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# An introduction to Leonard pairs and Leonard systems

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**Abstract**

Let $\mathcal{F}$ denote a field, and let $V$ denote a finite dimensional vector space over $\mathcal{F}$. We consider an ordered pair $(A, A^*)$, where $A$ and $A^*$ are $\mathcal{F}$-linear transformations from $V$ to $V$ that satisfy conditions (i), (ii) below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal, and the matrix representing $A^*$ is irreducible tridiagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is diagonal, and the matrix representing $A$ is irreducible tridiagonal.

We call such a pair a *Leonard pair* on $V$. We present a classification of Leonard pairs. We obtain Leonard pairs from irreducible representations of the quantum Lie algebra $U_q(sl_2)$. We show any Leonard pair satisfy two polynomial relations called the Askey-Wilson relations. We obtain Leonard pairs from five families of classical posets.

## 1 Introduction

Throughout this talk, $\mathcal{F}$ will denote an arbitrary field.

**Definition 1.1** *Let $V$ denote a finite dimensional vector space over $\mathcal{F}$. By a Leonard pair on $V$, we mean an ordered pair $(A, A^*)$, where $A$ and $A^*$ are $\mathcal{F}$-linear transformations from $V$ to $V$ satisfying (i), (ii) below.*

*(i) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is diagonal, and the matrix representing $A$ is irreducible tridiagonal.*

*(ii) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal, and the matrix representing $A^*$ is irreducible tridiagonal.*

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*(A tridiagonal matrix is said to be irreducible whenever all entries immediately above and below the main diagonal are nonzero).*

Here is an example of a Leonard pair. Set $V = \mathcal{F}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view $A$ and $A^*$ as linear transformations on $V$. We assume the characteristic of $\mathcal{F}$ is not 2 or 3, to insure $A$ is irreducible. Then $(A, A^*)$ is a Leonard pair on $V$. Indeed, condition (i) of Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix $P$ such that $P^{-1}AP$ is diagonal, and such that $P^{-1}A^*P$ is irreducible tridiagonal. Put

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, so $P^{-1}$ exists. Also by matrix multiplication,

$$AP = PA^*.$$

Apparently $P^{-1}AP$ equals $A^*$, and is therefor diagonal. By the above line, and since $P^{-1}$ is a scalar multiple of $P$, we find $P^{-1}A^*P$ equals $A$, and is therefor irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of $P$.

Referring to the above example, apparently the eigenvalues of $A^*$ (and $A$) are $3, 1, -1, -3$, and we observe these are distinct. This will always be the case. In fact, it is an easy exercise to show the following.

**Lemma 1.2** *With reference to Definition 1.1, let $(A, A^*)$ denote a Leonard pair on $V$. Then the eigenvalues of $A$ are distinct, and contained in $\mathcal{F}$. Moreover, the eigenvalues of $A^*$ are distinct, and contained in $\mathcal{F}$.*

When studying Leonard pairs, it is often convenient to consider a related and somewhat more abstract object, which we call a *Leonard system.* To define this, we need a few terms. Let $d$ denote a nonnegative integer, and let $\mathrm{Mat}_{d+1}(\mathcal{F})$ denote the $\mathcal{F}$-algebra consisting of all $d+1$ by $d+1$ matrices with entries in $\mathcal{F}$. We view the rows and columns as indexed by $0, 1, \ldots, d$. For the rest of this talk, $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathcal{F})$. An element $A \in \mathcal{A}$ will be called *multiplicity-free* whenever it has $d+1$ distinct eigenvalues, all of which are in $\mathcal{F}$. Assume $A$ is multiplicity free, and let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Then $\mathcal{D}$ has a basis $E_0, E_1, \ldots, E_d$ such that

$$E_iE_j = \delta_{ij}E_i \qquad (0 \le i, j \le d),$$

$$\sum_{i=0}^{d} E_i = I.$$

The elements $E_0, E_1, \ldots, E_d$ are unique up to permutation, and are called the *primitive idempotents* of $A$.

