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Projective Planes and Hadamard Designs

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1. Definition

Let $\mathcal{D}$ be a symmetric (v, k, λ) design, then $4n-1 \leq v \leq n^2+n+1$, where $n = k-\lambda$(the order of $\mathcal{D}$). If $v = 4n-1$ (therefore $k = 2n-1, \lambda = n-1$), then $\mathcal{D}$ is a Hadamard design of order n. If $v = n^2+n+1$(therefore $k = n+1, \lambda = 1$), then $\mathcal{D}$ is a projective plane of order n.

In the note, we show that there are some relations between projective planes with some small automorphism group and Hadamard designs.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order n, where $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set. Then an automorphism φ of Π is a *perspectivity* if φ fixes all poins on a line l of Π and all lines through a point P of Π. The line l is called the *axis*, and the point P the *center* of φ. If $P \in l$, we call φ an *elation*, and if $P \notin l$, we call φ a *homology*. Clearly, if φ is an elation, then $o(\varphi)|n$, and if φ is a homology, then $o(\varphi)|n-1$.

2. The Elation Type

Throughout this section we assume the following.

HYPOTHESIS 2.1. *Let* $\Pi = (\mathcal{P}, \mathcal{L})$ *be a projective plane of even order* $n(\geq 8)$. *Let* G *be an elation group of order* $\frac{1}{2}n$ *of* Π *with a common center* P_0 *and a common axis* l_0.

For example, all translation planes of even order $2^e(\geq 8)$ satisfy the hypothesis.

Let $(P_0) = \{l_0, l_1, \cdots, l_n\}$(the set of lines through the point P_0) and $(l_0) = \{P_0, P_1, \cdots, P_n\}$ (the set of points on the line l_0). Let $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{3n}$ be G-orbits on $\mathcal{P}$ and $\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_{3n}$ G-orbits on $\mathcal{L}$. Then we may assume the following:

$$\mathcal{P}_i = \{P_i\}, \quad \mathcal{L}_i = \{l_i\} \quad (0 \le i \le n),$$
$$|\mathcal{P}_i| = |\mathcal{L}_i| = \frac{1}{2}n \quad (n+1 \le i \le 3n),$$
$$(l_i) = \mathcal{P}_{n+i} \cup \mathcal{P}_{2n+i}, \quad (\mathcal{P}_i) = \mathcal{L}_{n+i} \cup \mathcal{L}_{2n+i} \quad (1 \le i \le n).$$

If Ω is a point orbit and Δ is a line orbit of G, set $(\Omega\ \Delta) = |\Omega \cap (l)|$, where l is a line in Δ. Here we remark that the number $(\Omega\ \Delta)$ depends only on Ω and Δ, not on l. For a subset H of G, set $\widehat{H} = \sum_{\mu \in H} \mu (\in Z[G])$ and $H^{-1} = \{\mu^{-1} | \mu \in H\}$.

Set $m_{ij} = (\mathcal{P}_j\ \mathcal{L}_i)$ for $i, j, 0 \le i, j \le 3n$, $M = (m_{ij})_{0 \le i,j \le 3n}$ and $L = (m_{ij})_{n+1 \le i,j \le 3n} = (l_{ij})_{0 \le i,j \le 2n-1}$. Choose a point $P_i \in \mathcal{P}_i$ and a line $l_i \in \mathcal{L}_i$ for $i, n+1 \le i \le 3n$. Set $D_{ij} = \{\mu \in G | P_j{}^{\mu} \in (l_i)\}$ for $i, n+1 \le i \le 3n$. Clearly $|D_{ij}| = m_{ij}$.

