states at which an player j is ε-rational and R^ε the intersection $\cap_{j\in N}R_j^\varepsilon$. I simply denote R_j^0 by R_j and R^0 by R.

Example 2. For each Nash equilibrium (σ_j) of G, I can construct the standard model of knowledge $\mathcal{A} =< \Omega, (K_i) >$ such that G is an $\mathcal{A}$-game with normal conjectures (i.e., with 0-perturbed conjectures) as follows:

- Ω is the product of all the supports of σ_j with j of N ; that is,

$$\Omega = \mathrm{Supp}(\sigma_1) \times \cdots \times \mathrm{Supp}(\sigma_n) ;$$

[2]The expectation Exp^ε is defined by

$$\mathrm{Exp}^\varepsilon(g_i(t_i, \mathbf{s}_{-i}); \omega) := \sum_{s_{-i}\in S_{-i}} g_i(t_i, s_{-i})\, \phi_i^\varepsilon(\omega)(s_{-i}) .$$

· Π_i is the i's information partition defined by

$$\Pi_i(\omega) = \mathrm{Supp}(\sigma_1) \times \cdots \times \mathrm{Supp}(\sigma_{i-1}) \times \{\omega_i\} \times \mathrm{Supp}(\sigma_{i+1}) \times \cdots \times \mathrm{Supp}(\sigma_n)$$

for $\omega = (\omega_1, \ldots, \omega_i, \ldots, \omega_n)$ of Ω ;

· K_i is the i's knowledge operator defined by

$$K_i F = \{\omega \in \Omega | \Pi_i(\omega) \subseteqq F\}$$

· μ is the probability measure on Ω defined by

$$\mu(\omega) = \sigma_1(\omega_1)\sigma_2(\omega_2) \cdots \sigma_n(\omega_n) .$$

I define the mapping $\mathbf{s}_i : \Omega \to S_i$ as $\omega = (\omega_1, \ldots, \omega_i, \ldots, \omega_n) \mapsto \omega_i$ and define the mapping $\mathbf{g}_i : \Omega \to \{g_i\}$ as $\omega \mapsto g_i$. I can plainly observe that G is an $\mathcal{A}$–game with normal conjectures. Furthermore, let K_C be the common-knowledge operator of $< \Omega, (K_i) >$ and let $\phi_i(s_{-i})$ be defined by $\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_{i-1}(s_{i-1})\sigma_{i+1}(s_{i+1})\cdots\sigma_n(s_n)$. I observe that $\phi(s_{-i})$ coincides with the normal conjecture $\phi_i(\omega, s_{-i}) = \mu([s_{-i}]|P_i(\omega))$ for every ω of Ω. In these circumstances, I note that for every player i, $K_i([s_i]) = [s_i]$ and $[\phi] = [g] = R_i = \Omega$, so that $K_E([g]) = K_E(R) = K_C([\phi]) = \Omega$. □

Aumann and Brandenburger (1995) succeeded in giving sufficient epistemic conditions for Nash equilibrium in the standard model of knowledge:

Proposition 2. *Let $\mathcal{A}$ be the standard model of knowledge with the trivial awareness operator. In an $\mathcal{A}$-game G with normal conjectures having a common-prior, if the player's payoff-functions and their rationality are mutually known and if their conjectures about other players' actions are commonly known then the n-tuple of conjectures is a mixed strategy Nash equilibrium of G.*

Proof. See Theorem B in Aumann and Brandenburger (1995). □

Remark 4. Aumann and Brandenburger (1995) proved the proposition for the model of belief with probability 1 , $< \Omega, (K_i) >$, defined by

$$K_i F = \{\omega \in \Omega | \mu(F|\Pi_i(\omega)) = 1\} ,$$

where (Π_i) is a given information partition. The knowledge operator satisfies Axioms **N** and **M**, thus an player i in this model has logically omniscient ability.

Common-knowledge (common-belief) of the players' overall conjectures seems a rather strong assumption with respect to its infinite recursion. The following example shows a possibility of getting away with less.

