Analysis of Colored Symmetrical Patterns
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Introduction
The study and classification of colored symmetrical patterns continues to be of interest in color symmetry today. A meaningful analysis of colored symmetrical patterns involves the symmetry group $G$ of the uncolored pattern as well as the symmetry group $K$ of the pattern when it is colored. In certain instances, not all elements of $G$ permute the colors and we also consider the subgroup $H$ of elements of $G$ which effect color permutations. This subgroup $H$ contains $K$ as a normal subgroup of elements of $H$ which fix the colors.

A coloring of a symmetrical pattern may be perfect or non-perfect. Perfect colorings occur whenever all the elements of $G$ permute the colors that is, $H = G$; otherwise we have non-perfect colorings.

Perfect colorings have been studied extensively before in [9]. The problem however lies on how to study non-perfect colorings systematically. In the paper “A Framework for Coloring Symmetrical Patterns” by De Las Peñas, Felix and Quilinguin[1], a framework was presented for analyzing both perfect and non-perfect colorings. Moreover, using the framework, all colorings of a symmetrical pattern were determined for which the elements of a given subgroup $H$ of the symmetry group $G$ of the uncolored pattern permute the colors and the elements of a given subgroup $K$ of $G$ fix the colors. In this paper, we shed more light to the study of perfect and non-perfect colorings by giving an alternative proof of this result. For the colorings obtained using the framework, we also find the subgroup $H^*$ consisting of elements of $G$ permuting the colors and the subgroup $K^*$ consisting of elements of $G$ fixing the colors. In [1], the case where the index of $H$ in $G$ is a prime $p$ was considered. In this paper, we present an additional situation where the index of $H$ in $G$ is not prime. Specifically we look at the case where the index of $H$ in $G$ is the smallest composite 4.

Setting for Coloring Symmetrical Patterns
We first explain the setting in which we will color symmetrical patterns. Consider $G$ to be the symmetry group of an uncolored pattern. We start with a fundamental domain for $G$
and a subset $R$ of this fundamental domain. The set $\{g(R) : g \in G\}$ will be referred to as the $G$-orbit of $R$. We assume that the given pattern can be obtained as the $G$-orbit of some subset $R$ of a fundamental domain for $G$. Then the assignment $g \mapsto g(R)$ defines a one-to-one correspondence between the group $G$ and the $G$-orbit of $R$. We then can label the set $g(R)$ by $g$ and by giving a color to each $g \in G$, we give a color to each set $g(R)$. This assignment of colors is what we will call a coloring of the pattern. Since this results in a partition of $G$ wherein the elements assigned the same color form one set in the partition, a coloring may be treated as simply a partition of the group $G$ or a decomposition of $G$ into non-empty disjoint subsets. Hence, a coloring of a pattern with symmetry group $G$ will be equivalent to a partition of $G$ or a decomposition of $G$.

We give an example which will illustrate the above concepts. Consider the uncolored pattern in Figure 1.1 which has symmetry group $G = D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ where $a$ is a $60^\circ$-counterclockwise rotation about the center of the hexagon and $b$ is a reflection in the horizontal line through the center of the hexagon. If $R$ is the triangular region labeled “e” in Figure 1.2, then for each $g \in G$, the triangular region $g(R)$ is labeled “g”. Let us partition $G$ into the sets $\{e, a^2, a^4, ab, a^3b, a^5b\}$, and $\{a, a^3, a^5, b, a^2b, a^4b\}$, and assign white and black to the first and second sets respectively. Consequently, we obtain the coloring in Figure 1.3.

