数理解析研究所講究録
1109 巻 1999 年 182-191

The most symmetric non-singular plane curves of degree $n<8$

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0 Introduction

Throughout this paper k stands for the complex number field $\mathbf{C}$. A homogeneous polynomial $f(x,y,z)\in k[x,y,z]$ defines a plane algebraic curve $f=0$, or $C(f)$ in the projective plane $\mathbf{P}^{2}$. A non-singular matrix $A\in GL(3,k)$ defines a projectivity (A) sending a point P with the homogeneous coordinates (x) to a point $(A)P$ with the homogeneous coordinates $(x(^{t}A))$. Denote by $PGL(3,k)$ the group of projectivities in $\mathbf{P}^{2}$. Denote by $\mathrm{Aut}(f)$ the projective automorphism group of f, namely $\mathrm{Aut}(f)=\{(A)\in PGL(3,k);\ f_{A}$ is proportional to $f\}$, where $f_{A}(x,y,z)=f((x,y,z)(^{t}A^{-1}))$. When $C(f)$ is non-singular and of degree n, i.e. $\deg f=n$, then $C(f)$ is a compact Riemann surface of genus $g=(n-1)(n-2)/2$. In this case we can consider the holomorphic automorphism group $\mathrm{AUT}(f)$ of the Riemann surface $C(f)$. Clearly $\mathrm{Aut}(f)$ is a subgroup of $\mathrm{AUT}(f)$. If $\deg f\geq 4$ and $C(f)$ is non-singular, then $\mathrm{Aut}(f)=\mathrm{AUT}(f)$ [7, p.372], and $|\mathrm{AUT}(f)|\leq 84(g-1)$ [5]. Therefore $|\mathrm{Aut}(f)|$ is bounded above when $C(f)$ runs through *non-singular* plane curve of degree $n\geq 4$, n being fixed. As will be shown in the next section, the same is true for non-singular plane cubics.

Let an f in $k[x,y,z]$ be homogeneous. We call f singular or non-singular according as the curve $C(f)$ has a singualr point or not. A non-singular curve $C(f)$ of degree $n(n\geq 3)$ is the most symmetric, if it attains the maximum order of the projective automorphism groups for non-singular plane algebraic curves of degree $n(n\geq 3)$. We often identify the polynomial f and the curve $C(f)$.

Our main results are the following Theoreıns 1, 3, and 5. Theorem 2 is well known [3, pp.348–349].

Theorem 1 *Let f be a non-singular plane cubic.*
(1) $|\mathrm{Aut}(f)|\leq 54$.
(2) $|\mathrm{Aut}(f)|=54$ *if and only if f is projectively equivalent to* $x^{3}+y^{3}+z^{3}$.

Theorem 2 *Let f be a non-singular plane quartic.*
(1) $|\mathrm{Aut}(f)| \leq 168$.
(2) $|\mathrm{Aut}(f)| = 168$ *if and only if f is projectively equivalent to the Klein quartic* $x^3y + y^3z + z^3x$.

Theorem 3 *Let f be a non-singular plane quintic.*
(1) $|\mathrm{Aut}(f)| \leq 150$.
(2) $|\mathrm{Aut}(f)| = 150$ *if and only if f is projectively equivalent to* $x^5 + y^5 + z^5$.

Theorem 4 ([1]) *Let f be a non-singular plane sextic.*
(1) $|\mathrm{Aut}(f)| \leq 360$.
(2) $|\mathrm{Aut}(f)| = 360$ *if and only if f is projectively equivalent to the Wiman sextic* $10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6$.

Theorem 5 *Let f be a non-singular plane septic.*
(1) $|\mathrm{Aut}(f)| \leq 294$.
(2) $|\mathrm{Aut}(f)| = 294$ *if and only if f is projectively equivalent to* $x^7 + y^7 + z^7$.