Example 3. Here the 'overall' conjectures are public belief, the individual conjectures are agreed on, rationality is public belief, and there is a common prior, and then we get Nash equilibrium even when the overall conjectures are not commonly

	h	t
H	1, 0, 2	0, 1, 2
T	0, 1, 2	1, 0, 2

W

	h	t
H	1, 0, 3	0, 1, 0
T	0, 1, 0	1, 0, 3

E

FIGURE 1

	h_1	t_1	h_2	t_2	h_3	t_3
H_1	ω_1 W_1	ω_2 E_1				
T_1	ω_3 E_1	ω_4 W_1				
H_2			ω_5 W_2	ω_6 W_3		
T_2			ω_7 W_3	ω_8 W_2		
H_3					ω_9 W_3	ω_{10} W_2
T_3					ω_{11} W_2	ω_{12} W_3

FIGURE 2

believed. Consider the three person game of Figure 1. We can plainly observe that $(\frac{1}{2}\mathrm{H}+\frac{1}{2}\mathrm{T}, \frac{1}{2}\mathrm{h}+\frac{1}{2}\mathrm{t}, \mathrm{W})$ is the unique Nash equilibrium.

Consider now this game as $\mathcal{A}$-game with normal conjectures, in which $\mathcal{A}$ is the awareness structure $< \Omega, (A_i)_{i=1,2,3}, (B_i)_{i=1,2,3} >$ with common-prior μ as follows:

- $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{12}\}$ and $\Pi_i (i = 1, 2, 3)$ is the partition of Ω in Figure 2;
- B_1 and B_2 are respectively the knowledge operators induced by Π_1, Π_2, and letting $F_0 := \{\omega_5, \omega_6, \ldots, \omega_{12}\}$, B_3 is defined by

$$B_3E := F_0 \cup \{\omega_1\} \quad \text{if } E = F_0,$$
$$B_3E := F_0 \quad \text{if } E = F_0 \cup \{\omega_1\},$$
$$B_3E := \{\omega \in \Omega | \Pi_3(\omega) \subseteqq E\} \qquad \text{if } E \text{ otherwise};$$

- $A_i(E) := B_i(E) \cup B_i(-B_i(E))$ for $i = 1, 2, 3$;
- $\mu(\omega) = 1/12$ for every $\omega \in \Omega$.
- $\mathbf{g}_i(\omega) = \Omega$ for every $\omega \in \Omega$, and $\mathbf{s}_i$ is defined in Figure 2.

Consider the state $\omega = \omega_5$. Player 1's overall conjecture ϕ_1 is $\frac{1}{2}$hW + $\frac{1}{2}$tW, player 2's overall conjecture ϕ_2 is $\frac{1}{2}$HW + $\frac{1}{2}$TW and player 3's overall conjecture ϕ_3 is $\frac{1}{4}$Hh + $\frac{1}{4}$Ht + $\frac{1}{4}$Th + $\frac{1}{4}$Tt. Players 1 and 2 agree on the conjecture $\frac{1}{2}$W + $\frac{1}{2}$E about player 3. Similarly, players 1 and 3 agree on the conjecture $\frac{1}{2}$h + $\frac{1}{2}$t about player 2, and players 2 and 3 agree on the conjecture $\frac{1}{2}$H + $\frac{1}{2}$T about player 1. Hence the individual conjectures derived from player 1's overall conjecture are $\sigma_2 = \frac{1}{2}$h + $\frac{1}{2}$t for player 2 and $\sigma_3 =$ W for player 3. These are the same as player 2's and player 3's conjectures about each other; in particular their individual conjectures about player 1 are the same distribution $\sigma_1 = \frac{1}{2}$H + $\frac{1}{2}$T.