**Definition 1.3** *Let $d$ denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard System in $\mathcal{A}$, we mean a sequence*

$$\Phi = (A;\ E_0,\ E_1,\ \ldots,\ E_d;\ A^*;\ E_0^*,\ E_1^*,\ \ldots,\ E_d^*) \tag{1}$$

*that satisfies (i)–(v) below.*

*(i)* $A,\ A^*$ *are both multiplicity-free elements in* $\mathcal{A}$.

*(ii)* $E_0,\ E_1,\ \ldots,\ E_d$ *is an ordering of the primitive idempotents of* $A$.

*(iii)* $E_0^*,\ E_1^*,\ \ldots,\ E_d^*$ *is an ordering of the primitive idempotents of* $A^*$.

*(iv)* $E_iA^*E_j = \begin{cases} 0, & \text{if } |i-j|>1; \\ \neq 0, & \text{if } |i-j|=1 \end{cases} \qquad (0 \le i,j \le d).$

*(v)* $E_i^*AE_j^* = \begin{cases} 0, & \text{if } |i-j|>1; \\ \neq 0, & \text{if } |i-j|=1 \end{cases} \qquad (0 \le i,j \le d).$

*We refer to $d$ as the diameter of $\Phi$, and say $\Phi$ is over $\mathcal{F}$.*

To see the connection between Leonard pairs and Leonard systems, observe conditions (ii), (iv) above assert that with respect to an appropriate basis consisting of eigenvectors for $A$, the matrix representing $A^*$ is irreducible tridiagonal. Similarily, conditions (iii), (v) assert that with respect to an appropriate basis consisting of eigenvectors for $A^*$, the matrix representing $A$ is irreducible tridiagonal.

**Definition 1.4** *Let the Leonard system $\Phi$ be as in (1). We let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp. $E_i^*$), for $0 \le i \le d$. We call $\theta_0, \theta_1, \ldots, \theta_d$ the eigenvalue sequence of $\Phi$. We call $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the dual eigenvalue sequence of $\Phi$.*

Given a Leonard system

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*),$$

we can get more Leonard systems. For example

$$\begin{aligned} \Phi^* &:= (A^*;\ E_0^*,\ E_1^*,\ \ldots,\ E_d^*;\ A;\ E_0,\ E_1,\ \ldots,\ E_d), \\ \Phi^{\downarrow} &:= (A;\ E_0,\ E_1,\ \ldots,\ E_d;\ A^*;\ E_d^*,\ E_{d-1}^*,\ \ldots,\ E_0^*), \\ \Phi^{\Downarrow} &:= (A;\ E_d,\ E_{d-1},\ \ldots,\ E_0;\ A^*;\ E_0^*,\ E_1^*,\ \ldots,\ E_d^*) \end{aligned}$$

are Leonard systems. Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$\begin{gathered} *^2 = \downarrow^2 = \Downarrow^2 = 1, \\ \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \Downarrow\downarrow . \end{gathered}$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the above relations is the dihedral group $D_4$. We recall $D_4$ is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of $D_4$ on the set of all Leonard systems. We say two Leonard systems are *relatives* whenever they are in the same orbit of this $D_4$ action.

In view of our above comments, when we discuss Leonard systems, we are often not interested in the orderings of the primitive idempotents involved; we just care how $A$ and $A^*$ interact. This brings us back to the notion of a Leonard pair.

**Definition 1.5** *Let $d$ denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard pair in $\mathcal{A}$, we mean an ordered pair $(A, A^*)$ such that*

*(i) $A, A^*$ are both multiplicity free elements of $\mathcal{A}$, and*

*(ii) There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $A$, and there exists an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of $A^*$, such that $(A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ is a Leonard System.*

## 2 A classification of Leonard systems

When studying a Leonard system $\Phi$, it is often useful to examine a second Leonard system that is isomorphic to $\Phi$ but in a particularily nice form. We present such a 'canonical form'. To describe it, we use the following notation. Let

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$$

denote a Leonard system in $\mathcal{A}$, and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathcal{F}$-algebras. Then we write

$$\Phi^\sigma := (A^\sigma; E_0^\sigma, E_1^\sigma, \ldots, E_d^\sigma; A^{*\sigma}; E_0^{*\sigma}, E_1^{*\sigma}, \ldots, E_d^{*\sigma}),$$

and observe $\Phi^\sigma$ is a Leonard system in $\mathcal{A}'$.