Our definitions and notaions are as follows. Let A, $B \in GL(3,k)$, and $f \in k[x_1, x_2, x_3]$. We define $f_A \in k[x_1, x_2, x_3]$ as $f_A(x_1, x_2, x_3) = f([x_1, x_2, x_3]({}^tA^{-1}))$ so that $(f_A)_B = f_{BA}$. Let G be a subset of the group $PGL(3,k)$ of projectivities of the projective plane P^2. A homogeneous $f \in k[x,y,z]$ is called G- invariant, if $f_A \sim f$ for any $(A) \in G$. More generally, let H be an abstract group. By abuse of notation we call f is H-invariant, if there is a subgroup G of $PGL(3,k)$ such that 1) G and H are isomorphic, and 2) f is G-invariant. For a homogeneous $f \in k[x_1.x_2, x_3]$ Hess(f) denotes the Hessian of f: $\mathrm{Hess}(f) = \det[\frac{\partial^2}{\partial x_i \partial x_j} f]$. It is well known that, if f is non-singular, then the intersection $f \cap h$ coincides with the set of all flexes. It is also known that $\mathrm{Aut}(f) \subset \mathrm{Aut}(\mathrm{Hess}(f))$. Finally $E_3 = [e_1, e_2, e_3]$ denotes the unit matrix of $GL(3,k)$, where e_j stands for the j-th column of E_3. When two quantities a and b such as functions and matrices, $a \sim b$ means that a and b are proportional.

The cases of cubics, quintics, and septics are discussed in §1, §2, and §3 respectively. Proofs are not given in princile to make our report short.

1 Cubics

In this section we will prove Theorem 1. We begin with

Theorem 1. 1 ([8], [6]) *Let* $f = x^n + y^n + z^n (n \geq 3)$. *Then* $|\mathrm{Aut}(f)| = 6n^2$.

Theorem 1. 2 *Let f be a non-singular plane cubic.*
(1) $|\mathrm{Aut}(f)| \leq 54$.
(2) $|\mathrm{Aut}(f)| = 54$ *if and only if f is projectively equivalent to* $x^3 + y^3 + z^3$.

Proof. As is known, f has a flex P. Without loss of generality we may assume that $P = (0,1,0)$ and that the tangent there is z. Namely $f(x,1,z) = z + 2z(ax+bz) + Ax^3 + Bx^2z + Cxz^2 + Dz^3$, or equivalently $f = y^2z + 2yz(ax+bz) + Ax^3 + Bx^2z + Cxz^2 + Dz^3$. Substituting y for $y + ax + bz$, we get $f = y^2z + Ax^3 + Bx^2z + Cxz^2 + Dz^3$. So we may assume that $f = y^2z + x^3 + Bx^2z + Cxz^2$. As can be seen easily, f is non-singular if and only if $C(B^2 - 4C) \neq 0$. Let $G_P = \{(A) \in \mathrm{Aut}(f);\ (A)P = P\}$, and assume $(A) \in G_P$. Since (A) fixes the tangent z at P as well, the rows of A take the form $[a_1, 0, c_1]$, $[a_2, 1, c_2]$, and $[0, 0, c_3]$ respectively up to constant multiplication. Since $f_{A^{-1}}$ contains none of monomials of degree 1 with respect to y, $a_2 = c_2 = 0$. Now $f_{A^{-1}} \sim f$, if and only if ${a_1}^3/c_3 = 1$, $3{a_1}^2c_1/c_3 + B{a_1}^2 = B$, $3a_1{c_1}^2/c_3 + 2a_1c_1B + a_1C/c_3 = C$ and ${c_1}^3/c_3 + {c_1}^2B + c_1c_3C = 0$. From the first and the second equalities of these four equalities, we get $c_3 = {a_1}^3$ and $c_1 = a_1(1 - {a_1}^2)B/3$. So the third equality can be written as $({a_1}^4 - 1)(-B^2/3 + C) = 0$. If $C \neq B^2/3$, then $|G_P| \leq 4$. If $C = B^2/3$, then the fourth equality can be written as $(1 - {a_1}^2)(1 + {a_1}^2 + {a_1}^4)B^3 = 0$. Note that $y^2z + x^3$ is singular. Hence, only when $C = B^2/3 \neq 0$, f is non-singular and $|G_P| = 6$. Since $|f \cap h| \leq 9$ by Bezout's theorem,

$$|\mathrm{Aut}(f)|/|G_P| = |\mathrm{Aut}(f)P| \leq 9.$$

So $|\mathrm{Aut}(f)| \leq 54$, and the equlity holds, if and only if $|G_P| = 6$ and $|\mathrm{Aut}(f)P| = 9$. We have shown that $|G_P| = 6$ if and only if $C = B^2/3 \neq 0$, namely $f = y^2z + x^3 + Bx^2z + B^2xz^2/3$ with $B \neq 0$, which is projectively equivalent to $f' = y^2z + x^3 + x^2x + xz^2/3$. Consequently, if there exists a non-singular cubic f with $|\mathrm{Aut}(f)| = 54$, then f is projectively equivalent to f'. This means the uniqueness of non-singular cubics satisfying $|\mathrm{Aut}(f)| = 54$. On the other hand there exists such a cubic by Theorem 1.1