LEMMA 2.2. *Let* $n+1 \le i, i' \le 3n$. *Then* $\sum_{n+1 \le j \le 3n} \widehat{D_{ij}^{-1}} \widehat{D_{i'j}}$

$$= \begin{cases} n & \text{if } i = i', \\ 0 & \text{if } \{i, i'\} \in \{\{n+1, 2n+1\}, \{n+2, 2n+2\}, \cdots, \{2n, 3n\}\}, \\ \widehat{G} & \text{otherwise.} \end{cases}$$

Proof. Set $\Phi = \sum_{n+1 \le j \le 3n} \widehat{D_{ij}^{-1}} \widehat{D_{i'j}}$. Assume that $i = i'$. Then $\Phi = \sum_{n+1 \le j \le 3n, P_j{}^{\mu} \in (l_i), \mu \in G} \mu^{-1}\mu = |(\mathcal{P}_{n+1} \cup \mathcal{P}_{2n+1}) \cap (l_i)| + |(\mathcal{P}_{n+2} \cup \mathcal{P}_{2n+2}) \cap (l_i)| + \cdots + |(\mathcal{P}_{2n} \cup \mathcal{P}_{3n}) \cap (l_i)| = n$.

Assume that $\{i, i'\} \in \{\{n+1, 2n+1\}, \{n+2, 2n+2\}, \cdots, \{2n, 3n\}\}$. Then there don't exist $\mu, \xi \in G$ such that $P_j{}^{\mu} \in (l_i)$ and $P_j{}^{\xi} \in (l_{i'})$. Therefore $\Phi = \sum_{n+1 \le j \le 3n, P_j{}^{\mu} \in (l_i), P_j{}^{\xi} \in (l_{i'}),\ \mu, \xi \in G} \mu^{-1}\xi = 0$.

Assume that $i \ne i'$ and $\{i, i'\} \notin \{\{n+1, 2n+1\}, \cdots, \{2n, 3n\}\}$. Let $\eta \in G$. Then there exists $(j, \xi) \in \{n+1, n+2, \cdots, 3n\} \times G$ such that $l_i{}^{\eta} \cap l_{i'} = P_j{}^{\xi}$. Set $\mu = \xi\eta^{-1}$. Since $P_j{}^{\mu} = P_j{}^{\xi\eta^{-1}} \in (l_i)$ and $P_j{}^{\xi} \in (l_{i'})$, we get $\mu \in D_{ij}, \xi \in D_{i'j}$ and $\eta = \mu^{-1}\xi$. Next suppose that $\mu' \in D_{ij'}, \xi' \in D_{i'j'}$ and $\eta = \mu'^{-1}\xi'$. These yield $l_i{}^{\eta} \cap l_{i'} = P_{j'}{}^{\xi'}$. Therefore $P_j{}^{\xi} = P_{j'}{}^{\xi'}$. Hence $j = j', \xi = \xi'$ and $\mu = \mu'$. Thus $\Phi = \widehat{G}$.

We get the following lemma by considering the action of the trivial character of G on the equations of Lemma 2.2.

LEMMA 2.3. *Let* $n+1 \le i, i' \le 3n$. *Then* $\sum_{n+1 \le j \le 3n} m_{ij} m_{i'j}$

$$= \begin{cases} n & \textit{if } i = i', \\ 0 & \textit{if } \{i, i'\} \in \{\{n+1, 2n+1\}, \{n+2, 2n+2\}, \cdots, \{2n, 3n\}\}, \\ \frac{1}{2}n & \textit{otherwise.} \end{cases}$$

LEMMA 2.4. (i) $l_{i\ j} = 0$ *or* 1 *for* $0 \le i, j \le 2n-1$.
(ii) *If* $0 \le i \le 2n-1, 0 \le j \le n-1$ *and* $(i,j) \notin \{(k+n-1) | 0 \le k \le n-1\}$ *then* $l_{i\ j} + l_{i\ j+n-1} = 1$.
(iii) *If* $0 \le i \le n-1, 0 \le j \le 2n-1$ *and* $(i,j) \notin \{(k, k+n-1) | 0 \le k \le n-1\}$ *then* $l_{i\ j} + l_{i+n-1\ j} = 1$.