In the analysis of a coloring, three groups play a significant role. These groups are:

$G =$ symmetry group of the uncolored pattern

$H =$ subgroup of elements of $G$ which permute the colors

$K =$ subgroup of elements of $G$ which fix the colors

We will refer to $H$ as the subgroup of color transformations and $K$ as the symmetry group of the colored pattern. The groups $G, H, K$ are such that $K \leq H \leq G$. Given a color, its stabilizer in $G$ will lie between $H$ and $K$. Since $H$ acts on the set $C$ of colors of the pattern, this action induces a homomorphism $f : H \to A(C)$, where $A(C)$ is the group of permutations of the set $C$ of colors of the pattern. For $h \in H$, $f(h)$ is the permutation of the colors that $h$ induces. An element $h$ is in the kernel of $f$ if and only if $f(h)$ is the identity permutation, that is, $h$ fixes all the colors. Thus the kernel of $f$ is $K$ and the resulting group of color permutations $f(h)$ is isomorphic to $H/K$. Consequently, $K$ is a normal subgroup of $H$. 
Enumerating Colorings of Symmetrical Patterns

In this part of the paper, we determine all colorings of an uncolored pattern with symmetry group $G$ such that the elements of a given subgroup $H$ of $G$ permute the colors and the elements of a given subgroup $K$ of $G$ fix the colors where $K \leq H \leq N_G(K)$.

The assumptions we are to consider in determining the colorings will be as follows. Let $G$ be a group and $H$ a subgroup of $G$. Let $P$ be a partition of $G$. Since a partition of $G$ corresponds to a coloring, we refer to the set $P$ as the set of colors.

**Definition 1** Let $G$ be a group, $H \leq G$, $Y$ a complete set of right coset representatives of $H$ in $G$, $\bigcup_{i \in I} Y_i$ a decomposition of $Y$ and for each $i \in I$, $J_i \leq H$. Then the coloring or decomposition $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_i Y_i$ or the partition of $G$, $P = \{hJ_i Y_i : i \in I, h \in H\}$ is called a $(Y_i, J_i)$-$H$ coloring.

**Lemma 2** A $(Y_i, J_i)$-$H$ coloring defines an $H$-invariant partition of $G$.

**Proof.** If $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_i Y_i$ is a $(Y_i, J_i)$-$H$ coloring, then it defines an $H$-invariant partition since for $h' \in H$, $h'G = \bigcup_{i \in I} \bigcup_{h \in H} h'J_i Y_i = \bigcup_{i \in I} \bigcup_{h \in H} hJ_i Y_i$ since premultiplication by $h' \in H$ simply permutes the elements of $H$.

Also, if $K \leq G$ such that $H \leq N_G(K)$ and $K \leq J_i$ for each $i$, then the elements of $K$ fix each of the sets $hJ_i Y_i$ because if $k \in K$ then $khJ_i Y_i = hkJ_i Y_i = hJ_i Y_i$.

**Lemma 3** If $P = \{P_i : i \in I\}$ is a $G$-invariant partition of the group $G$, then $P$ is the partition of $G$ consisting of left cosets of some subgroup $S$ of $G$. This subgroup is the set in the partition containing $e$. Moreover, the subgroup of elements of $G$ fixing $P = \{P_i : i \in I\}$ is $\text{core}_G S$.

**Proof.** Let $e \in P_1$ and $P_1$ an arbitrary element of $P$. If $g \in P_1$, then $g^{-1}g \in g^{-1}P_1$ and $e \in g^{-1}P_1$. Thus, $g^{-1}P_1 = P_1$ or $P_1 = gP_1$. This means that any element of $P$, $P_1$, can be expressed as $gP_1$ for some $g \in P_1$. If we can show that $P_1$ is a subgroup of $G$, then we are done. Now, $g \in G_{P_1}$, the stabilizer of $P_1$ under left multiplication by elements of $G \Leftrightarrow gP_1 = P_1 \Leftrightarrow g \in P_1$ because $e \in P_1$. Thus, $P_1$ is the stabilizer of $P_1$ and $P_1$ is a subgroup of $G$.

If we consider $a \in G$, and take any $P_1$ of $P$ where $P_1 = gP_1$ for some $g \in P_1$, $a$ fixes $P_1 = gP_1$ or $a(gP_1) = g(P_1)$ if and only if $(g^{-1}ag)P_1 = P_1$ so that $g^{-1}ag \in P_1$ and $a \in gP_1g^{-1}$. Thus the subgroup of elements of $G$ fixing the colors in $\text{core}_G P_1$. ■
Lemma 4 Let $G$ be a group, $X$ a non-empty subset of $G$ and $K$ a subgroup of $G$. Then $kX = X$ for all $k$ in $K$ if and only if $X$ is a union of right cosets of $K$ in $G$.