2 Quintics

In this section we will specify the most symmetric non-singular quintics (Theorems 2.2 and 2.22).

Theorem 2. 1 (Hurwitz) *Denote by* $\mathrm{AUT}(C)$ *the holomorphic automorphism group of a compact Riemann surface* C *of genus* $g \geq 2$. *Let* $g' = g - 1$. *The posssible values of the order* $d = |\mathrm{AUT}(C)|$ *are*

$$\begin{array}{llllllll} 84g', & 48g', & 40g', & 36g', & 30g', & \frac{132}{5}g', & 24g', & \frac{156}{7}g', \\ 21g', & 20g', & \frac{96}{5}g', & \frac{56}{3}g', & \frac{204}{11}g', & 18g' & & \textit{or less.} \end{array}$$

Proof. The author of [5] cites values down to $36g'$. For our purposes, however, other possible values are necessary. The idea of the proof given below is entirely due to [5]. According to [5] there exist integers $\hat{g} \geq 0$, $s \geq 3$, and $m_1 \geq m_2 \geq \ldots \geq m_s \geq 2$ such that

$$2g' = d\{2(\hat{g} - 1) + \sum_{j=1}^{s}(1 - \frac{1}{m_j})\}.$$

If $\hat{g} \geq 2$, then $d \leq g'$. If $\hat{g} = 1$, then $d \leq 4g'$. Suppose $\hat{g} = 0$. Note that $2g' \geq d\{-2+s/2\}$. If $s \geq 5$, then $d \leq 4g'$. If $s = 4$, then $m_1 \geq 3$ so that $2g' \geq d\{-2+(1-1/3)+3/2\} = d/6$, namely $d \leq 12g'$. Assume $s = 3$.

Suppose $m_3 \geq 4$. Then $2g' \geq d(1-3/4) = d/4$, namely $d \leq 8g'$. Suppose $m_3 = 3$. Then $m_1 \geq 4$. If $m_1 \geq 5$, then $2g' \geq d(1-1/5-1/3-1/3) = 2d/15$, namely $d \leq 15g'$. If $m_1 = 4$ and $m_2 = 4$, then $2g' = d(1-1/2-1/3) = d/6$, namely $d = 12g'$. If $m_1 = 4$ and $m_2 = 3$, then $2g' = d(1-1/4-2/3) = d/12$, namely $d = 24g'$. Suppose $m_3 = 2$. Then $m_2 \geq 3$. If $m_2 \geq 6$, then $2g' \geq d(1-2/6-1/2) = d/6$, namely $d \leq 12g'$.
Let $m_2 = 5$. If $m_1 \geq 6$, then $2g' \geq d(1-1/6-1/5-1/2) = 2d/15$, namely $d \leq 15g'$.
If $m_1 = 5$, then $2g' = d(1-2/5-1/2) = d/10$, namely $d = 20g'$.
Let $m_2 = 4$. Then $m_1 \geq 5$. If $m_1 \geq 8$, then $2g' \geq d(1-1/8-1/4-1/2) = d/8$, namely $d \leq 16g'$.
If $m_1 = 7$, then $2g' = d(1-1/7-3/4) = 3d/28$, namely $d = 56g'/3$.
If $m_1 = 6$, then $2g' = d(1-1/6-3/4) = d/12$, namely $d = 24g'$.
If $m_1 = 5$, then $2g' = d(1-1/5-3/4) = d/20$, namely $d = 40g'$.
Let $m_2 = 3$. Then $m_1 \geq 7$. If $m_1 \geq 19$, then $2g' \geq d(1/6-1/19) = 13d/114$, namely $d \leq 228g'/13$.
If $m_1 = 18$, then $2g' = d(1/6-1/18) = 2d/18$, namely $d = 18g'$.
If $m_1 = 17$, then $2g' = d(1/6-1/17) = 11d/102$, namely $d = 204g'/11g'$.
If $m_1 = 16$, then $2g' = d(1/6-1/16) = 5d/48$, namely $d = 96g'/5$.
If $m_1 = 15$, then $2g' = d(1/6-1/15) = d/10$, namely $d = 20g'$.
If $m_1 = 14$, then $2g' = d(1/6-1/14) = 2d/21$, namely $d = 21g'$.
If $m_1 = 13$, then $2g' = d(1/6-1/13) = 7d/78$, namely $d = 156g'/7$.
If $m_1 = 12$, then $2g' = d(1/6-1/12) = d/12$, namely $d = 24g'$.
If $m_1 = 11$, then $2g' = d(1/6-1/11) = 5d/66$, namely $d = 132g'/5$.
If $m_1 = 10$, then $2g' = d(1/6-1/10) = d/15$, namely $d = 30g'$.
If $m_1 = 9$, then $2g' = d(1/6-1/9) = d/18$, namely $d = 36g'$.
If $m_1 = 8$, then $2g' = d(1/6-1/8) = d/24$, namely $d = 48g'$.
If $m_1 = 7$, then $2g' = d(1/6-1/7) = d/42$, namely $d = 84g'$.