Proof. (i) follows from Lemma 2.3. Since G is an elation group with the center P_0 and the axis l_0, we get (ii) and (iii).

We may assume that $l_{0\ 0} = l_{0\ 1} = \cdots = l_{0\ n-1} = 1$ and $l_{1\ 0} = l_{2\ 0} = \cdots = l_{n-1\ 0} = 1$ by changing the labels of $\mathcal{P}_i$'s and $\mathcal{L}_j$'s appropriately. Therefore $l_{0\ n} = l_{0\ n+1} = \cdots = l_{0\ 2n-1} = 0$ and $l_{n\ 0} = l_{n+1\ 0} = \cdots = l_{2n-1\ 0} = 0$. Set $N = (l_{i\ j})_{1 \le i,j \le n-1}$.

LEMMA 2.5. *Let* $n \equiv 0 (mod\ 4)$. *Set* $k = \frac{1}{2}n - 1$ *and* $\lambda = \frac{1}{4}n - 1$. *Then*

$$N\ {}^tN = (k - \lambda)I + \lambda J$$

,where I *is the identity matrix and* J *is the all* 1 *matrix.*

Proof. Let $1 \le i \le n-1$. By considering the inner product of the 0 th row $\underbrace{1\cdots1}_{n}\underbrace{0\cdots0}_{n}$ of L and the i the row of L, we get $|\{1 \le k \le n-1 | l_{i\ k} = 1\}| = \frac{1}{2}n - 1$ by Lemma 2.3. Therefore the number of 1's contained in each row of N is $\frac{1}{2}n - 1$.

Next let $1 \le i < j \le n-1$. Then we want to show that the inner product of the i th row of N and j th row of N is $\frac{1}{4}n - 1$. By changing the labels of $\mathcal{P}_r$'s we may assume that the i th row of N is $\underbrace{1\cdots1}_{\frac{1}{2}n-1}\underbrace{0\cdots0}_{\frac{1}{2}n}$ and the j th row of N is $\underbrace{1\cdots1}_{r_1}\ \underbrace{0\cdots0}_{\frac{1}{2}n-1-r_1}\ \underbrace{1\cdots1}_{r_2}\ \underbrace{0\cdots0}_{\frac{1}{2}n-r_2}$. Then the i th row of L is $\underbrace{1\cdots1}_{\frac{1}{2}n}\ \underbrace{0\cdots0}_{n}\ \underbrace{1\cdots1}_{\frac{1}{2}n}$ and the j th row of L is $\underbrace{1\cdots1}_{r_1+1}\ \underbrace{0\cdots0}_{\frac{1}{2}n-1-r_1}\ \underbrace{1\cdots1}_{r_2}\ \underbrace{0\cdots0}_{\frac{1}{2}n-r_2+1+r_1}\ \underbrace{1\cdots1}_{\frac{1}{2}n-1-r_1}\ \underbrace{0\cdots0}_{r_2}$

$\underbrace{1\cdots 1}_{\frac{1}{2}n-r_2}$. By the above argument, we have

$$(\star)\ \ r_1 + r_2 = \frac{1}{2}n - 1.$$

By considering the inner product of the i th row of L and j th row of L, we get $1 + r_1 + \frac{1}{2}n - r_2 = \frac{1}{2}n$. Therefore

$$(\star\star)\ \ r_1 - r_2 = -1.$$

From $(\star\star)$ and $(\star)$, we have $r_1 = \frac{1}{2}n - 1$. Therefore $(l_{i\,1}\ l_{i\,2} \cdots l_{i\,n-1})\ {}^t(l_{j\,1}\ l_{j\,2} \cdots l_{j\,n-1}) = r_1 = \frac{1}{2}n - 1$. Thus the lemma holds.