Let us say a matrix $X \in \mathrm{Mat}_{d+1}(\mathcal{F})$ is *lower di-diagonal* whenever

$$X_{ij} \neq 0 \quad \to \quad i - j \in \{0, 1\} \qquad (0 \leq i, j \leq d).$$

That is, $X$ is lower di-diagonal whenever each nonzero entry lies either on or immediately below the main diagonal. We say $X$ is *upper di-diagonal* whenever the transpose $X^t$ is lower di-diagonal.

Let $\Phi$ denote the Leonard system in (1). We say $\Phi$ is in *split canonical form* whenever (i)–(iii) hold below.

(i) $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathcal{F})$.

(ii) $A$ is lower di-diagonal, with $A_{i,i-1} = 1$ for $1 \leq i \leq d$, and $A_{ii} = \theta_i$ for $0 \leq i \leq d$, where $\theta_i$ denotes the eigenvalue of $A$ associated with $E_i$.

(iii) $A^*$ is upper di-diagonal, with $A^*_{ii} = \theta^*_i$ for $0 \leq i \leq d$, where $\theta^*_i$ denotes the eigenvalue of $A^*$ associated with $E^*_i$.

We show there there exists a unique isomorphism of $\mathcal{F}$-algebras $\heartsuit : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathcal{F})$ such that $\Phi^\heartsuit$ is in split canonical form. Apparently

$$A^\heartsuit = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\heartsuit} = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & 0 \\ & \theta^*_1 & \varphi_2 & & \\ & & \theta^*_2 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta^*_d \end{pmatrix},$$

where $\varphi_1, \varphi_2, \ldots, \varphi_d$ are appropriate scalars in $\mathcal{F}$. We call $\varphi_1, \varphi_2, \ldots, \varphi_d$ the *$\varphi$-sequence* of $\Phi$. Let $\phi_1, \phi_2, \ldots, \phi_d$ denote the $\varphi$-sequence for $\Phi^\Downarrow$. Then abbreviating $\diamondsuit := \heartsuit(\Phi^\Downarrow)$, we have

$$A^\diamondsuit = \begin{pmatrix} \theta_d & & & & 0 \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \theta_0 \end{pmatrix}, \qquad A^{*\diamondsuit} = \begin{pmatrix} \theta^*_0 & \phi_1 & & & 0 \\ & \theta^*_1 & \phi_2 & & \\ & & \theta^*_2 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \phi_d \\ 0 & & & & & \theta^*_d \end{pmatrix}.$$

We call $\phi_1, \phi_2, \ldots, \phi_d$ the *$\phi$-sequence* of $\Phi$.

We obtain the following classification of Leonard systems.

**Theorem 2.1** *[7] Let $d$ denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let*

$$\begin{array}{ll} \theta_0, \theta_1, \ldots, \theta_d; & \theta^*_0, \theta^*_1, \ldots, \theta^*_d; \\ \varphi_1, \varphi_2, \ldots, \varphi_d; & \phi_1, \phi_2, \ldots, \phi_d \end{array}$$

*denote scalars in $\mathcal{F}$. Then there exists a Leonard System $\Phi$ over $\mathcal{F}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, $\varphi$-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and $\phi$-sequence $\phi_1, \phi_2, \ldots, \phi_d$ if and only if (i)–(v) hold below.*

*(i)* $\varphi_i \neq 0, \quad \phi_i \neq 0 \qquad (1 \leq i \leq d),$

*(ii)* $\theta_i \neq \theta_j, \quad \theta^*_i \neq \theta^*_j \quad$ *if* $\; i \neq j, \qquad (0 \leq i, j \leq d),$

*(iii)* $\displaystyle \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d),$

*(iv)* $\displaystyle \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d),$

*(v) The expressions*

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} \tag{2}$$

*are equal and independent of* $i$*, for* $2 \le i \le d-1$.