Let f be a non-singular plane quintic, hence $C(f)$ is a compact Riemann surface of genus $g = 6$. From now on let $g' = g - 1 = 5$ throughout this section. Then possible values of $|\mathrm{Aut}(f)|$ are

$$84g' = 4\cdot 3\cdot 5\cdot 7,\ 48g' = 16\cdot 3\cdot 5,\ 40g' = 8\cdot 5^2,\ 36g' = 4\cdot 3^2\cdot 5,\ 30g' = 2\cdot 3\cdot 5^2 \text{ or less.}$$

We will prove the following theorem by showing that $|\mathrm{Aut}(f)|$ cannot be equal to none of $84g'$, $48g'$, $40g'$, and $36g'$.

Theorem 2. 2 *If f is a non-singular plane quintic, then* $|\mathrm{Aut}(f)| \leq 150$.

A proof of this theorem will be given after a series of lemmas and propositions.

Let ε be a primitive n-th root of $1 (n \geq 3)$. A cyclic subgroup G_n of order n in $PGL(3,k)$ is clearly conjugate to either $G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >$ or $G_{ij} =< (\mathrm{diag}[1,\varepsilon^i,\varepsilon^j]) >$

for some $1 \leq i < j \leq n-1$ satisfying the greatest common divisor $(i, j, n) = 1$.

Lemma 2. 3 *Let notations be as above. Suppose that $1 \leq i < j \leq n-1$, $1 \leq i' < j' \leq n-1$, and $(i, j, n) = (i', j', n) = 1$. Then G_{ij} is conjugate to $G_{i'j'}$ if and only if there exists an $1 \leq m \leq n-1$ with $(m, n) = 1$ and a permutation $\sigma \in S_3$ such that*

$$\mathrm{diag}[\varepsilon_{\sigma(1)}, \varepsilon_{\sigma(2)}, \varepsilon_{\sigma(3)}] \sim \mathrm{diag}[1, \varepsilon^{i'}, \varepsilon^{j'}],$$

where $[\varepsilon_1, \varepsilon_2, \varepsilon_3] = [1, \varepsilon^{im}, \varepsilon^{jm}]$.

Lemma 2. 4 *Let ε be a primitive 7-th root of 1. A subgroup G_7 of $PGL(3, k)$ is isomorphic to $\mathbf{Z}_7$ if and only if G_7 is conjugate to one of the following subgroups of $PGL(3, k)$: $G_{01} =< (\mathrm{diag}[1, 1, \varepsilon]) >$, $G_{12} =< (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >$, $G_{13} =< (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >$.*

Lemma 2. 5 *Let f_1, ..., f_n be non-zero homogeneous polynomials of the same degree such that $f_{jA} = \lambda_j f_j$ $(j = 1, 2, ..., n)$ for an $A \in GL(3, k)$ with mutually distinct λ_j. Then a linear combination $f = c_1 f_1 + ... + c_n f_n \neq 0$ satisfies $f_A = \lambda f$ for some $\lambda \in k$ if and only if $c_j \neq 0$ except for just one value of j.*

The following proposition implies that $|\mathrm{Aut}(f)| = 84g' = 4 \cdot 3 \cdot 5 \cdot 7$ is impossible for any non-singular quintic f.