EXAMPLE 2.6. *If $n = 8$, then for example*

$$L = \begin{pmatrix}
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&0&0&0&1&0&1&0&0&1&1&1&0&1&0\\
1&1&1&0&0&0&1&0&0&0&0&1&1&1&0&1\\
1&0&1&1&0&0&0&1&0&1&0&0&1&1&1&0\\
1&1&0&1&1&0&0&0&0&0&1&0&0&1&1&1\\
1&0&1&0&1&1&0&0&0&1&0&1&0&0&1&1\\
1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1\\
1&0&0&0&1&0&1&1&0&1&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&1&1&1&0&1&0&1&1&0&0&0&1&0&1\\
0&0&0&1&1&1&0&1&1&1&1&0&0&0&1&0\\
0&1&0&0&1&1&1&0&1&0&1&1&0&0&0&1\\
0&0&1&0&0&1&1&1&1&1&0&1&1&0&0&0\\
0&1&0&1&0&0&1&1&1&0&1&0&1&1&0&0\\
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0\\
0&1&1&1&0&1&0&0&1&0&0&0&1&0&1&1
\end{pmatrix}.$$

Lemma 2.5 yields the following theorem.

THEOREM 2.7. *Let Π be a projective plane of order $n(\geq 8)$. If $n \equiv 0(mod\ 4)$ and Π has an elation group G of order $\frac{1}{2}n$ with a common center P_0 and a common axis l_0, then there exists a Hadamard design with parameters $(n -$*

$1, \frac{1}{2}n-1, \frac{1}{4}n-1)$.

Ko and Ray-Chaudhuri [4] and Arasu and Jungnickel [1] gave some correspondences from affine difference sets of order n with $n \equiv 0 (mod\ 4)$ to Hadamard designs with parameters $(n-1, \frac{1}{2}n-1, \frac{1}{4}n-1)$. But the correspondence used in the proof of Theorem 2.7 is differ from these correspondences. Since translation planes of even order satisfy the assumption of Theorem 2.7, we get the following result.

COROLLARY 2.8. *Let $\mathcal{S} = \{W_i | -1 \leq i \leq n-1\}$ be a spread of a $2m$ dimensional vector space V over $GF(q)$(that is, $\mathcal{S}$ be a set of $q^m + 1$ mutually disjoint m-dimensional subspaces of V), where $q = 2^e, me \geq 3$ and $n = q^m$. Let U be a subgroup of $(W_{-1}, +)$ of order $\frac{1}{2}n$. Then $N = (l_{i\,j})_{1 \leq i,j \leq n-1}$ is an incidence matrix of a Hadamard design with parameters $(n-1, \frac{1}{2}n-1, \frac{1}{4}n-1)$, where* $l_{i\,j} = \begin{cases} 1 & \text{if } v_j \in U + W_i, \\ 0 & \text{otherwise} \end{cases}$ *for* $i, j, 1 \leq i, j \leq n-1$.

3. n=12

In this section we consider the converse statement of Theorem 2.7. We get the following theorem from section 2.

THEOREM 3.1. *Let $n \equiv 0 (mod\ 4) (n \geq 8)$, G be a group of order $\frac{1}{2}n$ and $N = (l_{ij})_{1 \leq i,j \leq n-1}$ be an incidence matrix of a Hadamard design with parameters $(n-1, \frac{1}{2}n-1, \frac{1}{4}n-1)$. Set $S = (s_{i\,j})_{0 \leq i,j \leq n-1}$, where $s_{i\,j} = l_{i\,j}$ for $i, j, 1 \leq i, j \leq n-1$ and $s_{0\,i} = s_{i\,0} = 1$ for $i, 0 \leq i \leq n-1$. Set $S' = (s_{i\,j}{}')_{0 \leq i,j \leq n-1}$ where $s_{i\,j} + s_{i\,j}{}' = 1$ for $i, j, 0 \leq i, j \leq n-1$ and*

$$L = \begin{pmatrix} S & S' \\ S' & S \end{pmatrix} = (l_{i\,j})_{0 \leq i,j \leq 2n-1}.$$