*Moreover, if (i)–(v) hold above then* $\Phi$ *is unique up to isomorphism of Leonard Systems.*

From the above theorem, we routinely obtain the following corollary.

**Corollary 2.2** *[7] Let* $d$ *denote a nonnegative integer, and let* $\mathcal{F}$ *denote a field. Let* $A$ *and* $A^*$ *denote any matrices in* $Mat_{d+1}(\mathcal{F})$ *of the form*

$$A = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & 0 \\ & \theta^*_1 & \varphi_2 & & \\ & & \theta^*_2 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta^*_d \end{pmatrix}.$$

*Then the following are equivalent.*

*(i)* $(A, A^*)$ *is a Leonard pair.*

*(ii) There exist scalars* $\phi_1, \phi_2, \ldots, \phi_d$ *in* $\mathcal{F}$ *such that conditions (i)–(v) hold in Theorem 2.1.*

# 3 The quantum Lie algebra $U_q(sl_2)$

In this section, we obtain Leonard pairs from irreducible representations of the quantum Lie algebra $U_q(sl_2)$. Throughout this section, we assume our ground field $\mathcal{F}$ is algebraically closed with characteristic zero. We let $q$ denote a nonzero element in $\mathcal{F}$, and assume $q$ is not a root of 1.

Recall $U_q(sl_2)$ is the associative $\mathcal{F}$-algebra with 1 generated by symbols $e, f, k, k^{-1}$ subject to the relations

$$kk^{-1} = k^{-1}k = 1, \tag{3}$$

$$ke = q^2 ek, \qquad kf = q^{-2} fk, \tag{4}$$

$$ef - fe = \frac{k-k^{-1}}{q-q^{-1}}. \tag{5}$$

Let $d$ denote a nonnegative integer, and put

$$E = \begin{pmatrix} 0 & [d] & & & & \mathbf{0} \\ & 0 & [d-1] & & & \\ & & 0 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & [1] \\ \mathbf{0} & & & & & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & & & & & \mathbf{0} \\ [1] & 0 & & & & \\ & [2] & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & [d] & 0 \end{pmatrix},$$

where

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}} \qquad (\forall i \in \mathbb{Z}).$$

Also put

$$K = \operatorname{diag}(q^d, q^{d-2}, q^{d-4}, \ldots, q^{-d}).$$

Then $K$ is invertible, and $E$, $F$, $K$ satisfy the equations (4), (5), so they support a representation of $U_q(sl_2)$. It can be shown the representation is irreducible.

Let $\alpha$ and $\alpha^*$ denote nonzero elements in $\mathcal{F}$ such that $\alpha\alpha^*$ is not a power of $q$, and put

$$\begin{aligned} A &= \alpha F + \frac{K}{q - q^{-1}}, \\ A^* &= \alpha^* E + \frac{K^{-1}}{q - q^{-1}}. \end{aligned}$$

We claim $(A, A^*)$ is a Leonard pair. To see this, let $\sigma$ denote the automorphism of $\mathrm{Mat}_{d+1}(\mathcal{F})$ satisfying

$$X^\sigma = D^{-1} X D \qquad (\forall X \in \mathrm{Mat}_{d+1}(\mathcal{F})),$$

where $D$ is the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathcal{F})$ with entries

$$D_{ii} = [1][2]\cdots[i]\alpha^i \qquad (0 \le i \le d).$$

Then

$$A^\sigma = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\sigma} = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix},$$

where

$$\theta_i = \frac{q^{d-2i}}{q - q^{-1}}, \qquad (0 \le i \le d), \tag{6}$$

$$\theta^*_i = \frac{q^{2i-d}}{q - q^{-1}}, \qquad (0 \le i \le d), \tag{7}$$

$$\varphi_i = [i][d-i+1]\alpha\alpha^* \qquad (1 \le i \le d). \tag{8}$$

Set

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d).$$

Evaluating this using (6)–(8), we obtain

$$\phi_i = [i][d-i+1](\alpha\alpha^* - q^{2i-d-1}) \qquad (1 \leq i \leq d).$$

One readily checks the above scalars $\theta_i, \theta^*_i, \varphi_i, \phi_i$ satisfy the conditions of Theorem 2.1, so $(A^\sigma, A^{*\sigma})$ is a Leonard pair by Corollary 2.2. Applying $\sigma^{-1}$, we find $(A, A^*)$ is a Leonard pair. For this example, it turns out