Proposition 2. 6 *A $\mathbf{Z}_7$-invariant quintic has a singular point.*

Proof. Let ε be a primitive 7-th root of 1, and denote by $A_j (j = 1, 2, 3)$ the matrices $\mathrm{diag}[1, 1, \varepsilon]$, $\mathrm{diag}[1, \varepsilon, \varepsilon^2]$ and $\mathrm{diag}[1, \varepsilon, \varepsilon^3]$ respectively. Then a quintic satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \leq n \leq 6$ turns out to be singular. Indeed, let $f'(x, y, z)$ be a homogeneous polynomial of degree $d \geq 2$. Then $(1, 0, 0)$ is a singular point of $C(f)$, if and only if none of monomials x^d, $x^{d-1}y$ and $x^{d-1}z$ appears in f'. We summarize the values i such that $m_{A_j^{-1}} = \varepsilon^i m$ for each j and the special nine monomials m in the following table.

	x^5	x^4y	x^4z	y^5	y^4x	y^4z	z^5	z^4x	z^4y
(1)	0	0	1	0	0	1	5	4	4
(2)	0	1	2	5	4	6	3	1	2
(3)	0	1	3	5	4	0	1	5	6

From this table we can easily see that a quintic $C(f)$ satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \leq n \leq 6$ has a singular point $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$.

A finite group of order $48g'$ or $40g'$ contains a subgroup of order 8. Such a group is isomorphic to one of the following five groups [4, p.51–52]:
1) $\mathbf{Z}_8$
2) $\mathbf{Z}_2 \times \mathbf{Z}_4$
3) $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$
4) Q_8, which is generated by a and b such that $a^4 = 1$, $b^2 = a^2$, and $ba = a^{-1}b$.
5) D_8, which is generated by a and b such that $a^4 = 1$, $b^2 = 1$, and $ba = a^{-1}b$.

We may safely omit the proof of

Lemma 2. 7 *Let ε be a primitive 8-th root of 1. A subgroup G_8 of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_8$, if and only if G_8 is conjugate to one of the following 4 subgroups of $PGL(3,k)$:*

$$G_{01} =< (\text{diag}[1,1,\varepsilon]) >, \quad G_{12} =< (\text{diag}[1,\varepsilon,\varepsilon^2]) >,$$
$$G_{13} =< (\text{diag}[1,\varepsilon,\varepsilon^3]) >, \quad G_{14} =< (\text{diag}[1,\varepsilon,\varepsilon^4]) > .$$

Proposition 2. 8 *Let f be a $\mathbb{Z}_8$-invariant quintic.*
(1) f is non-singular if and only if it is projectively equivalent to $f' = x^5+Bx^3z^2+xz^4+y^4z$ with $B^2-4 \neq 0$.
(2) $|\text{Aut}(f')| \leq 148$.

Lemma 2. 9 *Let $p \neq 3$ be a prime and ε be a primitive p-th root of 1. Then a subgroup G of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ if and only if G is conjugate to $G_{p^2} =< (\text{diag}[1,\varepsilon,1]),\ (\text{diag}[1,1,\varepsilon]) >$.*

The following lemma is due to Hiroaki Taniguchi.

Lemma 2. 10 (Taniguchi) *Let p be a prime, let ε be a primitive p-th root of 1 and let G_{p^2} be as in Lemma 2.9. If $f(x,y,z)$ is a G_{p^2}-invariant homogeneous polynomial of degree d with $p \not| \ d$, then f is reducible.*

Proof. Let $A = \text{diag}[1,\varepsilon,1]$, and $B = \text{diag}[1,1,\varepsilon]$. Assume $f_A = \varepsilon^i f$ and $f_B = \varepsilon^j f$ for some $i,j \in \{0,1,...,p-1\}$. If $i > 0$, then y divides f. Similarly if $j > 0$, then z divides f. If $i = j = 0$,then x dives f, becasue f is a linear combination of monomials $x^{d_1}y^{d_2}z^{d_3}$ with $d_2 \equiv d_3 \equiv 0 \bmod p$ so that $d_1 = n - d_2 - d_2 n \not\equiv 0 \bmod p$.