If

$$D_{i\,j} = \begin{cases} \{g_{i\,j}\} \text{ for some } g_{i\,j} \in G & \text{if } l_{i\,j} = 1, \\ \phi & \text{otherwise} \end{cases}$$

$$(0 \leq i, j \leq 2n-1)$$

satisfy the condition

$$(\star) \quad \sum_{0 \leq j \leq 2n-1} \widehat{D_{i\,j}}^{-1} \widehat{D_{i'\,j}} = \widehat{G}$$

for i, i' such that $0 \le i \ne i' \le 2n-1$ and $\{i,i'\} \notin \{\{0,n\},\{1,n+1\},\cdots,\{n-1,2n-1\}\}$, then there exists a projective plane Π of order n such that G induces an elation group of Π of order $\frac{1}{2}n$ with a common center and a common axis.

In the rest of this section, we consider the case $n = 12$. Suppose that there exist a group G, a matrix $L = \begin{pmatrix} S & S' \\ S' & S \end{pmatrix} = (l_{i\ j})_{0 \le i,j \le 23}$ and $D_{i\ j}$ $(0 \le i, j \le 23)$ satisfying the conditions of Theorem 3.1(see [3]). Since the Hadamard design with parameters $(11, 5, 2)$ exists only one, we may assume that

$$S = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&0&1&1&1&0&0&0&1&0\\
1&0&0&1&0&1&1&1&0&0&0&1\\
1&1&0&0&1&0&1&1&1&0&0&0\\
1&0&1&0&0&1&0&1&1&1&0&0\\
1&0&0&1&0&0&1&0&1&1&1&0\\
1&0&0&0&1&0&0&1&0&1&1&1\\
1&1&0&0&0&1&0&0&1&0&1&1\\
1&1&1&0&0&0&1&0&0&1&0&1\\
1&1&1&1&0&0&0&1&0&0&1&0\\
1&0&1&1&1&0&0&0&1&0&0&1\\
1&1&0&1&1&1&0&0&0&1&0&0
\end{pmatrix}.$$

We consider the case that G is abelian. Then $(G, \cdot) \cong (Z_6, +)$. We want to determine $D_{i\ j}$'s $(0 \le i \le 7, 0 \le j \le 23)$. By changing the base points and the base lines and the generator of G, if necessary, we can write $(D_{i\ j})_{0 \le i \le 7, 0 \le j \le 23} =$

$$\left(\begin{array}{cccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&&&&&&&&&&&&\\
0&&a_0&&a_1&a_2&a_3&&&&a_4&&&0&&0&&&&0&0&0&&0\\
0&&&b_0&&b_1&b_2&b_3&&&&b_4&&b_5&0&&0&&&&b_6&b_7&0&\\
0&c_0&&&c_1&&c_2&c_3&c_4&&&&&&c_5&c_6&&0&&&&c_7&c_8&c_9\\
0&&d_0&&&d_1&&d_2&d_3&d_4&&&&d_5&&d_6&d_7&&0&&&&d_8&d_9\\
0&&&e_0&&&e_1&&e_2&e_3&e_4&&&e_5&e_6&&e_7&e_8&&e_9&&&&e_{10}\\
0&&&&f_0&&&f_1&&f_2&f_3&f_4&&f_5&f_6&f_7&&f_8&f_9&&f_{10}&&&\\
0&g_0&&&&g_1&&&g_2&&g_3&g_4&&&g_5&g_6&g_7&&g_8&g_9&&g_{10}&&
\end{array}\right)$$

, where $a_0 \in \{1,2,3\}$. Here we omit entries $D_{i\ j}$'s if they are empty sets and the notation $\{\ \}$ of each set $D_{i\ j} = \{*\}$. For example, $\sum_{0 \le j \le 23} \widehat{D_{2\ j}}^{-1} \widehat{D_{1\ j}} = \widehat{G}$ means $\{0, a_2 - b_1, a_3 - b_2, -b_5, -b_6, -b_7\} = Z_6$. But it follows that there is no $\{a_i\}, \{b_i\}$,
$\{c_i\}, \{d_i\}, \{e_i\}$, $\{f_i\}$ and $\{g_i\}$ satisfying the condition $(\star)$ of Theorem 3.1, using programs of sortings. Therefore we have the following theorem.