It can be observed that all pay-off functions $[g]$ is Ω which is public belief at every state, and that both rationality R and the overall conjectures $[\phi]$ are the same event F_0 that is public belief at ω_5 but is not commonly believed since $B_3B_3(F_0) = \emptyset$. Nevertheless, the triple of the individual conjectures $(\sigma_1, \sigma_2, \sigma_3)$ form the Nash equilibrium of the game. □

The main result in this lecture is as follows:

Theorem 2. *Let $\mathcal{A}$ be a finite awareness structure with a common-prior μ and G an $\mathcal{A}$-game with ε-perturbed conjectures. Let ϕ be an n-tuple of conjectures (ϕ_i). If for every sufficiently small positive number ε, it is public belief at some state that all players are rational, that $\boldsymbol{\phi} = \phi$ and that $\mathbf{g} = g$ then for each j, all the conjectures ϕ_i of players i other than j induce the same probability distribution σ_j on the j's actions. Moreover the profile (σ_j) constitutes a mixed strategy Nash equilibrium of G.*

Proof. For each ε, I set $[\boldsymbol{\phi}^\varepsilon = \phi]$ by F and $[\mathbf{s}_i = s_i] \cap F$ by H_i. Let ω be a state such that $\omega \in [g] \cap R^\varepsilon \cap F$. We note that F is P_i-invariant and that $\mu(F) \neq 0$.

I set the probability distribution Q on S by $Q(s) = \mu([s]|F)$. Let $Q(s_i)$ denote the marginal of Q on S_i and $Q(s_{-i})$ the marginal of Q on S_{-i}. I define a probability distribution σ_j on S_j by $\sigma_j(s_j) = Q(s_j)$ for each j. Let $\mathrm{Supp}(\sigma_j)$ denote the support of σ_j. I note that for every player i, if s_j belongs to $\mathrm{Supp}(\sigma_j)$ then $H_j = [\mathbf{s}_j = s_j] \cap F$ is non-empty and is P_j-invariant. Denote the conjecture q_i by

$$q_i(s_{-i}) := \frac{1}{1-\varepsilon}\{\boldsymbol{\phi}_i^\varepsilon(s_{-i}) - \varepsilon p_i(s_{-i})\} ,$$

and I note $q_i(s_{-i}) = \mu([s_{-i}]|P_i(\omega))$.

(α) For every player i, all conjectures q_j with $j \neq i$ induce the same distribution σ_i on S_i.

Proof of (α). For every player j and for every s of S with $s_j \in \mathrm{Supp}(\sigma_j)$, I obtain by Fundamental Lemma that

$$\mu([s_{-j}]|H_j) = q_j(s_{-j}) .$$

Dividing by $\mu(F)$ yields that $\mu([s]|F) = q_j(s_{-j})\mu([s_j]|F)$ and that

$$Q(s) = q_j(s_{-j})Q(s_j) \ . \tag{1}$$

Summing up over s_j I obtain that for every s_{-j} of S_{-j},

$$Q(s_{-j}) = q_j(s_{-j}) \ . \tag{2}$$

Therefore I can plainly observe that for each $i \neq j$, $q_j(s_i) = Q(s_i) = \sigma_i(s_i)$; that is, for all j the conjecture $q_j(s_i)$ about $i \neq j$ induced by ϕ_j is the same distribution σ_i which is independent of j, in completing the proof of (α).

(β) The n-tuple (σ_j) is a mixed strategy Nash equilibrium of G.

Proof of (β). By (1) and (2) it immediately follows that for every j and for all s_j of Supp(σ_j), $Q(s) = Q(s_{-j})Q(s_j)$. From this I can verify by induction on players $j = 1, 2, \dots, n$ that the distribution q_j is the product of σ_j ; that is,

$$q_j(s_{-j}) = \sigma_1(s_1) \cdots \sigma_{j-1}(s_{j-1})\sigma_{j+1}(s_{j+1}) \cdots \sigma_n(s_n) \ . \tag{3}$$