Proposition 2. 11 *A $\mathbb{Z}_2 \times \mathbb{Z}_4$-invariant quintic is singular.*

Proof. A $\mathbb{Z}_2 \times \mathbb{Z}_4$-invariant quintic is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant quintic. Such a quintic is reducible by Lemma 2.9 and Lemma 2.10.

Proposition 2. 12 *No subgroup of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Lemma 2. 13 *Let G_8 be a subgroup of $PGL(3,k)$.*
(1) G_8 is isomorphic to Q_8 if and only if it is conjugate to

$$< (\text{diag}[1,\sqrt{-1},\sqrt{-1}^3]),\ ([e_1,e_3,e_2]\text{diag}[1,\sqrt{-1},\sqrt{-1}]) > .$$

(2) G_8 is isomorphic to D_8 if and only if it is conjugate to

$$< (\text{diag}[1,\sqrt{-1},\sqrt{-1}^3]),\ ([e_1,e_3,e_2]) > .$$

Proposition 2. 14 (1) *A Q_8-invariant quintic, if any, is singualr.*
(2) *A D_8-invariant quintic, if any, is singualr.*

A group of order $36g'$ contains a subgroup of order 9 by Sylow's theorem. Such a group is isomorphic to either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ [4]. By Lemma 2.3 we get

Lemma 2. 15 *Let ε be a primitive* 9*-th root of* 1*. A subgroup G_9 of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_9$, if and only if it is conjugate to one of the following three subgroups:*

$$G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >, \ G_{12} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^2]) >, \ G_{13} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^3]) > .$$

Proposition 2. 16 *A $\mathbb{Z}_9$-invariant quintic is singular.*

Lemma 2. 17 *Let ω be a primitive third root of* 1*. A subgroup G_9 of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ if and only if it is conjugate to one of the following two groups:*

$$G_{01} =< (\mathrm{diag}[1,1,\omega]), (\mathrm{diag}[1,\omega,1]) >, \quad G_{12} =< (\mathrm{diag}[1,\omega,\omega^2]), ([e_2,e_3,e_1]) > .$$

Proposition 2. 18 *A $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant quintic is singular.*

Proof of Theorem 2.2 Let f be a non-singular quintic, and let $d = |\mathrm{Aut}(f)|$. Recall that

$$84g' = 4 \cdot 3 \cdot 5 \cdot 7, \quad 48g' = 16 \cdot 3 \cdot 5, \quad 40g' = 8 \cdot 25, \quad 36g' = 4 \cdot 5 \cdot 9.$$

By Proposition 2.6 we get $d \neq 84g'$. The inequalities $d \neq 48g'$, $40g'$ follow from Propositions 2.8, 2.11, 2.12 and 2.14. Finally Propositions 2.16 and 2.18 imply $d \neq 36g'$.

We note that $30g' = 2 \cdot 3 \cdot 25$. A group of order 25 is isomorphic to $\mathbb{Z}_{25}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$ [4].

Lemma 2. 19 *Let ε be a primitive* 25*-th root of* 1*. A subgroup G_{25} of $PGL(3,k)$ is isomorphic to $\mathbb{Z}_{25}$ if and only if it is conjugate to one of the following subgroups:*

$$\begin{array}{lll} G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >, & G_{12} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^2]) >, & G_{13} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^3]) >, \\ G_{14} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^4]) >, & G_{15} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^5]) >, & G_{1,10} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^{10}]) > . \end{array}$$

Proof. By Lemma 2.3 we can classify subgroups $G_{ij} =< (\mathrm{diag}[1,\varepsilon^i,\varepsilon^j]) > (1 \leq i < j \leq 24$ with the greatest common divisor $(i,j,5) = 1)$ up to conjugacy, using computer.