Proof. Assume $kX = X$ for all $k$ in $K$. Then $X = \bigcup_{x \in X} \{x\}$ is contained in $\bigcup_{x \in X} k_x$. Now $a \in \bigcup_{x \in X} K_x$ implies $a = kx$ for some $k \in K$ and $x \in X$. Therefore $a \in X$. Hence $X = \bigcup_{x \in X} k_x$.

On the other hand, if $X$ is a union of right cosets of $K$ in $G$, say $X = \bigcup_{g \in A} k_g$, where $A$ is a subset of $G$, then $kX = \bigcup_{g \in A} k_gk = \bigcup_{g \in A} k_g = X$. ■

Theorem 5 Let $G$ be a group and $H$ a subgroup of $G$. If $P$ is an $H$-invariant partition of $G$, then $P$ corresponds to a decomposition of $G$ in the form $G = \bigcup_{i \in I} hJ_i Y_i$ where $\bigcup_{i \in I} Y_i = Y$ is a complete set of right coset representatives of $H$ in $G$ and $J_i \leq H$ for every $i \in I$. If in addition $K \leq H$ and $K$ fixes the elements of $P$, then $K \leq J_i$ for every $i \in I$.

Proof. Since $P$ is an $H$-invariant partition of $G$, $H$ acts on $P$ by left multiplication. Consider the orbits under the action of $H$. Let $C_i$ be a color in the $i$th orbit. Moreover, let $J_i$ be the stabilizer in $H$ of $C_i$ so that $J_i C_i = C_i$. By Lemma 4, $C_i$ is a union of right cosets of $J_i$, say $C_i = J_i Y_i$ where $Y_i$ is a set consisting of one representative for each right coset of $J_i$ contained in $C_i$. Hence the $i$th orbit is the set $\{hJ_i : h \in H\}$. So $G = \bigcup_{i \in I} hJ_i Y_i$. Note that $\bigcup_{h \in H} hJ_i Y_i = \bigcup_{h \in H} hJ_i Y_i = H Y_i$ so that $G = \bigcup_{i \in I} H Y_i$. This implies that $Y = \bigcup_{i \in I} Y_i$ is a complete set of right coset representatives of $H$ in $G$. If $K \leq H$ and $K$ fixes all elements of $P$ then $K$ fixes $C_i$. This means that $K \leq J_i$. ■

The above theorem characterizes all partitions of a group $G$ which are invariant under multiplication on the left by elements of a subgroup $H$ of $G$ and whose elements are left fixed by multiplication on the left by elements of a subgroup $K$ of $H$. It should be mentioned that distinct complete sets of coset representatives of $H$ in $G$ may give rise to the same partition. This situation is addressed in [1].

The Subgroup $H^*$ Permuting the Colors and the Subgroup $K^*$ Fixing the Colors

Based on the previous theorem, we have determined all colorings of an uncolored pattern with symmetry group $G$ such that the elements of a subgroup $H$ of $G$ permute the colors and the
elements of a subgroup $K$ of $G$ fix the colors. The next step is to actually determine for these colorings the subgroup $H^*$ consisting of elements of $G$ permuting the colors and the subgroup $K^*$ of elements of $G$ fixing the colors. At this point, all we can say is that $H$ is contained in $H^*$ and $K$ is contained in $K^*$.

In the next theorem, given a $(Y_i, J_i) - H$ coloring, we establish the condition for determining when a coloring is perfect, that is, $H^* = G$ and for the special case where $[G : H] = p$ we compute for $K^*$.

1. The subgroup $H^*$ permuting the colors.

**Theorem 6** Let $G$ be a group, $H \leq G$, $Y$ a complete set of right coset representatives of $H$ in $G$, $\bigcup_i Y_i$ a decomposition of $Y$ and for each $i \in I$, $J_i \leq H$. If $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_i Y_i$ is a given $(Y_i, J_i) - H$ coloring, then this coloring is perfect if and only if $J_i Y_i$ is a subgroup of $G$ and for each $i, \in I$ there is a $y_i \in Y_i$ such that $y_i J_i Y_i = J_i Y_i$.