THEOREM 3.2. *There is no projective plane of order* 12 *with an elation of order* 6.

4. The Homology Type

Throught this section we assume the following.

HYPOTHESIS 4.1. *Let* $\Pi = (\mathcal{P}, \mathcal{L})$ *be a projective plane of odd order* $n (\geq 7)$. *Let* G *be a homology group of* Π *of order* $\frac{1}{2}(n-1)$ *with a common center* P_0 *and a common axis* l_0.

Let $(l_0) = \{P_1, P_2, \cdots, P_{n+1}\}$ and $l_i = P_0 P_i$ $(1 \leq i \leq n+1)$. Let $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{3n+3}$ be G-orbits on $\mathcal{P}$ and $\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_{3n+3}$ G-orbits on $\mathcal{L}$. Then we may assume the following:

$$\mathcal{P}_i = \{P_i\}, \quad \mathcal{L}_i = \{l_i\} \quad (0 \leq i \leq n+1),$$

$$|\mathcal{P}_i| = |\mathcal{L}_i| = \frac{1}{2}(n-1) \quad (n+2 \leq i \leq 3n+3).$$

$$(l_i) = \mathcal{P}_{n+i+1} \cup \mathcal{P}_{2n+i+2}, \quad (P_i) = \mathcal{L}_{n+i+1} \cup \mathcal{L}_{2n+i+2} \quad (1 \leq i \leq n+1).$$

If Ω is a point orbit and Δ is a line orbit of G, set $(\Omega\ \Delta) = |\Omega \cap (l)|$, where l is a line in Δ.

Set $m_{i\ j} = (\mathcal{P}_j\ \mathcal{L}_i)$ for $i, j, 0 \leq i, j \leq 3n+3$, $M = (m_{i\ j})_{0 \leq i,j \leq 3n+3}$ and $L = (m_{i\ j})_{n+2 \leq i,j \leq 3n+3} = (l_{i\ j})_{0 \leq i,j \leq 2n+1}$. Choose a point $P_i \in \mathcal{P}_i$ and a line $l_i \in \mathcal{L}_i$ for i, $n+2 \leq i \leq 3n+3$. Set $D_{i\ j} = \{\mu \in G | P_j^{\mu} \in (l_i)\}$ for i, $n+2 \leq i \leq 3n+3$. Clearly $|D_{i\ j}| = m_{i\ j}$. We get the following three lemmas by the similar arguments as in section2.

LEMMA 4.2. *Let* $n+2 \leq i, i' \leq 3n+3$. *Then* $\sum_{n+2 \leq j \leq 3n+3} \widehat{D_{i\ j}}^{-1} \widehat{D_{i'\ j}}$

$$= \begin{cases} n & \text{if } i = i', \\ 0 & \text{if } \{i, i'\} \in \{\{n+2, 2n+3\}, \{n+3, 2n+4\}, \cdots, \{2n+2, 3n+3\}\}, \\ \widehat{G} & \text{otherwise.} \end{cases}$$

LEMMA 4.3. *Let* $n+2 \leq i, i' \leq 3n+3$. *Then* $\sum_{n+2 \leq j \leq 3n+3} m_{i\ j} m_{i'\ j}$

$$= \begin{cases} n & \text{if } i = i', \\ 0 & \text{if } \{i, i'\} \in \{\{n+2, 2n+3\}, \{n+3, 2n+4\}, \\ & \quad \cdots, \{2n+2, 3n+3\}\}, \\ \frac{1}{2}(n-1) & \text{otherwise.} \end{cases}$$