Proposition 2. 20 *A $\mathbb{Z}_{25}$-invariant quintic is singualr.*

Proposition 2. 21 *A $\mathbb{Z}_5 \times \mathbb{Z}_5$-invariant non-singular quintic is projectively equivalent to $x^5 + y^5 + z^5$.*

Theorem 2. 22 *A non-singular quintic f satisfying $|\mathrm{Aut}(f)| = 150$ is projectively equivalent to $x^5 + y^5 + z^5$.*

Proof. Propositions 2.20 and 2.21 imply the theorem.

3 Septics

Let $g = 15$, the genus of non-singular plane septic(i.e. a curve of degree 7), and let $g' = g - 1 = 14$. By Theorem 1.1 $|\mathrm{Aut}(x^7 + y^7 + z^7)| = 21g'$. If f is a non-singular plane septic, then $|\mathrm{Aut}(f)|$ may take values

$$\begin{array}{llll} 84g' = 8 \cdot 3 \cdot 49, & 48g' = 32 \cdot 3 \cdot 7, & 40g' = 16 \cdot 5 \cdot 7, & 36g' = 8 \cdot 9 \cdot 7, \\ 30g' = 4 \cdot 3 \cdot 5 \cdot 7, & 24g' = 16 \cdot 3 \cdot 7, & \frac{156}{7}g' = 8 \cdot 3 \cdot 13, & 21g' = 2 \cdot 3 \cdot 49 \end{array}$$

or less by Theorem 2.1. The eight values above are multiples of 8 except for $30g'$ and $21g'$. As we remarked in §2, a group of order 8 is isomorphic to one of the following five groups: $\mathbf{Z}_8$, $\mathbf{Z}_2 \times \mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, Q_8 and D_8. No subgroup of $PGL(3, k)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ by Proposition 2.12. As for a quintic we have following Propositions 3.1 and 3.2

Proposition 3. 1 *A $\mathbf{Z}_8$-invariant septic is singular.*

Proposition 3. 2 *A $\mathbf{Z}_2 \times \mathbf{Z}_4$-invariant septic is singular.*

Proposition 3. 3 (1) *A Q_8-invariant septic, if any, is singular.*
(2) *A D_8-invariant septic, if any, is singular.*

Theorem 3. 4 *The maximum value of $|\mathrm{Aut}(f)|$ for a non-singular septic f is equal to either $30g'$ or $21g'$.*

Proof. By Propositions 3.1, 3.2 and 3.3 the order $|\mathrm{Aut}(f)|$ does not belong to $\{84g', 48g', 40g', 36g', 30g', 24g', \frac{156}{7}g'\} \setminus \{30g'\}$. Meanwhile $|\mathrm{Aut}(x^7 + y^7 + z^7)| = 21g'$ by Theorem 1.1.

We will show that $|\mathrm{Aut}(f)| \neq 30g'$ for any non-singular septic. Note that $30g' = 4 \cdot 3 \cdot 5 \cdot 7$. As we notice in the proof of Proposition 3.2,

Proposition 3. 5 *A $\mathbf{Z}_2 \times \mathbf{Z}_2$-invariant septic is singular.*

Suppose that there exists a non-singular septic f' such that $|\mathrm{Aut}(f')| = 30g'$. Denote by G' the finite group $\mathrm{Aut}(f')$. By Proposition 3.5 Sylow 2-group of G' is isomorphic to $\mathbf{Z}_4$. So we can apply the following theorem to G'.

Theorem 3. 6 ([4, p.146]) *If the Sylow subgroups of a finite group G of order n are all cyclic, then it is generated by two elements a and b with defining relations:*

$$\begin{array}{l} a^i = 1, \;\; b^j = 1, \;\; b^{-1}ab = a^r, \\ ij = n, \\ \gcd(i,\; (r-1)j) = 1, \\ r^j \equiv 1 \bmod i. \end{array}$$

For our group G' of order $420 = 4 \cdot 3 \cdot 5 \cdot 7$, possible pairs of $\{i, j\}$ in Theorem 3.6 are the followings(note that $\gcd(i, j) = 1$ if $r > 1$):

$$\{1, 420\}, \{4, 105\}, \{3, 140\}, \{5, 84\}, \{7, 60\}, \{12, 35\}, \{20, 21\}, \{28, 15\}.$$

In particular G' has an element of order 10, 12 or 15.