**Proof.** Assume the coloring is perfect. Then each set $hJ_i Y_i$ is a left coset of some subgroup of $G$. This subgroup is the set $hJ_i Y_i$ containing $e$ which is $J_i Y_i$. Therefore, $J_i Y_i$ is a subgroup of $G$. Let $y_i \in Y_i$. Then $y_i J_i Y_i$ is one of the sets $hJ_i Y_i$ since the coloring is perfect. This set is $J_i Y_i$ since $y_i$ is in this set. Hence $y_i J_i Y_i = J_i Y_i$. Conversely, assume $J_i Y_i$ is a group of $G$ and for each $i \in I$ there is a $y_i \in Y_i$ such that $y_i J_i Y_i = J_i Y_i$. Then $hJ_i Y_i = h y_i J_i Y_i$ is a left coset of the subgroup $J_i Y_i$. Hence the coloring is perfect since all elements of $G$ permute the left cosets.

The next theorem looks at $H^*$ when there is only one orbit of colors under the action of $H$.

**Theorem 7** Let $G$ be a group, $H \leq G$, $Y$ a complete set of right coset representatives of $H$ in $G$, $e \in Y$, and $J \leq H$. Let $P = \{hJY : h \in H\}$ be a coloring and $H^*$ the subgroup of $G$ consisting of all elements of $G$ which permute the colors. Let $Y' \subseteq Y$.

(i) If $H^* = HY'$ then $y' JY = JY$ for all $y' \in Y'$.

(ii) If $y' \in N_G(H)$ and $y' JY = JY$ for all $y' \in Y'$ then $HY' \subseteq H^*$.

**Proof.** (i) Assume $H^* = HY'$. Since $y' \in Y' \subseteq HY'$, then $y'$ permutes the sets in $P$ and $y' JY$ is the set in $P$ containing $y'$. This set is $JY$, hence $y' JY = JY$. 

(ii) Assume $y' \in N_G(H)$ and $y'JY = JY$ for all $y' \in Y'$. We show $y'$ permutes the sets in $P$. Now, $y' \in N_G(H)$ implies that if $h \in H$, there is an $h' \in H$ such that $y'h = h'y'$. Hence $y'hJY = h'y'JY = h'JY$. Thus for all $y' \in Y'$, $y'$ permutes the elements in $P$. Since $H$ permutes the elements in $P$, so does $HY'$. Therefore, $HY' \subseteq H$. ■

In the following corollary, we specialize Theorem 7 to the case where the index of $H$ in $G$ is 4.

**Corollary 8** Let $G$ be a group, $H \leq G$ such that $[G : H] = 4$, $Y = \{y_1 = e, y_2, y_3, y_4\}$ a complete set of right coset representatives of $H$ in $G$ and $J \leq H$. Suppose $P = \{hJY : h \in H\}$ is the given coloring or partition.

(i) The coloring is perfect if and only if $JY$ is subgroup of $G$.

(ii) If $H^* \neq G$ then for $i = 2, 3, 4$, $H^* = H \cup Hy_i$ if and only if $H \cup Hy_i$ is a subgroup of $G$ and $y_iJY = JY$. Otherwise $H^* = H$.

**Proof.** (i) This is a consequence of Theorem 6 where $JY = J_1Y_1$.

(ii) This follows from Theorem 7 since $H$ is a normal subgroup of $H \cup Hy_i = H \{e, y_i\}$ when $H \cup Hy_i$ is a subgroup of $G$. ■

2. The subgroup $K^*$ fixing the colors

Now that we have established for certain cases the condition for determining $H^*$, the subgroup of $G$ consisting of elements of $G$ that permute the colors of the corresponding colored pattern, we can give for these cases the formulas for $K^*$, the subgroup of $G$ consisting of the elements of $G$ fixing the colors. Notice that $K^*$ is a subgroup of $H^*$ so that in determining $K^*$ we consider only the elements of $H^*$.