$$\begin{aligned} A^2A^* - (q^2 + q^{-2})AA^*A + A^*A^2 &= \omega A + \eta I, \\ A^{*2}A - (q^2 + q^{-2})A^*AA^* + AA^{*2} &= \omega A^* + \eta^* I, \end{aligned}$$

where

$$\eta = \alpha\alpha^* \frac{q + q^{-1}}{q - q^{-1}}, \qquad \eta^* = \alpha\alpha^* \frac{q + q^{-1}}{q - q^{-1}},$$
$$\omega = -1 - \alpha\alpha^*(q^{-d-1} + q^{d+1}). \qquad (9)$$

We comment there is a second Leonard pair associated with $U_q(sl_2)$. Let $\alpha, \alpha^*$ be as above, and put

$$\begin{aligned} B &= \alpha K - (1 - q^{-2})KE, \\ B^* &= \alpha^* K^{-1} - (1 - q^{-2})K^{-1}F. \end{aligned}$$

Then $(B, B^*)$ is a Leonard pair. The proof is similar, and omitted.

# 4 The Askey-Wilson relations

In the previous section, we obtained a Leonard pair whose elements $A$, $A^*$ satisfied two polynomial equations. It turns out every Leonard pair satisfies a similar pair of equations.

**Theorem 4.1** *[6] Let $d$ denote a nonnegative integer, let $\mathcal{F}$ denote any field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. Let $(A, A^*)$ denote a Leonard pair in $\mathcal{A}$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ from $\mathcal{F}$ such that*

$$\begin{aligned} A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* &= \gamma^* A^2 + \omega A + \eta I, \\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A &= \gamma A^{*2} + \omega A^* + \eta^* I. \end{aligned}$$

*The sequence is unique if $d \geq 3$.*

The above equations are known as the *Askey-Wilson* relations [1], [2], [3], [4], [5], [9], [10], [11].

Concerning the converse to the above theorem, we have the following.

**Theorem 4.2** *[6] Let $d$ denote a nonnegative integer, let $\mathcal{F}$ denote any field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. Let $A, A^*$ denote multiplicity free elements in $\mathcal{A}$, and assume the irreducible $\mathcal{A}$-module is irreducible as an $(A, A^*)$-module. Pick any scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ from $\mathcal{F}$, and assume $A, A^*$ satisfy the corresponding Askey-Wilson relations. Assume further that none of the following (i)–(iii) occur:*

*(i)* $q$ *is a primitive* $d+1^{st}$ *root of* $1$, *where* $q+q^{-1}=\beta$.

*(ii)* $\beta = 2$ *and* $d+1 = char(\mathcal{F})$.

*(iii)* $\beta = -2$ *and* $d+1 = 2\,char(\mathcal{F})$.

*Then $(A, A^*)$ is a Leonard pair in $\mathcal{A}$.*

# 5 Leonard pairs from the classical posets

There is a way to obtain Leonard pairs from the following classical posets: (i) the subset lattice, (ii) the subspace lattice, (iii) the Hamming semi-lattice, (iv) the attenuated spaces, (v) the classical polar spaces. For the definitions of these posets, see [8]. The argument in each case is similar. To illustrate it, we will consider the attenuated spaces in some detail.