Lemma 3. 7 *Let ε be a primitive 10-th root of 1. A subgroup G_{10} of $PGL(3, k)$ is isomorphic to Z_{10} if and only if G_{10} is conjugate to one of the following subgroups:*

$$\begin{array}{ll} < (\mathrm{diag}[1, 1, \varepsilon]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, \\ < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) > . \end{array}$$

Proposition 3. 8 *A Z_{10}-invariant septic f is singular.*

Lemma 3. 9 *Let ε be a primitive 12-th root of 1. A subgroup G_{12} of $PGL(3, k)$ is isomorphic to Z_{12} if and only if G_{12} is conjugate to one of the following subgroups:*

$$\begin{array}{lll} < (\mathrm{diag}[1, 1, \varepsilon]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, \\ < (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) > . \end{array}$$

Proposition 3. 10 *If f is a Z_{12}-invaraiant non-singular septic, then* $|\mathrm{Aut}(f)| \neq 30g' = 420$.

Lemma 3. 11 *Let ε be a primitive 15-th root of 1. A subgroup G_{15} of $PGL(3, k)$ is isomorphic to Z_{15} if and only if it is conjugate to one of the following subgroups:*

$$\begin{array}{lll} < (\mathrm{diag}[1, 1, \varepsilon]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, \\ < (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) > . \end{array}$$

Proposition 3. 12 *A Z_{15}-invariant septic f is singular.*

Theorem 3. 13 $|\mathrm{Aut}(f)| \leq 21g' = 294$.

Proof. Propositions 3.8, 3.10, and 3.12 imply that $|\mathrm{Aut}(f)|$ cannot be equal to $30g'$. By Theorem 3.4 we get the desired inequality.

Finally we will show that non-singular septics f with $|\mathrm{Aut}(f)| = 21g' = 2 \cdot 3 \cdot 49$ are unique.

Lemma 3. 14 *Let ε be a primitive 49-th root of 1. A subgroup G_{49} of $PGL(3, k)$ is isomorphic to Z_{49}, if and only if it is conjugate to one of the following subgroups:*

$$\begin{array}{llll} < (\mathrm{diag}[1, 1, \varepsilon]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >, \\ < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^7]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^{14}]) >, \\ < (\mathrm{diag}[1, \varepsilon, \varepsilon^{18}) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^{19}]) >, & < (\mathrm{diag}[1, \varepsilon, \varepsilon^{21}]) > . & \end{array}$$

Proof. In view of Lemma 2.3 we can classify subgroups $< (\mathrm{diag}[1, \varepsilon^i, \varepsilon^j]) > (1 \leq i < j \leq 48)$ up to conjugacy, using computer.

Proposition 3. 15 *A* Z_{49}*-invariant septic* f *is singular.*

Proposition 3. 16 *A* $\mathrm{Z}_7 \times \mathrm{Z}_7$*-invariant septic* f *is non-singular if and only if* f *is projectively equivalent to* $x^7 + y^7 + z^7$.

Proof. Let $A = \mathrm{diag}[1, 1, \varepsilon]$ and $B = \mathrm{diag}[1, \varepsilon, 1]$. By Lemma 2.9 a subgroup G of $PGL(3, k)$ is isomorphic to $\mathrm{Z}_7 \times \mathrm{Z}_7$, if and only if G is conjugate to $< (A), (B) >$. A septic f satisfying $f_{A^{-1}} = \varepsilon^i f$ and $f_{B^{-1}} = \varepsilon^j f$, if any, is a singular except for the case $i = j = 0$. In the exceptional case f is a linear combination of x^7, y^7 and z^7.

Theorem 3. 17 *A non-singular plane septic* f *with* $|\mathrm{Aut}(f)| = 21g' = 2 \cdot 9 \cdot 2$ *is projectively equivalent to* $x^7 + y^7 + z^7$.

Proof. The theorem is a trivial consequence of Propositions 3.15 and 3.16.

Acknowledgement At the invitation of Professor G. Faina the first author was a visiting researcher at Department of Mathematics of Perugia University in September 1998, when this paper was written in collaboration with his collegues. The first author would like to express his sincere thanks to Professor Faina for his hospitalities and conveniences he generously offered.

We would like to thank Prof. S. Yoshiara at Osaka Kyoiku University, who suggested us the references concerning Theorem 3.6.

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