**Theorem 9** Let $G$ be a group, $H \leq G$ such that $[G : H] = p$ where $p$ is prime, $Y$ a complete set of right coset representatives of $H$ in $G$, $\bigcup_{i=1}^{t} Y_i$ a decomposition of $Y$ and for each $i \in \{1, 2, ..., t\}$, $J_i \leq H$. Suppose $G = \bigcup_{i=1}^{t} \bigcup_{h \in H} hJ_iY_i$ is a given $(Y_i, J_i)$-H coloring.

(i) If the coloring is perfect then $K^* = \text{core}_G(J_1Y_1)$.

(ii) If the coloring is non-perfect then $K^* = \bigcap_{i \in I} \text{core}_H(J_i)$.

**Proof.** (i) If the coloring is perfect, then the given $(Y_i, J_i)$-H coloring partitions $G$ into the sets of left cosets of $J_1Y_1$ in $G$. It follows that $K^* = \text{core}_G(J_1Y_1)$. 


(ii) On the other hand, if the coloring is non-perfect, then the subgroup \( H^* \) permuting the set of colors is \( H \) since \( |G : H| = p \) and \( H \leq H^* \leq G \) implies \( H^* = H \) or \( H^* = G \). Thus, in determining \( K^* \) we consider only elements of \( H \). Let \( a \in K^* \). Then \( ahJ_iY_i = hJ_iY_i \) for \( h \in H \), for all \( i \in \{1, 2, ..., t\} \). This implies that if \( Y_i = \{y_{i_1}, y_{i_2}, ..., y_{i_r}\} \), then \( ahJ_iy_{i_1} \cup ahJ_iy_{i_2} \cup ... ahJ_iy_{i_r} = hJ_iy_{i_1} \cup hJ_iy_{i_2} \cup ... hJ_iy_{i_r} \). Now \( a \in H \) so that \( ahJ_iy_{i_1} \subseteq H y_{i_1} \), \( ahJ_iy_{i_2} \subseteq H y_{i_2} \), ..., \( ahJ_iy_{i_r} \subseteq H y_{i_r} \). Since \( a \) fixes every color, then \( hJ_iy_{i_1} \) to itself in \( H y_{i_1} \), \( hJ_iy_{i_2} \) to itself in \( H y_{i_2} \) and so on. Thus for \( a \in K^* \), \( ahJ_iy_{i_1} = hJ_iy_{i_1} \) for all \( i \in \{1, 2, ..., t\} \), \( j \in \{1, 2, ..., r\} \). But \( ahJ_iy_{i_1} = hJ_iy_{i_1} \) implies \( a \in hJ_i \) or \( a \in hJ_ih^{-1} \) for \( h \in H \). That is, \( a \in \bigcap_{h \in H} hJ_ih^{-1} = \text{core}_H(J_i) \). Therefore \( K^* \subseteq \text{core}_H(J_i) \). The proof of the inclusion \( \bigcap_{i \in \{1, 2, ..., r\}} \text{core}_H(J_i) \subseteq K^* \) is straightforward. ■

**Theorem 10** Let \( G \) be a group, \( J \leq H \leq G \), \( Y \) a complete set of right coset representatives of \( H \) in \( G \) containing \( e \) and \( Y' \) a subset of \( Y \) containing \( e \). Let \( P = \{hJY : h \in H\} \) be a partition of \( G \). If \( H^* = HY' \) then \( K^* = \text{core}_{HY'}(JY') \).

**Proof.** Since \( H^* = HY' \), we limit our attention to \( H^* \). Now \( H^* \cap JY = JY' \) and the partition \( P \) induces the partition \( P^* = \{hJY' : h \in H\} \) on \( H^* \). Since \( P \) is \( H^* \)-invariant, it follows that \( P^* \) is \( H^* \)-invariant. Hence the induced coloring \( P^* \) is a perfect coloring and \( JY' \) is a subgroup of \( H^* \). Correspondingly, the subgroup of \( H^* \) fixing all the sets or colors in \( P^* \) is \( \text{core}_{H^*}(JY') \). Consequently, this is also the subgroup of elements of \( H^* \) which fix the sets in \( P \), that is, \( K^* = \text{core}_{HY'}(JY') \). ■

**Corollary 11** Let \( G \) be a group, \( H \leq G \) such that \( [G : H] = 4 \), \( Y = \{y_1 = e, y_2, y_3, y_4\} \) a complete set of right coset representatives of \( H \) in \( G \) and \( J \leq H \). Suppose \( P = \{hJY : h \in H\} \) is the given coloring or partition.