LEMMA 4.4. (i) $l_{i\ j} = 0$ *or* 1 *for* $0 \leq i, j \leq 2n+1$.
(ii) $l_{0\ 0} = l_{1\ 1} = \cdots = l_{2n+1\ 2n+1} = 0$,
$l_{0\ n+1} = l_{1\ n+2} = \cdots = l_{n\ 2n+1} = 0$,
$l_{n+1\ 0} = l_{n+2\ 1} = \cdots = l_{2n+1\ n} = 0$.
(iii) *If* $0 \leq i \leq 2n+1$, $0 \leq j \leq n$ *and* $(i,j) \notin \{(k,k),$
$(k+n+1,k) | 0 \leq k \leq n\}$ *then* $l_{i\ j} + l_{i\ j+n+1} = 1$.
(iv) *If* $0 \leq i \leq n$, $0 \leq j \leq 2n+1$ *and* $(i,j) \notin \{(k,k),$
$(k,k+n+1) | 0 \leq k \leq n\}$ *then* $l_{i\ j} + l_{i+n+1\ j} = 1$.

We may assume that $l_{0\ 1} = l_{0\ 2} = \cdots = l_{0\ n} = 1$ and $l_{1\ 0} = l_{2\ 0} = \cdots = l_{n\ 0} = 1$ by changing the labels of $\mathcal{P}_i$'s and $\mathcal{L}_j$'s appropriately. Therefore $l_{0\ n+1} = l_{0\ n+2} = \cdots = l_{0\ 2n+1} = 0$ and $l_{n+1\ 0} = l_{n+2\ 0} = \cdots = l_{2n+1\ 0}$. Set $N = (l_{i\ j})_{1 \leq i,j \leq n}$. Then we have the following lemma by the similar argument as in the proof of Lemma 2.5.

LEMMA 4.5. *Let* $n \equiv 3 (mod\ 4)$. *Set* $k = \frac{1}{2}(n-1)$ *and* $\lambda = \frac{1}{4}(n-3)$. *Then*

$$N\ {}^tN = (k-\lambda)I + \lambda L$$

, *where* I *is the identity matrix and* J *is the all* 1 *matrix.*

Lemmas 4.3, 4.4 and 4.5 yield the following lemma.

LEMMA 4.6. $(\star\star\star)$ $L\ {}^tL = \frac{1}{2}(n-1)J + \frac{1}{2}\begin{pmatrix} (n+1)I & (n-1)I \\ (n-1)I & (n+1)I \end{pmatrix}$, *where* J *is the all* 1 *matrix of degree* $2n+2$ *and* I *is the identity matrix of degree* $n+1$.

EXAMPLE 4.7. *If $n = 7$, then for example*

$$L = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

Lemma 4.5 yields the following theorem.

THEOREM 4.8. *Let Π be a projective plane of order $n(\geq 7)$. If $n \equiv 3(mod\ 4)$ and Π has a homology group G of order $\frac{1}{2}(n-1)$ with a common center P_0 and a common axis l_0, then there exists a Hadamard design with parameters $(n, \frac{1}{2}(n-1), \frac{1}{4}(n-3))$.*

Nearfield planes of order $n(\geq 7)$ with $n \equiv 3(mod\ 4)$ satisfy the assumtion of Theorem 4.8. Therefore we have the following corollary.

COROLLARY 4.9(cf. [2, p.97]). *Let $R = \{a_1, a_2, \cdots, a_q\}$ be a nearfield of order $q(\geq 7)$ with $q \equiv 3\ (mod\ 4)$ and S a subgroup of $(R - \{0\}, \cdot)$ of order $\frac{1}{2}(q-1)$. Set $m_{i\ j} = \begin{cases} 1 & \text{if } a_j - a_i \in S, \\ 0 & \text{otherwise} \end{cases}$ for $i, j, 1 \leq i, j \leq q$. Then $M = (m_{i\ j})_{1 \leq i,j \leq q}$ is an incidence matrix of a Hadamard design with parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-3))$.*

Especially, when R is a finite field, the Hadamard design defined in Corollary 4.9 is known as Paley type. Thus we have the following corollary.