**Definition 5.1** *Let $\mathcal{F}$ denote any field, let $V$ denote a finite dimensional vector space over $\mathcal{F}$, and let $A$ and $A^*$ denote $\mathcal{F}$-linear transformations from $V$ to $V$. We say $(A, A^*)$ is a generalized Leonard pair on $V$ whenever there exists a decomposition*

$$V = V_1 + V_2 + \cdots + V_n \qquad (\textit{direct sum}),$$

*such that*

$$AV_i \subseteq V_i, \qquad A^*V_i \subseteq V_i, \qquad (1 \leq i \leq n)$$

*and such that*

$$(A|_{V_i}, A^*|_{V_i}) \qquad \textit{is a Leonard Pair} \qquad (1 \leq i \leq n).$$

The posets mentioned above all support generalized Leonard pairs. In each case, the underlying vector space $V$ has the following form. Let $X$ denote a finite set. By $\mathcal{F}X$, we mean the vector space over $\mathcal{F}$ consisting of all formal sums

$$\sum_{x\in X} \alpha_x x,$$

where $\alpha_x \in \mathcal{F}$ for all $x \in X$.

We will be discussing posets, so let us recall some terms. let $P$ denote a poset. For all $x, y \in P$, we say $y$ *covers* $x$ whenever $x < y$, and there does not exist $z \in P$ such that $x < z < y$. In this case, we write $x \prec y$. Let $L$ denote the matrix in $\mathrm{Mat}_P(\mathbb{C})$ with entries

$$L_{xy} = \begin{cases} 1, & \text{if } x \prec y; \\ 0, & \text{if } x \not\prec y \end{cases} \qquad (\forall x, y \in P).$$

Viewing $L$ as a linear transformation on $\mathbb{C}P$,

$$Lx = \sum_{\substack{y \in P \\ y \prec x}} y \qquad (\forall x \in P).$$

We call $L$ the *lowering matrix* on $P$. Let $R$ denote the matrix in $\mathrm{Mat}_P(\mathbb{C})$ with entries

$$R_{xy} = \begin{cases} 1, & \text{if } y \prec x; \\ 0, & \text{if } y \not\prec x \end{cases} \qquad (\forall x, y \in P).$$

Viewing $R$ as a linear transformation on $\mathbb{C}P$,

$$Rx = \sum_{\substack{y \in P \\ x \prec y}} y \qquad (\forall x \in P).$$

We call $R$ the *raising matrix* on $P$. Now assume $P$ is ranked, with rank denoted $N$. For $0 \leq i \leq N$, let $F_i$ denote the diagonal matrix in $\mathrm{Mat}_P(\mathbb{C})$ with $yy$ entry

$$(F_i)_{yy} = \begin{cases} 1, & \text{if } \mathrm{rank}(y) = i; \\ 0, & \text{if } \mathrm{rank}(y) \neq i \end{cases} \qquad (\forall y \in P).$$

We refer to $F_i$ as the $i^{\text{th}}$ *projection matrix* of $P$. We observe

$$F_i F_j = \delta_{ij} F_i \qquad (0 \leq i, j \leq N),$$
$$F_0 + F_1 + \cdots + F_N = I.$$

Moreover,

$$F_i V = \mathrm{Span}\{x \in P \mid \mathrm{rank}(x) = i\} \qquad (0 \leq i \leq N),$$

where $V = \mathbb{C}P$.

For each of the five families of classical posets we mentioned at the outset, we obtain generalized Leonard pairs on $V = \mathbb{C}P$ of the form

$$A = \alpha R + \sum_{i=0}^{N} \theta_i F_i, \tag{10}$$

$$A^* = \alpha^* L + \sum_{i=0}^{N} \theta_i^* F_i, \tag{11}$$

where the $\alpha, \alpha^*, \theta_i, \theta_i^*$ are complex scalars.

To illustrate, we now restrict our attention to the attenuated space poset $A_q(N, M)$. This poset is defined as follows. Let $M$ and $N$ denote nonnegative integers, let $H$ denote a vector space of dimension $M + N$ over $GF(q)$, and fix a subspace $h \subseteq H$ of dimension $M$. Let $P$ denote the poset consisting of all subspaces $x$ of $H$ such that $x \cap h = 0$. The partial order on $P$ is

$$x \leq y \quad \text{whenever} \quad x \subseteq y \qquad (\forall x, y \in P).$$