(i) If the coloring is perfect then \( K^* = \text{core}_G(JY) \).

(ii) If \( H^* = H \) then \( K^* = \text{core}_H J \).

(iii) If \( H^* = H \cup Hy_i \) then \( K^* = \text{core}_{H \cup Hy_i}(J \cup Jy_i) \) for \( i = 2, 3, 4 \).

**Proof.** We obtain (i), (ii) and (iii) by taking \( Y' = Y \), \( \{e\} \) and \( \{e, y_i\} \) in Theorem 10 respectively. ■

We conclude the section by looking at the following examples. An illustration of Corollary 11 is given below.
Example 12 Let $G = D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ and $H, K$ subgroups of $G$ given by $H = \{e, a^2, a^4\}$, $K = \{e\}$.

Now $G = H \cup Hb \cup Ha \cup Hab$

$G = \{e, a^2, a^4\} \cup \{b, a^2b, a^4b\} \cup \{a, a^3, a^5\} \cup \{ab, a^3b, a^5b\}$ Among the possible $Y's$ are \{e, a^3, b, a^3b\}, \{e, a, ab, a^3b\}, \{e, a^5, a^2b, a^3b\}$ and $\{e, a, a^4b, a^5b\}$.

We give some colorings $P = \{hJY : h \in H\}$ of the hexagon in Figure 2 such that the elements of $H$ permute the colors and the elements of $K = \{e\}$ fix the colors. In the table below, we give $H^*$ and $K^*$ as well as the $Y$ used for each of the colorings. Note that for all colorings $J = \{e\}$ so that $JY = Y$. We use the following notation: $w$ for white, $s$ for striped and $b$ for black.

<table>
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<th>H</th>
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<th>Hab</th>
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<td>s</td>
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<table>
<thead>
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<th>Y used</th>
<th>H*</th>
<th>K*</th>
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</thead>
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<tr>
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<td>$Y = {e, a^3, b, a^3b}$</td>
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<td>${e, a^3}$</td>
</tr>
<tr>
<td>2</td>
<td>$Y = {e, a, ab, a^4b}$</td>
<td>$H$</td>
<td>$K$</td>
</tr>
<tr>
<td>3</td>
<td>$Y = {e, a^5, a^2b, a^3b}$</td>
<td>${e, a^2, a^4, b, a^2b, a^4b}$</td>
<td>$K$</td>
</tr>
<tr>
<td>4</td>
<td>$Y = {e, a, a^4b, a^5b}$</td>
<td>${e, a^2, a^4, ab, a^3b, a^5b}$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

Example 13 Consider the colored patterns in Figures 3, 4, 5, 6, 7, 8 which are assumed to repeat over the entire plane. For all the colored patterns, the symmetry group $G$ of the patterns with the colors disregarded is a hexagonal plane crystallographic group of type $p6m$ generated by $a, b, x$ and $y$ where $a$ is a $60^\circ$ - counterclockwise rotation about the indicated point $P$, $b$ is a reflection in a horizontal line through $P$ and $x, y$ are translations as indicated. These colored patterns have been obtained by choosing the subgroups $H = <a, x, y>$ and $K = <a^2, x, y>$ of
G. $H$ and $K$ are hexagonal plane crystallographic groups of types $p6$ and $p3$ respectively. $K$ is normal in $G$ so that $G = N_G(K)$. Observe that the colorings in Figure 7 and Figure 8 are the only non-perfect colorings, that is, $H^* = H$. Moreover, for these colorings, $K^* = K$. All the other colorings are perfect, so that $H^* = G$. For the perfect colorings in Figures 3, 4, 5, 6, $K^* = H, <a^2, b, x, y>, <a^2, ab, x, y> and K$ respectively. $<a^2, b, x, y > and <a^2, ab, x, y>$ are hexagonal plane crystallographic groups of types $p31m$ and $p3m1$ respectively.

References