COROLLARY 4.10. *The Hadamard design corresponding to Desarguesian plane of order* q *with* $q \equiv 3 \pmod 4$ *is Paley type with parameters* $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-3))$.

5. n=11

In this section we consider the converse statement of Theorem 4.8. We get the following theorem from section 4.

THEOREM 5.1. *Let* $n \equiv 3 (mod\ 4)(n \geq 7)$, G *be a group of order* $\frac{1}{2}(n-1)$ *and* $N = (l_{i\ j})_{1\leq i,j\leq n}$ *be an incidence matrix of a Hadamard design with parameters* $(n, \frac{1}{2}(n-1), \frac{1}{4}(n-3))$ *such that* $l_{i\ i} = 0$ *for* $i, 1 \leq i \leq n$. *Set* $S = (s_{i\ j})_{0\leq i,j\leq n}$, *where* $s_{i\ j} = l_{i\ j}$ *for* $i, j, 1 \leq i, j \leq n$, $s_{0\ 0} = 0$ *and* $s_{0\ i} = s_{i\ 0} = 1$ *for* $i, 1 \leq i \leq n$. it Set $S' = (s_{i\ j}{}')_{0\leq i,j\leq n}$ *where* $s_{i\ j} + s_{i\ j}{}' = 1$ *for* $i, j, 0 \leq i \neq j \leq n$ *and* $s_{i\ j}{}' = 0$ *for* $i, 0 \leq i \leq n$. *Suppose that the equation* $(\star\star\star)$ *of Lemma* 4.6. *Then if*

$$D_{i\ j} = \begin{cases} \{g_{i\ j}\} \text{ for some } g_{i\ j} \in G & \text{if } l_{i\ j} = 1, \\ \phantom{\{g_{i\ j}\} \text{ for some }} \phi & \text{otherewise} \end{cases}$$

$$(0 \leq i, j \leq 2n+1)$$

satisfy the condition

$$(\star) \sum\nolimits_{0\leq j\leq 2n+1} \widehat{D_{i\ j}^{-1}} \widehat{D_{i'\ j}} = \widehat{G}$$

for i, i' *such that* $0 \leq i \neq i' \leq 2n+1$ *and* $\{i, i'\} \notin \{\{0, n+1\}, \{1, n+2\}, \cdots, \{n, 2n+1\}\}$, *then there exists a projective plane* Π *of order* n *such that* G *induces a homology group of* Π *of order* $\frac{1}{2}(n-1)$ *with a common center and a common axis.*

In the rest of this section, we consider the case $n = 11$. Suppose that there exist a group $(G, \cdot) \cong (Z_5, +)$, a matrix $L = \begin{pmatrix} S & S' \\ S' & S \end{pmatrix} = (l_{i\ j})_{0\leq i,j\leq 23}$ and $D_{i\ j}(0 \leq i, j \leq 23)$ satisfying the conditions of Theorem 5.2. Then there is only one possibility for S up to equivalence.

$$S = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We want to determine $D_{i\ j}$'s satisfying the condition $(\star)$ of Theorem 5.2. By changing the base points, the base lines and the generator of G, we may assume that $D_{i\ 0} = D_{0\ i} = D_{i+11\ 0} = D_{0\ i+11} = \{0\}$ for $i, 1 \leq i \leq 11$, $D_{1\ 2} = D_{1\ 19} = D_{13\ 14} = \{0\}$ and $D_{1\ 3} = \{1\}$. Using a computer, we get exactly one possibility for $(D_{i\ j})_{0 \leq i,j \leq 23}$. Therefore we have the following theorem.

THEOREM 5.3. *The projective plane of order* 11 *with a homology of order* 5 *is desarguesian.*

References

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