I note that

$$[s_j] \cap [g_j] \cap R_j^\varepsilon \cap [\phi_j] \neq \emptyset \ .$$

Therefore I can observe that each action s_i with $q_j(s_i) = \sigma_i(s_i) > 0$ for some $i \neq j$ maximizes g_j against ϕ_j^ε since j is rational at some state of $[s_j] \cap [q_j] \cap R_j \cap [\phi_j]$. From this together with (3) I can conclude that (σ_i) is a $\frac{2\varepsilon}{1-\varepsilon}||g||_s$-equilibrium of G where $||g||_s := \max_{i \in N, s \in S} |g_i(s)|$, and therefore it constitutes a Nash equilibrium of G as ε tends to 0, in completing the proof of (β). □

Remark 5. I have showed in the above proof that: When there is a common-prior, public belief of ε-conjectures, the pay-off functions and of ε-rationality imply that the conjectures induce a $\frac{2\varepsilon}{1-\varepsilon}||g||_s$-equilibrium of G.

Remark 6. For normal conjectures, I can prove the following epistemic conditions for Nash equilibrium by the same way as above:

Let $\mathcal{A}$ be a finite awareness structure with a common-prior μ and G an $\mathcal{A}$-game with normal conjectures. If it is public belief at some state that all players are rational, that $\phi = \phi$ and that g $= g$, *then all the conjectures ϕ_i of i other than j induce the same probability distribution σ_j on the j's actions. Moreover the profile (σ_j) constitutes a mixed strategy Nash equilibrium of G.*

CONCLUSION

In this lecture I present the model of awareness and belief with emphasis on the logically non-omniscient point of view. With the common-prior assumption I extend the 'Agreeing to disagree' theorem of Aumann to the model and give an epistemic conditions for Nash equilibria in a finite game in strategic form without common-belief assumption.

Appendix

Proof of Fundamental Lemma

A similar lemma was proved by Matsuhisa and Kamiyama(1997, Section 11); I repeat the proof for the sake of importance and completeness. The idea of the proof comes from Samet(1990, Theorem 7.)

I can plainly observe that (P_i) is a *reflexive* and *transitive* information structure as follows: For each i and for every state ω,

(Reflexivity) $\quad \omega \in P_i(\omega)$;

(Transitivity) $\quad \xi \in P_i(\omega)$ implies $P_i(\xi) \subseteqq P_i(\omega)$.

I define the equivalence relation $\sim$ on the state-space Ω by

$$\xi \sim \omega \qquad \text{if and only if} \qquad P_i(\xi) = P_i(\omega)\ .$$

I denote by $\Pi_i(\omega)$ the equivalence class of a state ω. Since H is P_i-invariant, it immediately follows that H is decomposed into a disjoint union of components $\Pi_i(\xi)$ for $\xi \in H$. I can observe that each component $\Pi_i(\xi)$ is μ-measurable. I set by $\mathcal{S}$ the class of all the components $\Pi_i(\xi)$ of H such that $\mu(X \cap A_i(X)\ |\Pi_i(\xi)) = q_i$, and denote by S the union of all members of $\mathcal{S}$.

To prove the lemma it suffices to show that $S = H$. Suppose to the contrary that $S \subsetneqq H$. I observe the point that there exists a state $\omega_0 \in H \setminus S$ such that $P_i(\xi) \setminus S = P_i(\omega_0) \setminus S$ for every $\xi \in P_i(\omega_0) \setminus S$: For, if not then, noting that P_i is reflexive and transitive as above, I can plainly obtain an infinite sequence $\{\omega_n\}$ of states in H such that $\omega_{n+2} \in P_i(\omega_{n+1}) \setminus S \subsetneqq P_i(\omega_n) \setminus S \subseteqq H \setminus S$ for every $n = 0, 1, 2, \ldots$, in contradiction to the assumption that Ω is finite as required. Therefore, I can verify that $\Pi_i(\omega_0) = P_i(\omega_0) \setminus S$, and since $\omega_0 \in H \subseteqq [q_i]$ I conclude that $\Pi_i(\omega_0) \in \mathcal{S}$, in final contradiction. This establishes the Fundamental Lemma. □

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