The poset $P$ is ranked, with

$$\text{rank}(x) = \dim(x) \qquad (\forall x \in P).$$

Apparently, $P$ has rank $N$. For $0 \leq i \leq N$, each rank $i$ element of $P$ covers exactly

$$\frac{q^i - 1}{q - 1}$$

elements of $P$, and is covered by exactly

$$\frac{q^{N+M-i} - q^M}{q - 1}$$

elements in $P$. Moreover, it is shown in [8] that

$$\frac{q}{q+1} RL^2 \; - \; LRL \; + \; \frac{1}{q+1} L^2 R + f_i L \tag{12}$$

vanishes on $F_i V$, where $R$ and $L$ are the raising and lowering matrices, where $V = \mathbb{C}P$, and where

$$f_i = q^{N+M-i}. \tag{13}$$

Put

$$A \;=\; R + \sum_{i=0}^{N} \frac{q^i}{q-1} F_i, \tag{14}$$

$$A^* \;=\; \alpha^* L + \sum_{i=0}^{N} \frac{q^{-i}}{q-1} F_i, \tag{15}$$

where $\alpha^*$ is any scalar in $\mathbb{C}$ that is not one of $q^{-M-1}, q^{-M-2}, \ldots, q^{-M-N}$. We show $(A, A^*)$ is a generalized Leonard pair on $V$. Let $T$ denote the subalgebra of $\text{Mat}_P(\mathbb{C})$ generated by $R, L, F_0, F_1, \ldots, F_N$. Observe $R^t = L$, and each of $F_0, F_1, \ldots, F_N$ is symmetric, so $T$ is closed under the conjugate-transpose map. It follows $T$ is semi-simple, so $V$ is a direct sum of irreducible $T$-submodules. Let $W$ denote an irreducible $T$-submodule of $V$. The matrices $A$ and $A^*$ are contained in $T$ by (14), (15), so

$$AW \subseteq W, \qquad A^*W \subseteq W.$$

It remains to show that

$$(A|_W, A^*|_W)$$

is a Leonard pair on $W$. We do this as follows. Using (12), one can show there exists integers $r, p$ $(0 \le r \le p \le N)$ and a basis $w_r, w_{r+1}, \ldots, w_p$ for $W$ such that

(i) $w_i \in F_i V \qquad (r \le i \le p)$,

(ii) $Rw_i = w_{i+1} \qquad (r \le i < p), \quad Rw_p = 0$,

(iii) $Lw_i = x_i(r,p)w_{i-1} \qquad (r < i \le p), \quad Lw_r = 0$,

where

$$x_i(r,p) = \frac{q^{M+N-r-p-i+1}(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2} \tag{16}$$

for $r < i \le p$. Let $B$ (resp. $B^*$) denote the matrix representing $A$ (resp. $A^*$) with respect to the basis $w_r, w_{r+1}, \ldots, w_p$ . Apparently

$$B = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix}, \qquad B^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & 0 \\ & \theta^*_1 & \varphi_2 & & \\ & & \theta^*_2 & \ddots & \\ & & & \ddots & \varphi_d \\ 0 & & & & \theta^*_d \end{pmatrix},$$

where $d = p - r$, and where

$$\theta_i = \frac{q^{r+i}}{q-1}, \qquad \theta^*_i = \frac{q^{-r-i}}{q-1} \qquad (0 \le i \le d), \tag{17}$$

$$\varphi_i = \alpha^* x_{r+i}(r,p) \qquad (1 \le i \le d). \tag{18}$$

Set

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \le i \le d).$$

Evaluating this using (16), (17), and (18) we obtain

$$\phi_i = -\frac{(1-q^i)(1-q^{d-i+1})(1-\alpha^* q^{M+N+i-r-d})}{(q-1)^2 q^i} \qquad (1 \le i \le d).$$

One readily checks the above scalars $\theta_i, \theta^*_i, \varphi_i, \phi_i$ satisfy the conditions of Theorem 2.1, so $(B, B^*)$ is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$ by Corollary 2.2. It follows

$$(A|_W, A^*|_W)$$

is a Leonard pair on $W$. We have now shown $(A, A^*)$ is a generalized Leonard pair on $V$. We remark that by (12), (14), (15), we have

$$[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2] = 0,$$
$$[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2}] = 0$$

for this example.

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